Research Article

Conformal Metrics to a Product or Doubly Warped Product on $S^2 \times S^2$ and the Hopf Conjecture

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Hopf’s well-known conjecture states that there exists no metric of positive sectional curvature in the product manifold $S^2 \times S^2$. In this paper, we show that the Hopf conjecture is true for conformal metrics to the product metric or doubly warped products on $S^2 \times S^2$.

1. Introduction

A classical topic in Riemannian geometry is to study manifolds with positive sectional curvature. The most rigid classical curvature concept (consequently containing most information), which then gives rise to the other ones, is sectional curvature (see [1]). Let $(M, g_M)$ be a Riemannian manifold and let $T_p M$ denote the tangent vector space of $M$ at $p \in M$. The sectional curvature is the most natural generalization to higher dimensions of the Gaussian curvature of a surface, given that it controls the behavior of geodesics (see [2]). We will say that $(M, g_M)$ has positive sectional curvature if for every point $p \in M$, the sectional curvature $K(\Pi)$ of every 2-plane $\Pi \subset T_p M$ is positive. An example of such manifolds are the $n$-dimensional spheres of $S^n$ with the metric induced by $\mathbb{R}^{n+1}$. For 4-dimensional Riemannian manifolds, very few topological obstructions to positive sectional curvature are known, and many conjectures about this subject remain open, as, for example, the Hopf conjecture on $S^2 \times S^2$, which is one of the oldest conjectures in global Riemannian geometry (see [3]).

Conjecture (H. Hopf): $S^2 \times S^2$ does not admit a Riemannian metric of positive sectional curvature [4].

By Synge’s lemma (see [5]), one knows that on $\mathbb{R}P^2 \times \mathbb{R}P^2$ there is no Riemannian metric of positive sectional curvature. The only known examples of positively curved compact connected 4-manifolds are $S^4$, $\mathbb{R}P^4$, and $\mathbb{C}P^2$ (see [4]).

There have been various attempts to prove or disprove Hopf’s conjecture; one was to start with the standard product metric (which is nonnegatively curved) and try to deform it to a positively curved metric (see [1, 4, 6–8] for more details). Although one can make the curvature of mixed planes positive, there appear new planes of zero or even negative curvature; hence, this method seems not to answer the question. From work of Bourguignon [7], it is known that in the neighborhood of the product metric of $S^2 \times S^2$, there is no metric of positive curvature. In [9], Weinstein shows that a metric of positive sectional curvature on $S^2 \times S^2$ cannot be induced by an immersion in $\mathbb{R}^5$. Now, it follows from a result of Hopf that an embedding of such a Riemannian structure in $\mathbb{R}^5$ is not possible as then the manifold has to be a sphere.

Similarly, using the metric $g_0$, the Riemannian metric is induced by $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^1$ on $M = S^2 \times S^2$; the authors of [10] gave a positive answer to the conjecture in the class of conformal metrics $g = \exp(f)g_0$ where $f$ is a smooth function on $S^2 \times S^2$.

In [3, 11], Oscar Perdomo suggested an idea that may be useful for trying to prove Hopf’s conjecture. This idea relies on the main two theorems proven in [3]. The first one is a
small variation of a theorem by Nash that states that any Riemannian manifold can be isometrically embedded in some $\mathbb{R}^k$. The second one states that if $M = S_1 \times S_2$, in particular $k_1 + k_2 = n$ ($S_i \subset \mathbb{R}^{k_i}$ be two compact Riemannian surfaces with the metric induced by the euclidean spaces), then for any smooth function $F: M \to \mathbb{R}$, the manifold $\{(x, F(x)) \in \mathbb{R}^{k+1} : x \in M\}$ with the metric induced by $\mathbb{R}^{k+1}$ does not have positive sectional curvature. So, he makes the following remark:

**Remark 1.** If for any smooth map $F: S^2 \times S^2 \to \mathbb{R}^k$ the manifold $\{(x, F(x)) \in \mathbb{R}^k : x \in S^2 \times S^2\}$ with the metric induced by $\mathbb{R}^k$ does not have positive sectional curvature, then Hopf’s conjecture would be true.

The case $k = 1$ is solved in [3].

More generally, one might ask whether any nontrivial product manifold admits a metric with positive curvature. It is easy to see that if $M$ and $N$ admit nonnegative curvature metrics, then $M \times N$ with the product metric has non-negative curvature. This generalized conjecture is open. Thus, in [12], Amann and Kennard have proven that no product manifold $N \times N$ can carry a metric of positive sectional curvature admitting a certain degree of torus symmetry. Also, Hsiang and Kleiner proved that (see [13, 14]) $S^2 \times S^2$ does not admit a Riemannian metric with positive sectional curvature and an isometric circle action.

