Research Article

A Note on Tsuji's Criterion for Numerical Triviality

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In this study, we give an alternative and elementary proof to Tsuji's criterion for a Cartier divisor to be numerically trivial.

1. Introduction

In this article, every algebraic variety is proper over the field of complex numbers $\mathbf{C}$.

In 1970s, Iitaka [1] initiated the classification theory of higher dimensional algebraic varieties by using the pluri-canonical systems. In 1980s, Mori [2] deepened the Iitaka theory by cutting off the subvarieties of elliptic type.

In [3], Tsuji gave an interesting and useful criterion for a Cartier divisor to be numerically trivial.

Theorem 1 (Tsuji [[3], Lemma 5.1], cf. Bauer et al. [[4], Theorem 2.4]). *Let* $f: M \longrightarrow B$ *a surjective morphism between complete varieties. Let* L *be a nef Cartier divisor on* M *and* W *some subvarieties of* M, *such that* $f(W) = B$ *and* B_0 *a subset of* B *which is a union of countably many proper Zariski-closed subsets. Assume that*

(1) for some $b \in B$, $(L, C) = 0$ *for every curve* C *on* $f^{-1}(b)$

(2) $(L, C) = 0$ *for every irreducible curve* C *on* W, *such that* $f(C) \nsubseteq B_0$

Then, L *is numerically trivial.*

Tsuji's criterion for numerical triviality is one of the basic tools to decompose every algebraic variety into the varieties of elliptic type, of parabolic type, and of hyperbolic type by cutting off the varieties of parabolic type [5–7].

Tsuji's proof [3] is analytic and the proof [4] by Bauer et al. is algebraic.

In this research note, we give an alternative and elementary proof to Tsuji's criterion (Theorem 1). The argument (Subcase 7), which uses the following corollary of the Hodge index theorem, due to Bauer et al. is essential.

Lemma 1 (cf. [[4], Proposition 2.5]). *Let* $f: M \longrightarrow B$ *be a surjective morphism from a complete surface* M *to a complete curve* B. *Let* L *be a nef Cartier divisor on* M. *Assume that*

(1) For some $b \in B$, $(L, C) = 0$ *for every curve* C *on* $f^{-1}(b)$

(2) $(L, W) = 0$ *for some irreducible curve* W *on* M, *such that* $f(W) = B$

Then, L *is numerically trivial.*

Remark 1. In the statement of Lemma 1, condition (1) immediately implies that L is numerically trivial on every general fiber of the morphism f by considering the flattening. By the normalization, the Stein factorization, and the desingularization, the article ([4], Proposition 2.5), for an algebraically fibered surface, implies the assertion of Lemma 1.

2. Elementary Proof of Main Theorem 1

Proof. We prove the assertion by induction on $(\dim M, \dim B)$.

First, we take a commutative diagram as shown in Figure 1 with the following properties:

(1) M' and B' are nonsingular projective varieties

(2) μ is a birational morphism

(3) ν is a generically finite morphism

$$M' \xrightarrow{\quad g \quad} B'$$
$$\mu \downarrow \qquad\qquad \downarrow \nu$$
$$M \xrightarrow{\quad f \quad} B$$

FIGURE 1: The Stein factorization for f is as follows.

(4) g is a morphism with only connected fibers

There exists some irreducible component W' of $\mu^{-1}(W)$, such that $g(W') = B'$. We set $L' := \mu^* L$.

The locus $g(\cup \{C' | C'$ *is an irreducible curve on* M', g (C') *is a point, and the intersection number* $(L', C') > 0\})$ is included in a union of at most countably many proper Zariski-closed subsets of B' (Proposition 1). Thus, we obtain a union $B'_0 \supseteq \nu^{-1}(B_0)$ of countably many proper Zariski-closed subsets of B' with the following two properties:

(1) L' is numerically trivial on every fiber of g over $B' \backslash B'_0$

(2) $(L', C'_W) = 0$ for every irreducible curve C'_W on W', such that $g(C'_W) \not\subseteq B'_0$

It suffices to prove that $(L', C') = 0$ for every irreducible curve C' on M'. We fix an irreducible curve C' on M'.

