Research Article

Quasi-Regular Graphs Associated with Commutative Rings

Nasr Zeyada,1,2 Najat Muthana1, and Sultanah Al-Rashidi1

1Department of Mathematics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia
2Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt

Correspondence should be addressed to Nasr Zeyada; nzeyada@gmail.com

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One of the most important branches of mathematics is algebraic graph theory, which solves graph problems with algebraic methods. In graph theory, several algebraic properties of a ring can be represented. In this paper, we define an innovative graph on rings, explore its characteristics, and examine how it relates to other notions in the field. Let $S$ be a ring; the quasi-regular graph of $S$ is a graph with a vertex set of $S - \{0\}$ and any two different vertices $w$ and $z$ are adjacent if $1 - wz$ is a unit in $S$. We study this graph by providing different examples and proving some crucial characteristics. This study provides important results and paves the way for a lot of different inquiries and studies utilizing this novel approach.

1. Introduction

The relationship between graph theory and rings has been extensively studied and has produced amazing results in algebraic graph theory. The investigation of the theoretical aspects of a graph over a ring graph was first thought of by Beck in 1988 [1]. He proposed the concept of a zero-divisor graph $\Gamma(S)$ for a commutative ring $S$. He defined $\Gamma(S)$ as an undirected graph with components at its vertices $Z(S)^* = Z(S) - \{0\}$, the set of nonzero zero-divisors of $S$, where any two vertices $w$ and $z$ are adjacent if $wz = 0$ [1]. The first main rearrangements of Beck’s zero-divisor graph were presented by Anderson and Livingston [2].

Grimaldi was the first to study the unit graph for $\mathbb{Z}_n$ [3]. The unit graph of $S$, denoted by $G(S)$, is the graph produced by making all elements of $S$ vertices and defining different vertices $w$ and $z$ to be adjacent if and only if $w + z \in U(S)$ where $U(S)$ is the set of unit elements of $S$.

One sort of graph related to rings, the unit graph was first introduced in 2010 by Ashraf et al. [4]. Such graph extended the unit graph $G(\mathbb{Z}_n)$ to $G(S)$ and derived numerous characterization findings for finite commutative rings in terms of connectedness, chromatic index, diameter, girth, and planarity of $G(S)$. If we remove the phrase “distinct” from the definition, we get the closed unit graph $\overline{G}(S)$, which may contain loops. It is worth noting that if $2 \notin U(S)$, then $\overline{G}(S) = G(S)$.

A graph’s diameter is an important invariant. Many articles in this field are devoted to the diameter of the generated graph; see, for example, [5, 6]. In 2014, Su and Zhou [7] proved that, for any arbitrary ring $S$, the girth of $G(S)$ is 3, 4, 6, or $\infty$ and recently explored the diameter of $G(S)$ with $S/I(S)$ self-injective ring $S$ and provided a thorough characterization of the diameter of $G(S) = 1, 2, 3$ or $\infty$, respectively [8].

In this paper, we introduce a new concept that we call a quasiregular graph and symbolize it with $Q(S)$, where $S$ is a commutative ring with a nonzero identity. Furthermore, we determine when a quasiregular graph is isomorphic to several well-known graphs. We also provide some number theory approaches to prove some of the properties of a quasiregular graph over $S$.

We refer to [9–12] for the undefined notions in this paper.

2. Quasiregular Graph

Definition 1. A quasiregular $Q(S)$ is a graph with vertex set $V(Q(S)) = S\{0\}$ in which any two distinct vertices $w$ and $z$ are adjacent if and only if $1 - wz$ is a unit in $S$. Therefore, the
edge set of \( Q(S) \) is \( E(Q(S)) = \{ wz: 1 - wz \text{ is a unit in } S \} \) for \\
\( w, z \in S \).

**Remark 1.** Using the notions of Jacobson radical of ring \( J(S) = \cap \{m: m \text{ is a maximal ideal of } (S, wz \in E(Q(S)) \text{ if and only if } w \in J(S) \text{ for any } z \in S \} \).