Singly warped products were first introduced by Bishop and O’Neill in their attempt to construct a class of Riemannian manifolds with negative curvature [15]. Warped products have many applications, in general relativity [16], in the studies related to solutions of Einstein’s equations (see [17] and references therein).

The study of relativity theory demands a wider class of manifold, and the idea of doubly twisted product as a generalization of warped product was introduced and studied by many authors (see [18] and references therein).

Let $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ be two Riemannian manifolds of dimensions $m_1$ and $m_2$, respectively, and let $\pi_1: M_1 \times M_2 \to M_1$ and $\pi_2: M_1 \times M_2 \to M_2$ be the canonical projections. Also, let $f_1: M_1 \times M_2 \to \mathbb{R}^+$ and $f_2: M_1 \times M_2 \to \mathbb{R}^+$ be smooth functions. Then, the doubly twisted product $(M_1 \times (f_1, f_2)M_2)$ with twisting functions $f_1$ and $f_2$ is defined to be the product manifold $M = M_1 \times M_2$ with metric tensor $g_M = f_1^2 g_{M_1} \oplus f_2^2 g_{M_2}$, given by

$$g_M = f_1^2 \pi_1^* g_{M_1} + f_2^2 \pi_2^* g_{M_2}. \tag{1}$$

We denote this Riemannian manifold $(M, g_M)$ by $M_1 \times (f_1, f_2)M_2$. In particular, if $f_2 = 1$, then $M_1 \times (f_1, f_2)M_2$ is called the twisted product of $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ with twisting function $f_1$. Moreover, if $f_1$ only depends on the points of $M_1$, then $M_1 \times f_2 M_2$ is called the warped product of $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ with warping function $f_1$. Also, as a generalization of the warped product of the two Riemannian manifolds $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$, $M_1 \times (f_1, f_2)M_2$ is called the doubly warped product of Riemannian manifolds $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ with twisting functions $f_1$ and $f_2$ if $f_1$ and $f_2$ only depend on the points of $M_1$ and $M_2$, respectively (see [18]).

The Hopf conjecture is true for the singly warped product manifold (see [20]). In this paper, we study the Hopf conjecture in the class of doubly warped products and conformal metrics to the product metrics on $S^2 \times S^2$. Let $S^2 \times (f_1, f_2)S^2$ be a doubly twisted product manifolds with twisting functions $f_1$ and $f_2$. Next, we are going to prove the following results.

**Theorem 1.** A doubly warped product manifold $S^2 \times (f_1, f_2)S^2$ does not have positive sectional curvature.

**Theorem 2.** A conformal metric $g_{S^2 \times S^2}$ to a product metric $g_{S^2} \oplus g_{S^2}$ on $S^2 \times S^2$ does not have positive sectional curvature.

The paper is organized as follows: After preliminaries, Section 2 contains the basic notions of doubly twisted products, and the proofs of the two theorems above will been given in Section 3.

## 2. Preliminaries

Let $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ be two Riemannian manifolds with Levi-Civita connections $\nabla^{M_1}$ and $\nabla^{M_2}$, respectively. Denote by $\nabla$ the Levi-Civita connection and grad the gradient of the doubly twisted products $M_1 \times (f_1, f_2)M_2$ of $(M_1, g_{M_1})$ and $(M_2, g_{M_2})$ with twisting functions $f_1$ and $f_2$. Let $\sigma_1 = \log(f_1)$ and $\sigma_2 = \log(f_2)$. For a $Z$ vector field of $M_1$, the lift of $Z$ to $M_1 \times (f_1, f_2)M_2$ is the vector field $\tilde{Z}$ whose value at each $(p, q)$ is the lift of $Z_p$ to $(p, q)$. Thus, the lift of $Z$ is the unique vector field on $M_1 \times (f_1, f_2)M_2$, that is, $\pi_1$-related to $Z$ and $\pi_2$-related to the zero vector field on $M_2$ (see [19]). We denote the set of lifts of vector fields on $M_1$ and $M_2$ to $M_1 \times M_2$ by $\mathfrak{V}(M_1)$ and $\mathfrak{V}(M_2)$, respectively. The following two propositions are results from [18] (Proposition 1 and Proposition 2).

