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Research Article

Applying the Reproducing Kernel Method to Fractional Differential Equations with Periodic Conditions in Hilbert Space

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In this article, the reproducing kernel method is presented for the fractional differential equations with periodic conditions in the Hilbert space. This method gives an approximate solution to the problem. The approximate and exact solutions are displayed in the form of series in the reproduction kernel space. In addition, we provide an error analysis for this technique. The presented method is tested by some examples to show its precision.

1. Introduction

Fractional differential equations are needed to model and analyze large volumes of problems. FDEs are applied in large number of fields such as fluid mechanics, biology, chemistry, and diffusion [1–7]. Some methods for solving these equations are Laplace transforms [8], Fourier transform [9], Adomian decomposition method [10], finite difference method [11], variational iteration method [12, 13], collocation method [14], and other methods [15–19].

Many papers have worked on FDEs with periodic conditions, some of which are listed below. Belmekki et al. have discussed the existence and uniqueness of the solution in [20]. Wei et al. have reviewed the minimal and maximal solutions for periodic problems in [21]. In [22], authors have given monotone iterative techniques for existing solutions. In [23], Javidi and Saedshoar Heris have used the method fractional backward differentiation formulas for

$$\lambda_n D_t^{\alpha_n} y(t) + \lambda_{n-1} D_t^{\alpha_{n-1}} y(t) + \cdots + \lambda_1 D_t^{\alpha_1} y(t) + \lambda y(t-\tau) = f(t), \tag{1}$$

with periodic condition $y(0) = y(T)$.

In this work, we use reproducing kernel Hilbert space (RKHS) method to solve multiterm FDEs in the form as follows:

$$\mu_n D_{0^+}^{\eta_n} v(t) + \mu_{n-1} D_{0^+}^{\eta_{n-1}} v(t) + \cdots + \mu_0 D_{0^+}^{\eta_0} v(t) = g(t, v(t)), \quad t \in [0, T], \tag{2}$$

with periodic condition as follows:

$$v(0) = v(T), \tag{3}$$

where $0 \le \eta_0 < \eta_1 < \cdots < \eta_n < 1, T > 0, \quad \mu_j \in \mathbb{R}\,(j = 0, 1, \ldots, n), \mu_n \ne 0$ in Caputo sense. In [24], the existence and uniqueness of the solution have been proven to this problem by using green's function.

The reproducing kernel method was first used in research on boundary value problems in the early twentieth century. In 1907, Zarmba was the first to introduce the kernel of certain functions and to express their reproducing properties. Since 1980, with the efforts of Cui, the reproducing kernel functions of Hilbert space have been introduced in the form of very simple polynomials. They were able to use methods based on the reproducing kernel space [25–27]. Many researchers use the RKHS method to find approximate solutions to various problems [28–30], and also some new applications of reproducing kernel methods and neural networks in machine learning are found in [31–33]. Very recently, RKHS is applied on fractional differential equations [34–36]. In this paper, the reproducing kernel method is presented for the fractional differential equations with periodic conditions in the Hilbert space. The approximate solution obtained from this method is uniformly convergent to the exact solution.

This paper is arranged as follows. Section 2 provides some definitions. Analysis of the RKHS method is proposed in Section 3. The convergence of the approximate solution to the exact solution is given in Section 4. Examples are given in Section 5.

2. Basic Definitions

We describe some of the symbols and basic definitions used in this article. Let $C(I,\mathbb{R})$ represent the Banach space of all continuous functions of $I=[0,T]$ into $\mathbb{R}$, and $C^m(I,\mathbb{R})$ shows the real valued functions on I where the mth order derivative is continuous.

Definition 1. The fractional integral of $g\in C(I,\mathbb{R})$ of order $\eta>0$ is

$$I_{0^+}^{\eta} g(t)=\frac{1}{\Gamma(\eta)}\int_0^t \frac{g(s)}{(t-s)^{1-\eta}}\mathrm{d}s, \quad 0<t<T. \tag{4}$$

Definition 2. The Caputo fractional derivative of $g\in C^m(I,\mathbb{R})$ of order $\eta>0$ is

$$D_{0^+}^{\eta} g(t)=\begin{cases} I^{m-\eta}D^m g(t), & m-1<\eta<m, m\in\mathbb{N},\\ g^{(m)}(t), & \eta=m.\end{cases} \tag{5}$$

Definition 3 (see [37]). Suppose H is a function Hilbert space, including all real or complex value functions defined on a abstract space X, with the inner product $\langle .,.\rangle_H$. For each fixed $y\in X$, if there exist a function $R_y(.)\in H$ which satisfies

$$\langle f,R_y\rangle_H=f(y), \tag{6}$$

then $R_y(.)$ is called the reproducing kernel of H and the Hilbert space H is called the reproducing kernel space.

Remark 1. The real value function space

$$\begin{aligned} W_2^2[0,T]=\{v|v' &\text{ is absolutely continuous},\\ & v''\in L^2[0,T],\ v(0)=v(T)\},\end{aligned} \tag{7}$$

is a function Hilbert space with

$$\langle v,z\rangle_{W_2^2}=\sum_{i=0}^{1} v^{(i)}(0)z^{(i)}(0)+\int_0^T v^{(2)}(t)z^{(2)}(t)\mathrm{d}t, \tag{8}$$

$$\|v\|_{W_2^2}=\langle v,v\rangle_{W_2^2}^{1/2}, \tag{9}$$

where $L^2[0,T]$ denotes the set of square Lebesgue integrable functions on $[0,T]$.