Hence, the elements of Jacobson Radical of a ring \( S \) are adjacent to each vertex in the quasiregular graph [13].

The following examples show the difference between the notion of a unit graph and the notion of a quasiregular graph over the same ring \( Z_6 \).

**Example 1.** Figure 1, instant unit graph of the ring \( Z_6 \).

**Example 2.** The quasiregular graph of the ring \( Z_6 \) is shown in Figure 2.

We now illustrate some examples of quasiregular graphs over an infinite ring.

**Example 3.** For the ring of integers \( Z \). We find that the edge set of the quasiregular graph of \( Z \) contains edges \((1, 2), (1, -1), (2, -2),\) and otherwise, all elements in \( Z/\{1, -1, 2, -2\} \) of the graph will be isolated points.

**Example 4.** The quasiregular graph of rational numbers \( Q \). For all \( a \in Q^* \), \( a \) is adjacent to \( b \) for all \( b \neq 1/a \) in \( Q^* \).

**Example 5.** Let \( F_4 \) be the extension field over \( Z_2[x] \) by the irreducible polynomial \( P(x) = x^2 + x + 1 \) over \( Z_2 \). Hence, \( F_4 = \{0, 1, u, 1 + u\} \). The quasiregular graph of \( F_4 \) is a path shown in Figure 3.

**Example 6.** Let \( F_8 \) be an extension field over \( Z_2[x] \) by the irreducible polynomial \( P(x) = x^3 + x + 1 \) over \( Z_2 \). Hence, \( F_8 = \{0, 1, u, u^2, 1 + u, 1 + u^2, u + u^2, 1 + u + u^2\} \). The quasiregular graph of \( F_8 \) is shown in Figure 4.

**Proposition 1.** If \( S \cong R \) as rings, then \( Q(S) \cong Q(R) \) as graphs.

**Proof.** Assume that \( f: S \to R \) is an isomorphism of rings. This isomorphism induces an isomorphism \( h: Q(S) \to Q(R) \) where \( h(a) = f(a) \) for all \( a \in S^* \). Indeed, if \( a, b \in S \) are adjacent vertices in \( Q(S) \), then \( 1_a - ab \) is an invertible element in \( S \). Hence, \( h(1_a - ab) = 1_R - h(a)h(b) \) is an invertible element in \( R \) and \( Q(S) \cong Q(R) \) as graphs.

In the following example, we construct the quasiregular graphs of two isomorphic rings \( Z_2 \times Z_3 \cong Z_6 \).

**Example 7.** By Proposition 1, since \( Z_2 \times Z_3 \cong Z_6 \) as rings \( Q(Z_2 \times Z_3) \cong Q(Z_6) \) as graphs as shown in Figure 5.

**Example 8.** In Figure 6, the quasiregular graph over \( Z_3 \) is complete.

The concept in the following theorem is based on the work of Ashrafi et al. [4].

**Theorem 1.** Let \( S \) be a ring. If \( S \) is a division ring with \( \text{Char } (S) = 2 \), then the unit graph \( G(S) \) is complete.

**We prove the following result for quasiregular graphs over a ring \( S \).**

**Proposition 2.** For a commutative ring \( S \) with identity, the quasiregular graph \( Q(S) \) is a complete graph if and only if either \( S \cong Z_2 \) or \( Z_3 \).

**Proof.** Assume the commutative ring \( S \), which has the identity 1, and suppose that \( Q(S) \) is a complete graph. Such that \( |S| > 3 \); and \( a \in S/\{0, 1, -1\} \) so 1 is adjacent to \( 1 + a \). Thus, \( 1 - (1 + a) = -a \) is a unit. Therefore, \( S \) is a field.
Assume that $a^{-1} \neq a$, $a^{-1}$ is not adjacent to $a$. Hence, $a^2 = 1$ for all $a \in S^*$ and $2^2 = 1$ whenever $2 \neq 0$. In this case, char $S = 3$. Now, suppose $a \in S/(0, 1, -1)$; $1 = (a + 1)^2 = a^2 + 2a + 1$ and $1 + 2a = 0$, so $1 = -2a = a$ which is a contradiction. Therefore, $S \cong \mathbb{Z}_3$. 