Case 1. $g(C') \not\subseteq B'_0$. This case divides into Subcases 1 and 2.

Subcase 1. $g(C') \not\subseteq B'_0$ and $g(C')$ is a point.
We have $(L', C') = 0$ from (1).

Subcase 2. $g(C') \not\subseteq B'_0$ and $g(C')$ is a curve. This subcase divides into Subcases 3 and 7.

Subcase 3. $g(C') \not\subseteq B'_0$, $g(C')$ is a curve and $\dim B = 1$. This subcase divides into Subcases 4, 5, and 6.
We note that $g(W_1) = g(C') = B'$ for some irreducible curve W_1 on W'.

Subcase 4. $g(C') \not\subseteq B'_0$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M = 1$.
$M' = W' = W_1 = C'$. Thus, $(L', C') = 0$.

Subcase 5. $g(C') \not\subseteq B'_0$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M = 2$.
Because $(L', W_1) = 0$ from (2), Lemma 1 implies that L' is numerically trivial, and thus, $(L', C') = 0$.

Subcase 6. $g(C') \not\subseteq B'_0$, $g(C')$ is a curve, $\dim B = 1$, and $\dim M \geqq 3$.
Because the codimension $\operatorname{codim}(W_1 \cup C', M') \geqq 2$, we have an irreducible hyperplane section H of M' that includes W_1 and C' (Proposition 2). Then, $L'|_H$ is numerically trivial from the induction hypothesis. Consequently, $(L', C') = 0$.

Subcase 7 (cf. [[4], 2.1.2]). $g(C') \not\subseteq B'_0$, $g(C')$ is a curve, and $\dim B \geqq 2$.

Let $S := \{S_i\}$ be the set of irreducible components of $g^{-1}(g(C'))$. We note that $g(W_1) = g(C')$ for some irreducible curve W_1 on W'. Thus, $L'|_{S_1}$ is numerically trivial for some $S_1 \in S$, such that $S_1 \supseteq W_1$ from the property (2) and from the induction hypothesis.

If $g(\cup_{m\neq 1} S_m) = g(C')$, then $g(S_1 \cap (\cup_{m\neq 1} S_m)) = g(C')$ from the connectedness of fibers of g, and therefore, $g(S_1 \cap S_2) = g(C')$ for some $S_2 \in S$.

Thus, $\dim g(\cup_{m\neq 1} S_m) \leqq 0$ or $g(S_1 \cap S_2) = g(C')$.

If $g(S_1 \cap S_2) = g(C')$ and $g(\cup_{m\neq 1,2} S_m) = g(C')$, then $g((S_1 \cup S_2) \cap (\cup_{m\neq 1,2} S_m)) = g(C')$ from the connectedness of fibers of g, and therefore, $g((S_1 \cup S_2) \cap S_3) = g(C')$ for some $S_3 \in S$. From this argument, we obtain the following properties:

(1) $g(S_1) = g(S_2) = \cdots = g(S_k) = g(C')$

(2) $g((S_1 \cup S_2 \cup \cdots \cup S_{i-1}) \cap S_i) = g(C')$ for all i with $1 \leqq i \leqq k$

(3) $\dim g(\cup_{i\neq 1,2,\dots k} S_i) \leqq 0$

The fact that $L'|_{S_1}$ is numerically trivial and that $g(S_1 \cap S_2) = g(C')$ implies that $L'|_{S_2}$ is numerically trivial from the induction hypothesis. The fact that $L'|_{S_1 \cup S_2}$ is numerically trivial and that $g((S_1 \cup S_2) \cap S_3) = g(C')$ implies that $L'|_{S_3}$ is numerically trivial from the induction hypothesis. This argument implies that $L'|_{S_1 \cup S_2 \cup \cdots \cup S_k}$ is numerically trivial. Because $\dim g(\cup_{i\neq 1,2,\dots k} S_i) \leqq 0$, we have that $C' \subseteq S_1 \cup S_2 \cup \cdots \cup S_k$. Consequently, $(L', C') = 0$.

Case 2. $g(C') \subseteq B'_0$. This case divides into Subcases 8 and 11.

Subcase 8. $g(C') \subseteq B'_0$ and $\dim M = 2$. This subcase divides into Subcases 9 and 10.