Remark 2. The reproducing kernel function $R_y(.)$ in $W_2^2[0,T]$ can be written as

$$R_y(t)=\left\{\sum_{i=1}^{4} b_i(y)t^{i-1}, t\le y, \sum_{i=1}^{4} c_i(y)t^{i-1}, t>y.\right. \tag{10}$$

It is easy to prove that $R_y(.)$ is obtained as follows. From (8), we have

$$\langle v,R_y\rangle_{w_2^2}=\sum_{i=0}^{1} v^{(i)}(0)\frac{\partial^i R_y(0)}{\partial t^i}+\int_0^T v^2(t)\frac{\partial^2 R_y(t)}{\partial t^2}\mathrm{d}t. \tag{11}$$

Using several integration by part of $\int_0^T v^2(t)(\partial^2 R_y(t)/\partial t^2)\mathrm{d}t$, we obtain that

$$\begin{aligned}\langle v,R_y\rangle_{w_2^2}&=\sum_{i=0}^{1} v^{(i)}(0)\left[\frac{\partial^i R_y(0)}{\partial t^i}-(-1)^i\frac{\partial^{3-i}R_y(0)}{\partial t^{3-i}}\right]\\ &\quad+\sum_{i=0}^{1}(-1)^{1-i}v^{(i)}(T)\frac{\partial^{3-i}R_y(T)}{\partial t^{3-i}}+\int_0^T v^2(t)\frac{\partial^2 R_y(t)}{\partial t^2}\mathrm{d}t.\end{aligned} \tag{12}$$

If $R_y(.)\in W_2^2[0,T]$, then $R_y(0)=R_y(T)$; also if $v(.)\in W_2^2[0,T]$, then $v(0)=v(1)$. Therefore,

$$\begin{aligned}\langle v,R_y\rangle_{w_2^2}&=\sum_{i=0}^{1} v^{(i)}(0)\left[\frac{\partial^i R_y(0)}{\partial t^i}-(-1)^i\frac{\partial^{3-i}R_y(0)}{\partial t^{3-i}}\right]\\ &\quad+\sum_{i=0}^{1}(-1)^{1-i}v^{(i)}(T)\frac{\partial^{3-i}R_y(T)}{\partial t^{3-i}}\\ &\quad+\int_0^T v^2(t)\frac{\partial^2 R_y(t)}{\partial t^2}\mathrm{d}t+b_1(v(0)-v(1)).\end{aligned} \tag{13}$$

Therefore, $R_y(.)$ satisfies the following generalized differential equation:

$$\begin{cases}\dfrac{\partial^4 R_y(t)}{\partial t^4}=\delta(t-y),\\[2ex] \dfrac{\partial^2 R_y(T)}{\partial t^2}=0,\\[2ex] R_y(0)+\dfrac{\partial^3 R_y(0)}{\partial t^3}+b_1=0,\\[2ex] \dfrac{\partial R_y(0)}{\partial t}-\dfrac{\partial^2 R_y(0)}{\partial t^2}=0,\\[2ex] \dfrac{\partial^3 R_y(T)}{\partial t^3}+b_1=0,\end{cases} \tag{14}$$

where δ denotes the Dirac delta function. While $y\ne t$, $R_y(t)$ is the solution of the constant differential equation:

$$\frac{\partial^4 R_y(t)}{\partial t^4}=0, \tag{15}$$

with the boundary conditions

$$\begin{cases} \dfrac{\partial^2 R_y(T)}{\partial t^2} = 0, \\ R_y(0) + \dfrac{\partial^3 R_y(0)}{\partial t^3} + b_1 = 0, \\ \dfrac{\partial R_y(0)}{\partial t} - \dfrac{\partial^2 R_y(0)}{\partial t^2} = 0, \\ \dfrac{\partial^3 R_y(T)}{\partial t^3} + b_1 = 0. \end{cases} \tag{16}$$

The characteristic equation for (15) is $\lambda^4 = 0$. Therefore, the general solution can be written as (10), where coefficients $b_i(y)$ and $c_i(y), i = 1, 2, 3, 4$, are obtained by solving the following equations:

$$\begin{cases} \dfrac{\partial^m R_y(t+0)}{\partial t^m} = \dfrac{\partial^m R_y(t-0)}{\partial t^m}, \quad m = 0, 1, 2, \\ \dfrac{\partial^3 R_y(t+0)}{\partial t^3} - \dfrac{\partial^3 R_y(t-0)}{\partial t^3} = 1, \; \dfrac{\partial^2 R_y(T)}{\partial t^2} = 0, \\ R_y(0) + \dfrac{\partial^3 R_y(0)}{\partial t^3} + b_1 = 0, \dfrac{\partial R_y(0)}{\partial t} - \dfrac{\partial^2 R_y(0)}{\partial t^2} = 0, \\ \dfrac{\partial^3 R_y(T)}{\partial t^3} + b_1 = 0. \end{cases} \tag{17}$$

Therefore, the reproducing kernel function $R_y(.)$ in $W_2^2[0, T]$ is obtained as follows:

$$R_y(t) = \begin{cases} \dfrac{1}{12T^2(3+T)}\big(36T^2 - 6t^3T^2 + 12T^3 - 2t^3T^3 + 6t^3Ty + 12tT^3y, \\ + 6t^2T^3y + 3t^3Ty^2 - 18tT^2y^2 - 9t^2T^2y^2 - t^3y^3 + 6tTy^3 + 3t^2Ty^3\big), \quad y \le t, \\ \dfrac{1}{12T^2(3+T)}\big(36T^2 + 12T^3 + 6t^3Ty - 18t^2T^2y + 12tT^3y + 3t^3Ty^2, \\ -9t^2T^2y^2 + 6tT^3y^2 - t^3y^3 + 6tTy^3 + 3t^2Ty^3 - 6T^2y^3 - 2T^3y^3\big), \quad y > t. \end{cases} \tag{18}$$