Now, let char $S = 2, a \in S/(0, 1)$ implies that $a + 1 \neq 0$ and $(a + 1)^2 = 1$ so $a^2 + 1 = 1$ and $a^2 = 0$ which is a contradiction. Hence, $S \cong \mathbb{Z}_2$. The converse is clear.

Note that, as a result of Theorem 1 and Proposition 2, the completeness property indeed gives an important difference between the graphs $G(S)$ and $Q(S)$.

Example 9. In Figure 7, the quasiregular graph over $\mathbb{Z}_4$ is the path graph.

Proposition 3. For a commutative ring $S$ with 1, $Q(S)$ is a path graph if and only if $S \cong \mathbb{Z}_4$.

Proof. Assume that $Q(S)$ is a path $[a_1, a_2, \ldots]$. So $1 - a_1a_2$ and $1 - a_2a_3$ are units; $(1 - a_1a_2)(1 - a_2a_3) = 1 - a_1a_2 - a_2a_3 + a_1a_2a_3 = 1 - a_2(a_1 + a_3 - a_1a_2a_3)$ is a unit. Hence, $a_1 + a_3 - a_1a_2a_3 = a_3$ or $a_1$, or $a_2$, the first and second cases give a contradiction. Thus, $a_1 + a_2 - a_1a_2a_3 = a_2$.

Now, if $1$ has two adjacent elements $a_r, a_{r+2}$, then $1 - a_r$ and $1 - a_{r+2}$ are units and $a_r + a_{r+2} - a_r a_{r+2} = 1$. Since $1 - 1 = 0$ is a unit, $S = \mathbb{Z}_2$. Hence, $1$ is adjacent to $2$ and we may assume that $a_r = 2$. Thus, $1 - 2 = a_{r+2}(1 - 2)$ and $a_{r+2} = 1$ which is a contradiction. Hence, $1$ is adjacent to only one element and the graph must be started by $1$, $[1 - 2 - a_3 - \cdots]$. The element $1 - 2a_3$ is a unit. Thus, we get $1 - 2a_3 = 2 - a_3 = 1 - (a_3 - 1)$ is a unit.

It follows that $a_3 = 3$ and $1 + 3 - 6 = 2$. Thus, $-2 = 2$ and $4 = 0$, which means that char $S = 4$. Assume that $a \in S/(0, 1, 2, 3)$ with $a$ is adjacent to $3$, so $1 - 3a$ is a unit. Thus, $(1 - 3a)^2 = 1 - 6a + 9a^2 = 1 - 2a + a^2 = (1 + a)^2$ is a unit, and $1$ is adjacent to $a$, which is a contradiction. Therefore, $S \cong \mathbb{Z}_4$. The converse is clear from the preceding example.

Example 10. The quasiregular graph over $\mathbb{Z}_2$, is cyclic as seen in Figure 7.

Proposition 4. The quasiregular graph $Q(S)$ over a ring $S$ is cyclic if and only if $S \cong \mathbb{Z}_2$.

Proof. Let $Q(S)$ be a cyclic graph. Then, $1$ is adjacent to two elements $a_1, a_2$. Then, $1 - a_1$ and $1 - a_2$ are units and $(1 - a_1)(1 - a_2) = 1 - a_1 - a_2 + a_1a_2$ is a unit. It follows that $a_1 + a_2 - a_1a_2 = a_3$, $a_1 + a_2 - a_1a_2 = a_2$ or $a_1 + a_2 - a_1a_2 = 1$, which implies that $a_1(1 - a_1) = 0$ or $a_1(1 - a_1) = 0$ or $a_2(1 - a_1) = 1 - a_1$. Since $1 - a_1$ and $1 - a_2$ are units; So, $a_1 = 1$ the other possibility gives a contradiction. Thus, $1$ is adjacent to only one element if $S$ has more than two elements, which is a contradiction. Hence, $S \cong \mathbb{Z}_2$.