**Proposition 1** (see [18]). If $X, Y \in \mathfrak{V}(M_1)$ and $V \in \mathfrak{V}(M_2)$, then

$$\nabla_X Y = \nabla^{M_1}_X Y + X(\sigma_2)Y + Y(\sigma_2)X - g(X, Y)\text{grad}(\sigma_2), \tag{2}$$

$$\nabla_X V = V(\sigma_1)X + X(\sigma_1)V. \tag{3}$$

In a doubly twisted product $M_1 \times (f_1, f_2)M_2$, we can interchange the roles of the base manifold $(M_1, g_{M_1})$ and the fiber manifold $(M_2, g_{M_2})$. In the above proposition, we have the expression of the Levi-Civita connection for vector fields $X, Y \in \mathfrak{V}(M_1)$ in (2). Similarly, by making corresponding changes, one can easily see the expression of Levi-Civita connection for vector fields $V, W \in \mathfrak{V}(M_2)$ as in (2). Obviously, the analogue expressions hold for vector fields in $\mathfrak{V}(M_2)$ by making corresponding changes [18].

We will denote by $R_{M_1}$, $R_{M_2}$, and $R$ the curvature tensors of $(M_1, g_{M_1})$, $(M_2, g_{M_2})$, and $(M_1 \times (f_1, f_2)M_2$, respectively.
Lemma 1. Let \( M_1 \times (f_1, f_2) M_2 \) be a doubly twisted product manifold with metric tensor \( g_M \) denoted by \( \langle \cdot, \cdot \rangle \). For \( X, Y \in \mathfrak{X}(M_1) \), \( V \in \mathfrak{X}(M_2) \) such that \( X \) and \( V \) are orthonormal on some neighborhood \( \mathcal{U} \) of a point for the product metric \( g_{f_1 f_2} \). Then, at any point \( p \) of \( \mathcal{U} \), the sectional curvature of the mixed 2-plane spanned by \( X(p) \) and \( V(p) \) is given by

\[
K(X, V)(p) = \frac{1}{f_1 f_2} \left[ \langle XX, \nabla^M_X X \rangle (\sigma_1) - \langle VV, \nabla^M_V V \rangle (\sigma_2) \right] + \langle XX, \nabla^M_X X \rangle (\sigma_1) - 2 \langle X, \nabla^M_X X \rangle (\sigma_2) \|V\|^2
+ \frac{1}{f_1 f_2} \left[ \langle VV, \nabla^M_V V \rangle (\sigma_2) - 2 \langle V, \nabla^M_V V \rangle (\sigma_2) \|X\|^2 \right] + \frac{1}{f_1 f_2} \langle \nabla^M(\sigma_1), \nabla^M(\sigma_2) \rangle \|V\|^2 \|X\|^2 \quad \text{at } p.
\]

Proof. By definition,

\[
K(X, V)(p) = \frac{\langle R(X, V)X, V \rangle - \langle X, V \rangle^2}{\langle X, X \rangle \langle V, V \rangle}.
\]

By Proposition 2, we have that

\[
\langle R(X, V)X, V \rangle = \langle h^{12}_M(X, X) + (X(\sigma_1))^2 - 2X(\sigma_1)X(\sigma_2) \|V\|^2
+ \langle V, \nabla^M_V V \rangle (\sigma_2) - 2 \langle V, \nabla^M_V V \rangle (\sigma_2) \|X\|^2
\]

\[
+ \langle \nabla^M(\sigma_1), \nabla^M(\sigma_2) \rangle \|V\|^2 \|X\|^2 \quad \text{at } p.
\]

3. The Main Results

In this section, we give the proof of Theorem 1 and Theorem 2.

3.1. Proof of Theorem 1

Proof. We will assume that the sectional curvature of the manifold \( M_1 \times (f_1, f_2) M_2 \) is everywhere positive. We consider the case where \( M_1 \) and \( M_2 \) are the standard unit sphere \( S^2 \) and the doubly warped product manifolds \( S^2 \times (f_1, f_2) S^2 \) with the warping functions \( f_1 \) and \( f_2 \). We keep the hypotheses and notations of Lemma 1. Since \( S^2 \times (f_1, f_2) S^2 \) is a doubly warped product manifold, then

\[
K(X, V)(p) = \frac{\langle R(X, V)X, V \rangle - \langle X, V \rangle^2}{\langle X, X \rangle \langle V, V \rangle}.
\]
\[ X(\sigma_2) = V(\sigma_1) = 0, \quad \langle \text{grad}(\sigma_1), \text{grad}(\sigma_2) \rangle = (\text{grad}(\sigma_2))(\sigma_1) = 0. \] (13)

By using the above relations with (6) and the fact that the mixed sectional curvature \( K(X, V) \) is positive, we have
\[
\begin{align*}
XX(\sigma_1) - (V^{M}_X X)(\sigma_1) + (X(\sigma_2))^2 \|V\|^2 \\
+ [VV(\sigma_2) - (V^{M}_V V)(\sigma_2) + (V(\sigma_2))^2] \|X\|^2 > 0.
\end{align*}
\] (14)