Remark 3. The real value function space

$$W_2^1[0, T] = \{v \mid v \text{ is absolutely continuous}, \; v' \in L^2[0, T]\}, \tag{19}$$

is a function Hilbert space with inner product

$$\langle v, z\rangle_{W_2^1} = v(0)z(0) + \int_0^T v'(t)z'(t)\mathrm{d}t. \tag{20}$$

It can be proved that $W_2^1[0, T]$ is a reproducing kernel Hilbert space and

$$R_y(t) = \begin{cases} t + 1, & t \le y, \\ y + 1, & t > y. \end{cases} \tag{21}$$

3. Solution Procedure (2) by RKHS Method

Here, we will construct a linear differential operator and an orthogonal system in $W_2^1[0, 1]$. After that, the RKHS method for obtaining solution (2) with condition (3) is presented.

First, by introducing linear operator L: $W_2^2[0, T] \longrightarrow W_2^1[0, T]$ as

$$Lv(t) = \sum_{j=0}^{n} \mu_j D_{0^+}^{\eta_j} v(t), \tag{22}$$

then problem (2) will be converted into the following form:

$$\begin{cases} (Lv)(t) = g(t, v(t)), & 0 \le t \le T, \\ v(0) = v(T). \end{cases} \tag{23}$$

Theorem 1. *The operator L is a bounded linear operator.*

Proof. It can be easily shown that L is a linear operator. So, we only prove the boundary of L. From (20), we have

$$\|Lv\|_{W_2^1}^2 = \langle Lv, Lv\rangle_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lv)'(t)]^2\mathrm{d}t. \tag{24}$$

By reproducing property of $R_y(.)$, we have

$$\begin{cases} v(t) = \langle v(y), R_t(y)\rangle_{W_2^2}, \\ (Lv)(t) = \langle v(y), LR_t(y)\rangle_{W_2^2}, \\ (Lv)'(t) = \langle v(y), (LR_t(y)')\rangle_{W_2^2}. \end{cases} \tag{25}$$

By Schwarz inequality, we get

$$\begin{aligned} |(Lv)(t)| &= \left|\langle v(y), LR_t(y)\rangle_{W_2^2}\right| \le \|LR_t\|_{W_2^2}\|v\|_{W_2^2} = M_1\|v\|_{W_2^2}, \\ |(Lv)'(t)| &= \left|\langle u(y), (LR_t(y))'\rangle_{W_2^2}\right| \le \|(LR_t)'\|_{W_2^2}\|v\|_{W_2^2} = M_2\|v\|_{W_2^2}, \end{aligned} \tag{26}$$

where $M_1, M_2 > 0$ are positive constants and the proof is completed.

Thus, $[(Lv)(0)]^2 \le M_1^2\|v\|^2_{W_2^2}$, $\|[(Lv)'(t)]^2 \le M_2^2\|$ and $\int_0^1 [(Lv)'(t)]^2 dt \le M_2^2\|v\|^2_{W_2^2}$. That is,

$$\|(Lv)\|^2_{W_2^1} = [(Lv)(0)]^2 + \int_0^1 [(Lu)'(t)]^2 dx \le \left(M_1^2 + M_2^2\right)\|v\|_{W_2^2}, \tag{27}$$

where $M = M_1^2 + M_2^2 > 0$. We will construct a complete system of $\{\Psi_i(.)\}_{i=1}^{\infty}$ of $W_2^2[0,T]$ by setting $\Phi_i(t) = T_{t_i}(t)$ and $\Psi_i(t) = L^*\Phi_i(t)$, where $\{t_i\}_{i=1}^{\infty}$ is dense on $[0, T]$ and $L^*: W_2^1[0,T] \longrightarrow W_2^2[0,T]$ is conjugate operator of L. □

Lemma 1 (see [37]). *If $\{t_i\}_{i=1}^{\infty}$ is dense on [0, T], then $\{\Psi_i(.)\}_{i=1}^{\infty}$ is a complete system of $W_2^2[0,T]$ and $\Psi_i(t) = LR_y(t)|_{y=t_i}$.*

By using the Gram–Schmidt process of $\{\Psi_i(.)\}_{i=1}^{\infty}$ is obtained the orthonormal basis $\{\overline{\Psi}_i(.)\}_{i=1}^{\infty}$ of space $W_2^2[0,T]$, which satisfies

$$\overline{\Psi}_i(t) = \sum_{k=1}^{i} \rho_{jk}\Psi_k(t). \tag{28}$$

The coefficients ρ_{jk} are positive and given by

$$\begin{aligned} \rho_{11} &= \frac{1}{\|\Psi_1\|}, \\ \rho_{ii} &= \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} d_{ik}^2}}, \\ \rho_{ij} &= \frac{-\sum_{k=j}^{i-1} d_{ik}\rho_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} d_{ik}^2}}, \end{aligned} \tag{29}$$

where $d_{ik} = \langle \Psi_i, \overline{\Psi_k}\rangle_{W_2^2}$.