Definition 2. A star graph $S_r$ is a bipartite graph that is complete $K_{1,r}$ with $r + 1$ vertices [10], Figure 8.

Proposition 5. For a commutative ring $S$ with 1, the quasiregular graph $Q(S)$ is a star graph over the ring $S$ if and only if either $S \cong F_q$ or $S \cong \mathbb{Z}_4$.

Proof. Assume that $Q(S)$ is a star graph; there are two cases; 

Case I: 1 is adjacent to every $a$ in $S/(0, 1)$, so $1 - a$ is a unit for all $a$ in $S/(0, 1)$. If $a \in S/(0, 1)$, then $1 - a \not\in S/(0, 1)$ and $1 - (1 - a) = a$ is a unit. Hence $S$ is a field. Now, if $a \not\in S/(0, 1)$, then $1 - ab$ is not a unit and $1 = ab$. Therefore, $S$ is a field of four elements $0, 1, a, b$ with $ab = 1$ and $S \equiv F_4$, where $K$ is an expression filed of $\mathbb{Z}_2$ of degree 2 over the minimal polynomial $x^2 + x + 1$.

Case II: there is $e \in S/(0, 1)$ in which $e$ is adjacent to each element in $S/(0, e)$. In this case, $1 - ea$ is a unit for every $a \in S/(0, e)$. Assume that char $(S) = 2$, so $1 - ea$ is a unit for every $a \in S^*$ and $1 - e^2 = (1 + e)^2$ is a unit.
Thus, \( J(S) = \{0, e\} \) and \( e^2 = e \). It follows that
\[ e(1 - e) = e - e^2 = 0, \text{ but } 1 - e \text{ is a unit; So } e = 0 \text{ which is a contradiction. Hence, } \text{char } S \neq 2. \] Now, we have two subcases:

Case II.1: \( e = -1 \), which means that \( 1 + a \) is a unit for all \( a \in S/\{0, -1\} \). Thus 2 is a unit; Also, \( 1 + 2 = 3 \) is a unit and by induction \( \{1, 2, 3, \ldots\} \leq U(S) \). Thus \( 1 + 2 = 3 \) is a unit, which implies that \( 1 \) is adjacent to \(-2\) and \(-2 = -1\). Thus \( 1 = 0 \) which is a contradiction.

Case II.2: \( e \in S/\{0, 1, -1\} \), we know that \(-1\) is a unit and \( 1 - 2 = -1 \), So 1 is adjacent to 2 and \( 2 = e \). Thus \( 1 - 2(-1) = 3 \) is a unit and 1 is adjacent to \(-2\), which implies that \(-2 = 2 \) and \( 4 = 0 \). Hence char \( S \) = 4. Now, assume that \( a \in S/\{0, 1, 2, 3\} \), so \( 1 - 2a \) is a unit.

Thus \( 2a = 2 \) for every \( a \in S/\{0, 2\} \). Since \( a \neq 1 \) and \( a \neq 3 \), So \( 2(a + 1) = 2 \) and \( 2a + 2 = 2 \). It follows that \( 2 = 0 \) which is a contradiction. Therefore, \( S \equiv \mathbb{Z}_4 \). The converse is clear from Examples 5 and 9.

The diameter of the quasiregular graph for a commutative ring \( S \) is discussed in both cases: if \( J(S) \neq 0 \) and if \( J(S) = 0 \).

**Proposition 6.** For finite commutative ring \( S \) with 1, \( \text{diam}(Q(S)) \leq 3. \)

**Proof.** For the Jacobson radical \( J(S) \), there are two cases;

Case I: \( J(S) \neq 0 \); Let \( w \) be a nonzero element in \( J(S) \). Then, \( wz - 1 \) is a unit and \( w \) is adjacent to all \( z \in S \). If \( a \) and \( b \) are nonadjacent elements, then we have a path \( a - w - b \) of length 2 where \( w \) is a nonzero element in \( J(R) \). Therefore, \( \text{dim}(R) = 2 \).