Since \( \sigma_1 \) (respectively, \( \sigma_2 \)) depends only of the points of the left sphere \( M_1 \) of \( M \) (respectively, depends only of the points of the right sphere \( M_2 \) of \( M \)), we can define a smooth function \( \tilde{\sigma} \) in \( M_1 \) (respectively, a smooth function \( \bar{\sigma} \) in \( M_2 \)) such that \( \sigma_1 = \tilde{\sigma} \circ p_1 \) and \( \sigma_2 = \bar{\sigma} \circ p_2 \). So, we have the following remarks:

1. If a point \( p \in M_1 \) is a maximum for \( \tilde{\sigma} \), then for any \( y \in M_2 \), \( (p, y) \) is a maximum of \( \sigma_1 \).
2. If a point \( q \in M_2 \) is a maximum for \( \bar{\sigma} \), then for any \( z \in M_1 \), \( (z, q) \) is a maximum of \( \sigma_2 \).
3. Consequently, if \( p \) is a maximum for \( \tilde{\sigma} \) and \( q \) is a maximum for \( \bar{\sigma} \), then \( (p, q) \) is a maximum for \( \sigma_1 \) and \( \sigma_2 \).

Then, at the critical point \( x = (p, q) \) of \( \sigma_1 \) and \( \sigma_2 \), we have
\[
(X(\sigma_1))^2 = (V(\sigma_2))^2 = (V^{M}_X X)(\sigma_1) = (V^{M}_V V)(\sigma_2) = 0\] (15)
at \( x = (p, q) \).

Using (15)–(17) in (14), we get a contradiction.

3.2. Proof of Theorem 2

**Proof.** We take \( M_1 = S^2 \) and \( M_2 = S^2 \). If \( f_1 = f_2 = f \), \( M_1 \times_{(f, f)} M_2 \) is a doubly twisted product with the twisting function \( f \) and \( \sigma = \log(f) \). Then, the tensor metric \( g_M = f^2 \pi^*_1 g_{M_1} + f^2 \pi^*_2 g_{M_2} = f^2 (\pi^*_1 g_{M_1} + \pi^*_2 g_{M_2}) = f^2 g_{M_1} \otimes g_{M_2} \) is a conformal metric to the product metric \( g_{M_1} \otimes g_{M_2} \).

Now, for \( X \in \mathfrak{X}(M_1) \), \( V \in \mathfrak{X}(M_2) \) such that \( X \) and \( V \) are orthonormal with respect to the product metric \( g_{M_1} \otimes g_{M_2} \), on some neighborhood \( \mathcal{U} \) of a point, then at any point \( p \) of \( \mathcal{U} \) the sectional curvature of the mixed 2-plane spanned by \( X \) and \( V \) is given by
\[
K((X, V)\mathcal{p}) = \frac{1}{f^4} [XX(\sigma) - (V^{M}_X X)(\sigma) - (X(\sigma))^2] \|V\|^2 \\
+ \frac{1}{f^4} [VV(\sigma) - (V^{M}_V V)(\sigma) - (V(\sigma))^2] \|X\|^2 \\
+ \frac{1}{f^4} \|\text{grad}(\sigma)\|^2 \|V\|^2 \|X\|^2 \] (18)
at \( p \).

Finally, Since \( S^2 \times_{(f, f)} S^2 \) is a compact manifold, the smooth function \( \sigma = S^2 \times S^2 \rightarrow \mathbb{R} \) has a maximum \( q = (q_1, q_2) \). At the critical point \( q \) of \( \sigma \), we have \( (X(\sigma))^2 = (V^{M}_X X)(\sigma) = (V^{M}_V V)(\sigma) = 0 \). And the sectional curvature of the mixed 2-plane spanned by \( X(\sigma) \) and \( V(\sigma) \) is given by
\[
K(X, V) = \frac{1}{f^4} [XX(\sigma)\|V\|^2 + VV(\sigma)\|X\|^2].
\] (19)

Since the critical point \( q \) of \( \sigma \) is a maximum then, we have
\[
h^\sigma(X, X) = \langle H^\sigma(X), X \rangle = XX(\sigma)\leq 0 \quad \text{at } q, \quad h^\sigma(V, V) = \langle H^\sigma(V), V \rangle = VV(\sigma)\leq 0 \quad \text{at } q,
\] (20)

where \( H^\sigma \) denotes the Hessian of \( \sigma \) with respect to the metric \( f^2 (g_{M_1} \otimes g_{M_2}) \). In this case, the sectional curvature \( K(X, V) \) at \( q \) is not positive [21, 22].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

**References**


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