$$v(t) = \sum_{i=1}^{\infty} A_i \overline{\Psi}_i(t), \tag{30}$$

where $A_i = \sum_{k=1}^{i} \rho_{ik} g(t_k, v_{k-1}(t_k))$ are unknown and we will obtain A_i by using B_i. So, suppose $v_0(t) = 0$ and $v_n(t)$ is given by

$$v_n(t) = \sum_{i=1}^{n} B_i \overline{\Psi}_i(t), \tag{31}$$

where B_i of $\overline{\Psi}_i(t)$ is given by

$$\begin{aligned} B_1 &= \rho_{11} g(t_1, v_0(t_1)), \\ v_1(t) &= B_1 \overline{\Psi}_1(t), \\ B_2 &= \sum_{k=1}^{2} \rho_{2k} g(t_k, v_{k-1}(t_k)), \\ v_2(t) &= \sum_{i=1}^{2} B_i \overline{\Psi}_i(t), \\ &\vdots \\ B_n &= \sum_{k=1}^{n} \rho_{nk} g(t_k, v_{k-1}(t_k)). \end{aligned} \tag{32}$$

Theorem 2 (see [37]). *Let $\{t_i\}_{i=1}^{\infty}$ be dense set in $[0,T]$ and the exact solution $v(.)$ of (23) in space $W_2^2[0,T]$ be unique, then*

$$v(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \rho_{ik} g(t_k, v(t_k))\overline{\Psi}_i(t), \tag{33}$$

and for this problem, approximate solution nth order as follows:

$$v_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \rho_{ik} g(t_k, v(t_k))\overline{\Psi}_i(t). \tag{34}$$

Theorem 3. *If $v \in W_2^2[0,T]$, then there exists constant $C > 0$ such that*

$$\left|v^{(i)}(t)\right| \le C\|v\|_{W_2^2}, \quad i = 0, 1. \tag{35}$$

Proof. *For each $t \in [0,T]$, we obtain*

$$|v(t)| = \left|\langle v(\xi), R_t(\xi)\rangle_{W_2^2}\right| \le \|R_t(\xi)\|_{W_2^2}\|v(\xi)\|_{W_2^2} \le C_1\|v(\xi)\|_{W_2^2}, \tag{36}$$

and we also have

$$|v'(t)| = \left|\langle v(\xi), \frac{\partial R_t(\xi)}{\partial t}\rangle_{W_2^2}\right| \le \left\|\frac{\partial R_t(\xi)}{\partial t}\right\|_{W_2^2}\|v(\xi)\|_{W_2^2} \le C_2\|v(\xi)\|_{W_2^2}, \tag{37}$$

where C_1 and C_2 are positive constants and $C = \max\{C_1, C_2\}$.

Corollary 1. *The approximate solutions $v_n(.)$ and $v_n'(.)$ uniformly converge to the exact solutions $v(.)$ and $v'(.)$, respectively.*

Proof. From Theorem 3, for each $t \in [0,T]$, we obtain

$$\begin{aligned} \left|v_n^{(i)}(t) - v^{(i)}(t)\right| &= \left|\langle v_n(\xi) - v(\xi), \frac{\partial^i R_t(\xi)}{\partial t}\rangle_{W_2^2}\right|, \\ &\le \left\|\frac{\partial^i R_t(\xi)}{\partial t}\right\|_{W_2^2}\left\|v_n(\xi) - v(\xi)\right\|_{W_2^2}, \\ &\le C_i\left\|v_n(\xi) - v(\xi)\right\|_{W_2^2}, \end{aligned} \tag{38}$$

where $C_i, i = 0, 1$ are positive constants. Then, if $v_n(\xi) \xrightarrow{(.)_{W_2^2}} v(\xi)$ as $n \longrightarrow \infty$, the approximate solutions $v_n(t)$ and $v_n'(t)$ converge uniformly to $v(t)$ and $v'(t)$, respectively. □

Remark 4. We apply the following two cases to solve equations (2) and (3) by using the RKHS method.

Case 1. Let (2) be linear and (30) and (31) denote the exact and approximate solutions, respectively.

Case 2. Let (2) be nonlinear; in this case, the solution of (2) is as follows:

We can guarantee that $v_n(t)$ in equation (37) satisfies condition (3).

4. Convergence Analysis

In this section, we will show that approximate solution $v_n(.)$ of equation (37) is convergent to the exact solution $v(.)$ of equation (2). First, we express the following lemma.

Lemma 2. *If $v_n(t) \xrightarrow{(.)_{W_2^2}} v(t), t_n \longrightarrow y$, $(n \longrightarrow \infty)$ and $g(t, y)$ is continuous function with respect to $t \in [0, T]$ and $y \in (-\infty, \infty)$, then $g(t_n, v_{n-1}(t_n)) \longrightarrow g(y, v(y)) as n \longrightarrow \infty$.*

Proof. Observe that

$$\begin{aligned} |v_{n-1}(t_n) - v(y)| &= |v_{n-1}(t_n) - v_{n-1}(y) + v_{n-1}(y) - v(y)| \\ &\leq |v_{n-1}(t_n) - v_{n-1}(y)| + |v_{n-1}(y) - v(y)|. \end{aligned} \tag{39}$$

Reproducing property of $R_y(\xi)$ yields that

$$\begin{aligned} |v_{n-1}(t_n) - v_{n-1}(y)| &= \left|\langle v_{n-1}(\xi), R_{t_n}(\xi) - R_y(\xi)\rangle_{W_2^2}\right| \\ &\leq \|v_{n-1}\|_{W_2^2} \left\|R_{t_n}(\xi) - R_y(\xi)\right\|_{W_2^2}. \end{aligned} \tag{40}$$