Subcase 9. $g(C') \subseteq B'_0$, $\dim M = 2$, and $\dim B = 1$.
Lemma 1 implies that L' is numerically trivial, and thus, $(L', C') = 0$.

Subcase 10. $g(C') \subseteq B'_0$, $\dim M = 2$, and $\dim B = 2$.
Because $W' = M'$, we have that $(L', H) = 0$ for an irreducible hyperplane section H of M' from the property (2) of the divisor L'. The Hodge index theorem implies that L' is numerically trivial. Thus, $(L', C') = 0$.

Subcase 11. $g(C') \subseteq B'_0$ and $\dim M \geqq 3$.
Because the codimension $\operatorname{codim}(C', M') \geqq 2$, there exists an irreducible hyperplane section H of M' that includes C' (Proposition 2). We may assume that $g(H) \not\subseteq B'_0$. Note that, from Case 1, $(L, C'') = 0$ for every irreducible curve C'' on H, such that $g(C'') \not\subseteq B'_0$. Thus, $L'|_H$ is numerically trivial from the induction hypothesis. Consequently, $(L', C') = 0$. □

3. Appendix

In this appendix, we state two elementary propositions and their proofs, which are well known to the experts, for the readers' convenience.

Proposition 1. *Let* $f \colon M \longrightarrow B$ *be a surjective morphism between projective varieties and L a nef Cartier divisor on M.*

We assume that for some $b \in B$, the intersection number $(L, C) = 0$ for every irreducible curve C on $f^{-1}(b)$.

Then, the locus $f(\cup \{C | C \text{ is an irreducible curve on } M,$ $f(C) \text{ is a point, and the intersection number } (L, C) > 0\})$ is included in a union of at most countably many proper Zariski-closed subsets of B.

Proof. There exists some ample divisor A on B. Assume that C is an irreducible curve on M, such that $(f^*A, C) = 0$ (i.e., $f(C)$ is a point) and that $(L, C) > 0$. There exists some irreducible component W of the universal scheme for the Hilbert scheme $Hilb(M)$ of M, such that W includes $p_2^{-1}([C])$, where $[C]$ is the point $(\in Hilb(M))$, which represents the subscheme C of M, and Figure 2 shows the projections p_1 and p_2 and the property that $dim W = dim p_2(W) + 1$. We set $T := p_2(W)$.

First, we consider the normalization $n_1 \colon W_n \longrightarrow W$, $n_2 \colon T_n \longrightarrow T$, and $n_3 \colon W_n \longrightarrow T_n$ of the morphism $p_2|_W \colon W \xrightarrow{s_1} T$. Next, consider the Stein factorization $W_n \xrightarrow{s_1} T' \xrightarrow{s_2} T_n$ of the morphism $n_3 \colon W_n \longrightarrow T_n$.

Last, consider the flattening $f_1 \colon W'' \longrightarrow W_n$, $f_2 \colon T'' \longrightarrow T'$, and $f_3 \colon W'' \longrightarrow T''$ of the morphism $s_1 \colon W_n \longrightarrow T'$, where the morphism f_2 is birational and the variety T'' is nonsingular. We note that the morphism $f_3 \colon W'' \longrightarrow T''$ is flat and with only connected fibers.

We put $h := n_1 f_1$.

Thus, we have the commutative diagram, as shown in Figure 3.

From the flatness of the morphism $f_3 \colon W'' \longrightarrow T''$, the intersection number $(h^* p_1^* f^* A, F'') > 0$ for every fiber F'' of the morphism $f_3 \colon W'' \longrightarrow T''$ because $(p_1^* f^* A, p_2^{-1}([C])) = 0$. Thus, for every fiber F'' of the morphism $f_3 \colon W'' \longrightarrow T''$, the morphism f contracts $p_1 h(F'')$ to one point from the connectedness of F''. In other words, $p_1 h(F'')$ is included in some fiber of f.