Case II: \( J(S) = 0 \); If \( |S| > 6 \), then a direct product of finite fields is isomorphic to \( S \). It is clear to see that \( \text{diam}(Q(S)) = 2 \) when \( S \) is a field. Since 1 is adjacent to all elements in \( S/\{0, 1\} \). We can assume, without losing generality, that \( S = F_1 \times F_2 \), \( F_1 \) and \( F_2 \) are finite fields, respectively. Thus, \( (1, 1) \) is not adjacent to all elements \( (1, b) \) and \( (a, 1) \), where \( a \in F_1/\{1\} \) and \( b \in F_2/\{1\} \). The elements of the form \( (a, 1) \) are adjacent to \( (c, d) \) where \( c \neq 1 \neq d \) and \( c \neq a^{-1} \). The elements of the form \( (1, b) \) are adjacent to \( (0, d) \) where \( b \neq 1 \) \( d \neq 1 \). Note that, \( (1, 1) \) is adjacent to all elements of the form \( (c, d) \) where \( c \neq 1 \neq d \). (that case of \( |S| = 6 \)) Therefore; the diameter of \( Q(S) \leq 3. \)

From the previous proposition, it was found that there are four possibilities for the diameter of a quasiregular graph, if the diameter equals 0 then the example of that from the quasiregular graph over the ring \( \mathbb{Z}_2 \) is as drawn in Figure 7. In Figure 6, The quasiregular graph of ring \( \mathbb{Z}_3 \) is the case if \( di am = 1 \). If the quasiregular graph is complete, for example, \( Q(\mathbb{Z}_4) \) then \( di am = 2 \) as drawn in Figure 7. Finally; diam \( = 3 \) as shown in the graph of ring \( \mathbb{Z}_6 \) is drawn \( Q(\mathbb{Z}_6) \) in Figure 2.

**Remark 2.** For a given prime number \( p \), Let \( S \equiv \mathbb{Z}_p \); then we have the following formula to calculate the number of edges in the Quasiregular graph if \( Q(\mathbb{Z}_p) \).

\[
|E[Q(\mathbb{Z}_p)]| = \frac{p^2 - 4p + 5}{2}
\]

**Example 11.** \( |E[Q(\mathbb{Z}_7)]| = 13 \) and \( |E[Q(\mathbb{Z}_{11})]| = 41 \), See Figure 9.

**Proposition 7.** For a finite commutative ring \( S \) with identity, if \( Q(S) \) is not a star graph, then \( gr(Q(S)) = 3 \).

**Proof.** There are three possibilities for \( J(S) \):

Case I: \( |J(S)| \geq 3 \), there are nonzero elements \( w, z \in J(S) \) in this case, so \( w \) and \( z \) are adjacent. Also, both elements \( w \) and \( z \) are adjacent to \( 1 \). Thus, \( Q(S) \) has a 3-cycle and \( gr(Q(S)) = 3 \).

Case II: \( |J(S)| = 2 \), So \( J(S) \) has only one nonzero element \( a \). Thus, \( a \) is adjacent to all elements in \( S/\{0, a\} \). By an assumption, \( Q(S) \) is not a star graph. So, there exists \( b \neq c \in S/\{0, a\} \) with \( b \) and \( c \) adjacent. Thus, \( Q(S) \) has a 3-cycle and \( gr(Q(S)) = 3 \).
Case III: $|J(S)| = 1$, if $S$ is a finite field with $|S| > 4$ (otherwise $Q(S)$ is a star graph). For $a, b \in S^*$ with $ab \neq 1$, then $a$ and $b$ are adjacent. Hence, we get a 3-cycle $1 - a - b - 1$ and $gr(Q(S)) = 3$. Now, suppose that $|S| > 4$, and $S \cong F_1 \times F_2$, such that $F_1$ and $F_2$ are finite fields, respectively, with $|F_2| > 2$; So, $Q(S)$ has a 3-cycle $(0, 1) - (0, a) - (1, 0) - (0, 1)$ where $a \neq 1$; Thus $gr(Q(S)) = 3$. Given the fact that any finite commutative ring with zero radical is isomorphic to a direct sum of finite fields; our case is handled.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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