From the symmetry of $R_y(\xi)$, result is $\|R_{t_n}(\xi) - R_y(\xi)\| \longrightarrow 0$. Therefore, $|v_{n-1}(t_n) - v_{n-1}(y)| \longrightarrow 0$ as $t_n \longrightarrow y$, $(n \longrightarrow \infty)$. From Corollary 1, it holds that $|v_{n-1}(y) - v(y)| \longrightarrow 0 as n \longrightarrow \infty$. Then, $v_{n-1}(t_n) \xrightarrow{(.)_{W_2^2}} v(y) as n \longrightarrow \infty$. Because $g(.)$ is continuous functions, then $g(t_n, v_{n-1}(t_n)) \longrightarrow g(y, v(y)) as n \longrightarrow \infty$. □

Lemma 3. *For $v_n(t)$ in equation (37), we have*

$$Lv_n(t_j) = Lv(t_j) = g(t_j, v_{j-1}(t_j)). \tag{41}$$

Proof. Suppose $j \leq n$, therefore,

$$\begin{aligned} Lv_n(t_j) &= \sum_{i=1}^{n} B_i L\overline{\Psi}_i(t_j) = \sum_{i=1}^{n} B_i \langle L_s \overline{\Psi}_i(t), \Phi_j(t)\rangle_{W_2^1} \\ &= \sum_{i=1}^{n} B_i \langle \overline{\Psi}_i(t), L^* \Phi_j(t)\rangle_{W_2^2} = \sum_{i=1}^{n} B_i \langle \overline{\Psi}_i(t), \Psi_j(t)\rangle_{W_2^2}. \end{aligned} \tag{42}$$

By using orthogonality of $\{\overline{\Psi}_i(t)\}_{i=1}^{\infty}$, we obtain

$$\begin{aligned} \sum_{l=1}^{j} \rho_{jl} Lv_n(t_l) &= \sum_{i=1}^{n} B_i \langle \overline{\Psi}_i(t), \sum_{l=1}^{j} \rho_{jl} \Psi_l(t)\rangle_{W_2^2}, \\ &= \sum_{i=1}^{n} B_i \langle \overline{\Psi}_i(t), \overline{\Psi}_j(t)\rangle_{W_2^2}, \\ &= B_j = \sum_{l=1}^{j} \rho_{jl} g(t_l, v_{l-1}(t_l)). \end{aligned} \tag{43}$$

If $j = 1$, then

$$Lv_n(t_1) = g(t_1, v_0(t_1)). \tag{44}$$

Besides if $j = 2$, then

$$\rho_{21} Lv_n(t_1) + \rho_{22} Lv_n(t_2) = \rho_{21} g(t_1, v_0(t_1)) + \rho_{22} g(t_2, v_1(t_2)), \tag{45}$$

that is, $Lv_n(t_2) = g(t_2, v_1(t_2))$. By the same manner, it yields that

$$Lv_n(t_j) = g(t_j, v_{j-1}(t_j)). \tag{46}$$

Hence, $v(t) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(t)$ is obtained by taking the limit of equation (37). Therefore, $v_n(t) = P_n v(t)$, P_n is an orthogonal projector of $W_2^2[0, 1]$ to span $\{\Psi_1, \Psi_2, \ldots, \Psi_n\}$. Then,

$$\begin{aligned} Lv_n(t_j) &= \langle Lv_n(t), \Phi_j(t)\rangle_{W_2^1} = \langle v_n(t), L^* \Phi_j(t)\rangle_{W_2^2} \\ &= \langle P_n v(t), \Psi_j(t)\rangle_{W_2^2} \\ &= \langle v(t), P_n \Psi_j(t)\rangle_{W_2^2} = \langle v(t), \Psi_j(t)\rangle_{W_2^2} \\ &= \langle v(t), L^* \Phi_j(t)\rangle_{W_2^2} \\ &= \langle Lv(t), \Phi_j(t)\rangle_{W_2^1} = Lv(t_j). \end{aligned} \tag{47}$$

Theorem 4. *Let $\|v_n\|_{W_2^2}$ be bounded and $\{t_i\}_{i=1}^{\infty}$ is dense on $[0, T]$, then n-term approximate solutions $v_n(t)$ in equation (37) converge to*

$$v(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \rho_{ik} g(t_k, v_{k-1}) \overline{\Psi}_i(t). \tag{48}$$

Proof. Firstly, we prove that the convergence of $\{v_n\}_{n=1}^{\infty}$ in equation (37) is convergent in the sense of $\|.\|_{W_2^2}$. From equation (37), it is inferred that $v_{n+1}(t) = v_n(t) + B_{n+1} \overline{\Psi}_{n+1}(t)$. Since $\{\overline{\Psi}_i\}_{i=1}^{\infty}$ is orthogonal, hence,

$$\|v_{n+1}\|_{W_2^2}^2 = \|v_n\|_{W_2^2}^2 + (B_{n+1})^2 = \cdots = \sum_{i=1}^{n+1} (B_i)^2, \tag{49}$$

where $B_i = \sum_{k=1}^{i} \rho_{ik} g(t_k, v_{k-1}(t_k))$. It holds that $\|v_n\|_{W_2^2}^2 \leq \|v_{n+1}\|_{W_2^2}^2$.

Because $\|v_n\|_{W_2^2}^2$ is bounded, $\|v_n\|_{W_2^2}$ is convergent as $n \longrightarrow \infty$. Therefore, there exists constant c such that $\sum_{i=1}^{\infty} (B_i)^2 = c$. The implies that $B_i \in l^2 = \{B | \sum_{i=1}^{\infty} (B_i)^2 < \infty\}$.