There exists some ample divisor A' on W. Of course, $(A', F) > 0$ for every curve F on W. Because the morphism h is birational, we have that $h(F'')$ is not a point (i.e., $(h^* A', F'') > 0$) for some fiber F'' of $f_3 \colon W'' \longrightarrow T''$. From the flatness, $(h^* A', F'') > 0$ for every fiber F'' of $f_3 \colon W'' \longrightarrow T''$. Thus, every fiber F'' of $f_3 \colon W'' \longrightarrow T''$ cannot be contracted to a point by the morphism h.

There exists some fiber F_1'' of $f_3 \colon W'' \longrightarrow T''$, such that $h^{-1}(p_2^{-1}([C])) \cap F_1'' \neq \varnothing$. Then, $F_1'' \subset h^{-1}(p_2^{-1}([C]))$ because the morphism $p_2 h$ maps F_1'' to a point $[C] \in T$. Consequently, $h(F_1'') = p_2^{-1}([C])$ because F_1'' does not contract to a point by the morphism h. Thus, $(h^* p_1^* L, F_1'') > 0$. From the flatness of the morphism $f_3 \colon W'' \longrightarrow T''$, the intersection number is $(h^* p_1^* L, F'') > 0$ for every fiber F'' of the morphism $f_3 \colon W'' \longrightarrow T''$.

We note that every fiber of $f_3 \colon W'' \longrightarrow T''$ is mapped in some fiber of $p_2|_W \colon W \longrightarrow T$. In other words, every fiber of $p_2|_W$ is swept out by fibers of f_3.

So, for every fiber F of $p_2|_W \colon W \longrightarrow T$, the locus $p_1(F)$ is swept out by connected curves C', such that $f(C')$ is one
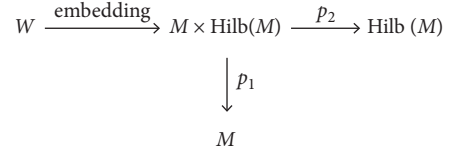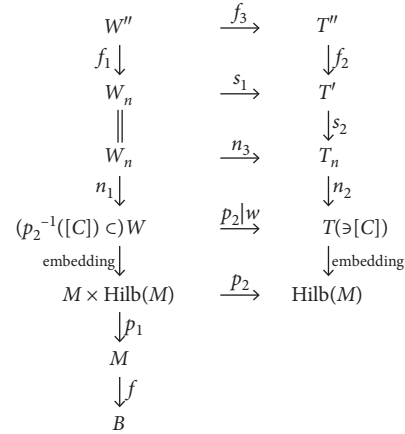

Figure 2: The deformation of C in M.

Figure 3: The flattening of the deformation.

point and that the intersection number $(L, C') > 0$. We note that we consider $p_1 h(F'')$ as C' and that $C' = p_1 h(F'') = Supp((p_1 h)_* F'')$ from the connectedness of F''. Thus, $b \notin f(p_1(F))$. Consequently, $b \notin f(p_1(W))$. In other words, $p_1(W)$ is disjoint with $f^{-1}(b)$.

The countability of the irreducible components of the Hilbert scheme $Hilb(M)$ of M implies the assertion. $\square$

Proposition 2. *Let M be a nonsingular projective variety and C a Zariski-closed subset with codimension $co\,dim(C, M) \geq 2$. Then, there exists some irreducible hyperplane section H, such that H^C.*

Proof. We take some ample divisor A on M. We have a birational morphism $f \colon M' \longrightarrow M$, such that M' is a nonsingular projective variety, that $f^{-1}(C)$ is divisorial with only simple normal crossings and that there exists an effective divisor C_0 with the property that $Supp(C_0) = f^{-1}(C)$ and $-C_0$ is f-ample. Then, $m f^* A - C_0$ is ample for a sufficiently large integer m. For a sufficiently large and divisible integer l, the divisor lmA is very ample, and there exists a member $H_0 \in |l(m f^* A - C_0)|$ which is very ample and irreducible. We put $H := f_*(H_0 + lC_0)$. Then, $H \in |lmA|$.

The locus $f^{-1}(H)$ coincides with $Supp(f^* H) = Supp(H_0 + lC_0)$. Thus, $f^{-1}(H)^{f^{-1}(C)}$. $\square$

Data Availability

No data were used to support this study.

Disclosure

Conflicts of Interest

Acknowledgments

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