Let $m > n$, then from the orthogonality of $v_{n+1}(t) - v_n(t)$, it follows that

$$\begin{aligned} \|v_m - v_n\|_{W_2^2}^2 &= \|v_m - v_{m-1} + v_{m-1} - \cdots + v_{n+1} - v_n\|_{W_2^2}^2, \\ &= \|v_m - u_{m-1}\|_{W_2^2}^2 + \cdots + \|v_{n+1} - v_n\|_{W_2^2}^2, \end{aligned} \tag{50}$$

because $\|v_m - v_{m-1}\|_{W_2^2}^2 = (B_m)^2$. Consequently,

$$\|v_m - v_n\|_{W_2^2}^2 = \sum_{i=n+1}^{m} (B_i)^2 \longrightarrow 0 as n, m \longrightarrow \infty. \tag{51}$$

Table 1: The absolute error for Example 1 with $n = 22$.

t_i	$v(t)$	$v_{22}(t)$	Absolute error
0.1	0.081	0.0808825	0.000223837
0.2	0.128	0.127862	0.000172347
0.3	0.147	0.14684	0.000157614
0.4	0.144	0.143828	0.000150239
0.5	0.125	0.12483	0.000143359
0.6	0.096	0.0958422	0.000134868
0.7	0.063	0.0628596	0.000124506
0.8	0.032	0.0318794	0.000112694
0.9	0.009	0.00890275	0.0000997064
1	0	-1.55353×10^{-15}	1.55353×10^{-15}

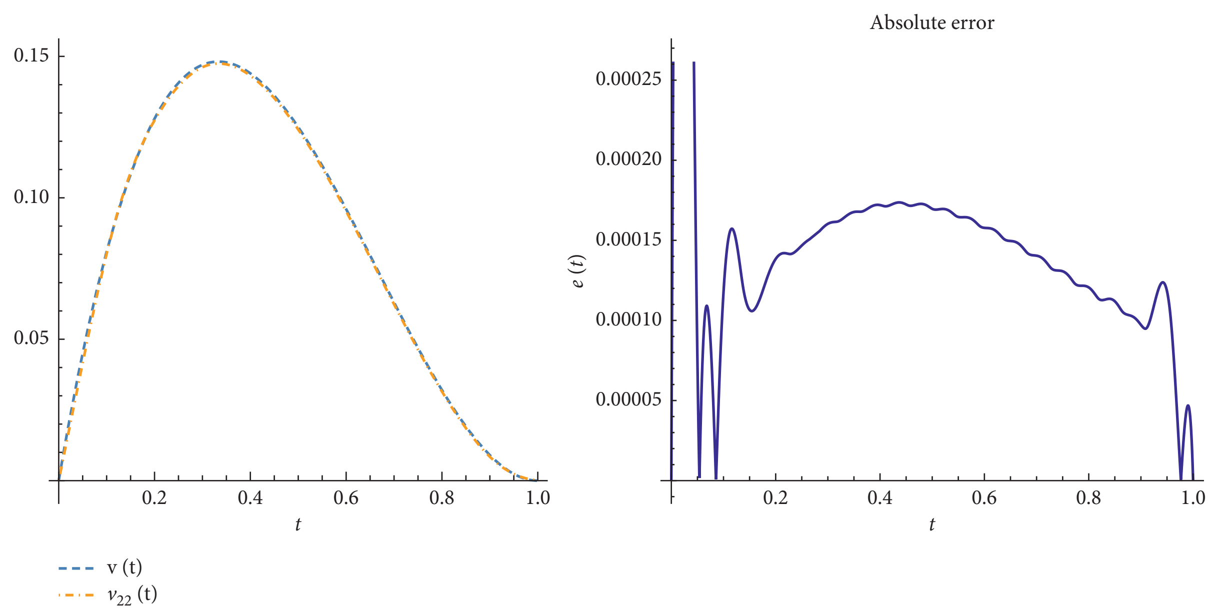

Figure 1: Graphs of numerical solution and absolute error with $n = 22$ for Example 1.

Hence, $W_2^2[0,T]$ is complete, and then $v_n(t) \xrightarrow{\langle . \rangle_{W_2^2}} v(t)$, as $(n \longrightarrow \infty)$.

Now, we prove that $v(t)$ is the solution of equation (23). Because $\{t_i\}_{i=1}^{\infty}$ is dense on $[0,T]$, for each $t \in [0,T]$, there exists a subsequence $\{t_{n_j}\}$ such that $\{t_{n_j}\} \longrightarrow t$ as $j \longrightarrow \infty$.

Since $\{t_j\}_{j=1}^{\infty}$ is dense on $[0,T]$, thus for all $t \in [0,T]$, there exists a subsequence $\{t_{n_j}\}$ such that $t_{n_j} \longrightarrow t$, as $i \longrightarrow \infty$. By Lemma 3, it follows that $Lv_n(t_{n_j}) = g(t_{n_j}, v_{n_j-1}(t_k))$. Hence, let $j \longrightarrow \infty$, from Lemma 2 and the continuity of g, we have $Lv(t) = g(t, v(t))$. □

Theorem 5. *Assume $r_n = \|v(t) - v_n(t)\|_{W_2^2}^2$, where $v_n(t)$ is derived from the RKHS method. Therefore, r_n is decreasing in $\|.\|_{W_2^2}$.*

Proof. Note that

$$
\begin{aligned}
r_n^2 &= \|v(t) - u_n(t)\|_{W_2^2}^2 = \left\| \sum_{j=n+1}^{\infty} \langle v(t), \overline{\Psi}_j(t) \rangle_{W_2^2} \overline{\Psi}(t) \right\|^2 = \sum_{j=n+1}^{\infty} \left(\langle v(t), \overline{\Psi}_j(t) \rangle_{W_2^2} \right)^2, \\
r_{n-1}^2 &= \|v(t) - v_{n-1}(t)\|_{W_2^2}^2 = \left\| \sum_{j=n}^{\infty} \langle v(t), \overline{\Psi}_j(t) \rangle_{W_2^2} \overline{\Psi}_j(t) \right\|_{W_2^2}^2 = \sum_{j=n}^{\infty} \left(\langle v(t), \overline{\Psi}_j(t) \rangle_{W_2^2} \right)^2,
\end{aligned} \tag{52}
$$

therefore, $\|r_n(t)\|_{W_2^2} \le \|r_{n-1}(t)\|_{W_2^2}$. □

Table 2: The absolute error for Example 2 with $n = 20$.

t_i	$v(t)$	$v_{20}(t)$	Absolute error
0.1	−0.09	−0.092247	0.00224696
0.2	−0.16	−0.161132	0.00113237
0.3	−0.21	−0.210852	0.000852089
0.4	−0.24	−0.240714	0.000713747
0.5	−0.25	−0.250521	0.00062149
0.6	−0.24	−0.240549	0.000549103
0.7	−0.21	−0.210485	0.000485426
0.8	−0.16	−0.160424	0.000423569
0.9	−0.09	−0.0903508	0.000350814
1	0	-7.51086×10^{-16}	7.51086×10^{-16}

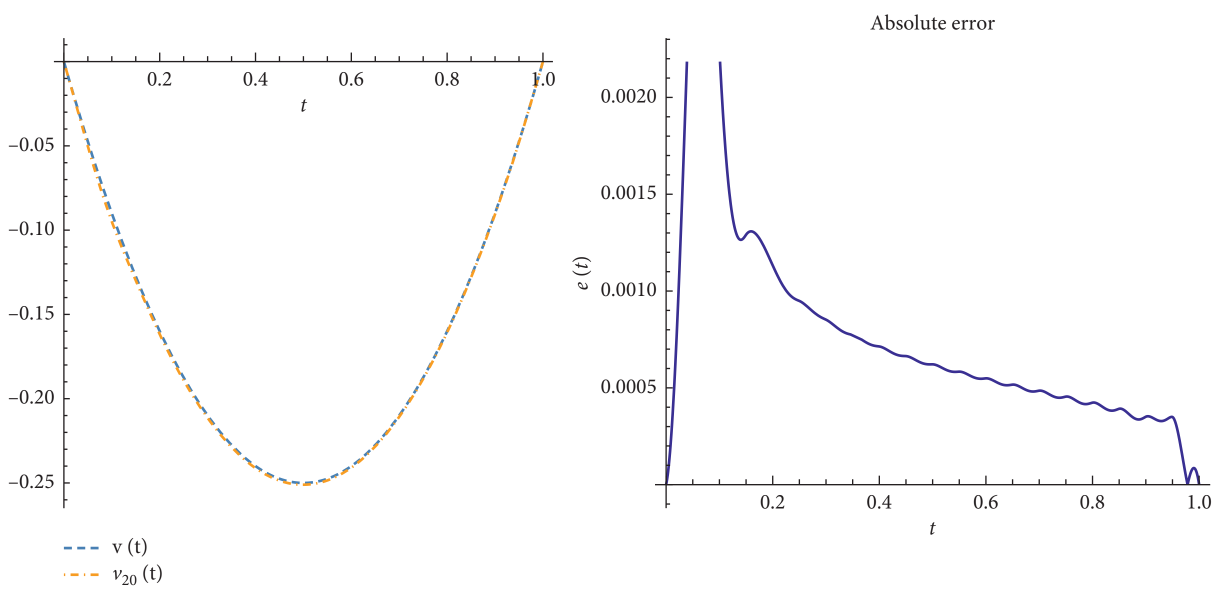

Figure 2: Graphs of numerical solution and absolute error with $n = 20$ for Example 2.

Table 3: Numerical solution for Example 3 with $n = 10$.

t_i	$v(t)$	$v_{10}(t)$	$\eta_0 = 0.4, \eta_1 = 0.5$	$\eta_0 = 0.5, \eta_1 = 0.6$	Absolute error
0.2	−0.36	−0.351811	−0.350017	−0.34599	0.00818945
0.4	−0.64	−0.637683	−0.636476	−0.633814	0.00231702
0.6	−0.84	−0.837922	−0.836963	−0.834929	0.00207829
0.8	−0.96	−0.958537	−0.95775	−0.956096	0.0014633
1	−1	−0.998869	−0.998235	−0.99692	0.00113105
1.2	−0.96	−0.959148	−0.958648	−0.957624	0.000852181
1.4	−0.84	−0.839392	−0.83902	0.83827	0.00060833
1.6	−0.64	−0.639624	−0.639379	−0.638901	0.000376393
1.8	−0.36	−0.359955	−0.359858	−0.359676	0.0000452721
2	0	3.3158×10^{-14}	1.37181×10^{-14}	3.4689×10^{-14}	3.3158×10^{-14}

5. Numerical Tests

We provide three examples to explain the content given, and we realize the validity and accuracy of the RKHS method.

Example 1. In this example, we consider FDE with periodic condition:

$$\begin{aligned} &D_{0^+}^{0.4}v(t) + D_{0^+}^{0.3}v(t) + v^2(t)e^{v(t)} \\ &= \frac{\Gamma(4)}{\Gamma(3.6)}t^{2.6} - 2\frac{\Gamma(3)}{\Gamma(2.6)}t^{1.6} + \frac{\Gamma(2)}{\Gamma(1.6)}t^{0.6} \\ &\quad + \frac{\Gamma(4)}{\Gamma(3.7)}t^{2.7} - 2\frac{\Gamma(3)}{\Gamma(2.7)}t^{1.7} + \frac{\Gamma(2)}{\Gamma(1.7)}t^{0.7} \\ &\quad + \left(t^3 - 2t^2 + t\right)^2 e^{t^3 - 2t^2 + t}, \\ &\quad \cdot v(0) = v(1), \end{aligned} \tag{53}$$

for $t \in [0, 1]$. The exact solution is $v(t) = t^3 - 2t^2 + t$. We choose 22 points in $[0, 1]$, and by using the proposed method, the approximate solution v_{22} is obtained. Absolute error values are reported in Table 1 for $T = 1$ and $t_i = i/n, i = 1, 2, \ldots, n$. The graphs of the absolute error and the numerical solution are plotted in Figure 1. Here,

$$\begin{aligned} L &= D_{0^+}^{0.4}v(t) + D_{0^+}^{0.3}v(t), \\ g(t, v(t)) &= \frac{\Gamma(4)}{\Gamma(3.6)}t^{2.6} - 2\frac{\Gamma(3)}{\Gamma(2.6)}t^{1.6} \\ &\quad + \frac{\Gamma(2)}{\Gamma(1.6)}t^{0.6} + \frac{\Gamma(4)}{\Gamma(3.7)}t^{2.7} \\ &\quad - 2\frac{\Gamma(3)}{\Gamma(2.7)}t^{1.7} + \frac{\Gamma(2)}{\Gamma(1.7)}t^{0.7} \\ &\quad + \left(t^3 - 2t^2 + t\right)^2 e^{t^3 - 2t^2 + t} \\ &\quad - v^2(t)e^{v(t)}, \end{aligned} \tag{54}$$

and

$$\Psi_i(t) = LR_y(t)|_{y=t_i} = D_{0^+}^{0.4}R_{t_i}(t) + D_{0^+}^{0.3}R_{t_i}(t), \tag{55}$$

finally $v(t)$ and $v_n(t)$ are obtained from (36) and (37), respectively.

Example 2. We consider FDE with periodic condition:

$$\begin{aligned} &D_{0^+}^{0.6}v(t) + D_{0^+}^{0.5}v(t) + D_{0^+}^{0.2}v(t) \\ &\quad + \frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5} - \frac{\Gamma(2)}{\Gamma(1.5)}t^{0.5} \\ &\quad - \frac{\Gamma(2)}{\Gamma(1.8)}t^{0.8} + \sinh\left(t^2 - t\right)(1 - t), \\ &v(0) = v(1), \end{aligned} \tag{56}$$

for $t \in [0, 1]$. The exact solution is $v(t) = t^2 - t$. We choose 20 points in $[0, 1]$, and by using the proposed method, we obtained the approximate solution v_{20}. Absolute error values are reported in Table 2 for $T = 1$ and $t_i = i/n, i = 1, 2, \ldots, n$. The graphs of the absolute error and the numerical solution are plotted in Figure 2.

Example 3. We consider FDE with periodic condition

$$\begin{aligned} D_{0^+}^{\eta_1}v(t) + D_{0^+}^{\eta_0}v(t) &= \frac{\Gamma(3)}{\Gamma(3-\eta_1)}t^{2-\eta_1} - 2\frac{\Gamma(2)}{\Gamma(2-\eta_1)}t^{1-\eta_1} \\ &\quad + \frac{\Gamma(3)}{\Gamma(3-\eta_0)}t^{2-\eta_0} - 2\frac{\Gamma(2)}{\Gamma(2-\eta_0)}t^{1-\eta_0}, \\ v(0) &= v(2), \end{aligned} \tag{57}$$

for $t \in [0, 2]$. The exact solution is $v(t) = t^2 - 2t$. Using this method, taking $t_i = i/n, i = 1, 2, \ldots, n$, for $n = 10$ and $T = 2$, the numerical results are given in Table 3 for $\eta_0 = 0.2$ and $\eta_1 = 0.5$. Three graphs of the approximate solution for η_1 and η_0 are drawn in Figure 3.

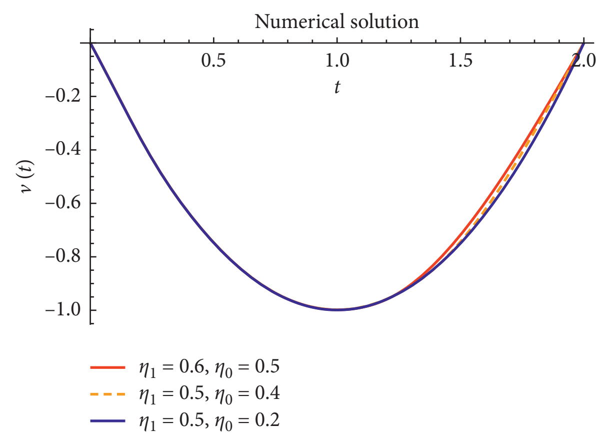

Figure 3: Graph of numerical solution with $n = 10$ for Example 3.

6. Conclusion

In this paper, we have proposed the RKHS method to solve fractional differential equations with periodic conditions. This method is a powerful technique for finding approximate solutions. The approximation error and convergence analysis are obtained in the RKHS. We illustrate the efficiency of the method with a few examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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