Hindawi
Journal of Mathematics
Volume 2022, Article ID 6323524, 11 pages
https://doi.org/10.1155/2022/6323524

Research Article

Monadic Effect Algebras

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Received 24 April 2022; Revised 21 June 2022; Accepted 20 July 2022; Published 19 September 2022

Academic Editor: Li Guo

The main goal of this paper is to introduce and investigate the related theory on monadic effect algebras. First, we design the axiomatic system of existential quantifiers on effect algebras and then use it to give the definition of the universal quantifier and monadic effect algebras. Then, we introduce relatively complete subalgebra and prove that there exists a one-to-one correspondence between the set of all the existential quantifiers and the set of all the relatively complete subalgebras. Moreover, we characterize and give the generated formula of monadic ideals and prove that Riesz monadic ideals and Riesz monadic congruences can be mutually induced. Finally, we study the strong existential quantifier and characterize monadic simple and monadic subdirectly irreducible effect algebras.

1. Introduction

Effect algebras have been introduced by Foulis and Bennett [1] for modeling unsharp measurement in the quantum mechanical system. They are a generalization of many structures which arise in the axiomatization of quantum mechanics (Hilbert space effects), noncommutative measure theory and probability (orthomodular lattices and posets), fuzzy measure theory, and many-valued logic (MV-algebras). Moreover, several attempts have been made to study ideals and congruences on effect algebras, staring points are in the paper [2, 3] in which it is proved that there exists an order isomorphism between Riesz ideals and Riesz congruences and developed in [4, 5].

The name "monadic" comes from the connection with predicate logics for languages having one placed predicate and a single quantifier. Monadic Boolean algebras are Boolean algebras with an additional unary operation which is an algebraic counterpart of the existential quantifier; they were introduced by Halmos [6] as an algebraic counterpart of the one-variable fragment of the classical predicate logic. Algebraic counterparts of the existential or universal quantifiers have also been consequently studied for certain nonclassical logics. For example, monadic MV-algebras were introduced and studied in [7], which is an algebraic model of the one-element fragment of Łukasiewicz predicate logic. Also, the theory of monadic MV-algebras has been developed in [8, 9]. Then, monadic noncommutative GMV-algebras having the same sense for noncommutative Łukasiewicz logic were introduced and studied in [10]. Moreover, monadic Heyting algebras were introduced in [11] as an algebraic model of the one-variable fragment of the intuitionistic predicate logic. From then on, the notion of monadic has been extended to other logical algebras such as monadic BL-algebras [12–14], monadic residuated lattices [15], monadic R_0-algebras [16], monadic hoop algebras [17], monadic equality algebras [18], monadic quantum B-algebras [19, 20], and their noncommutative cases. Note that, since both MV-algebras and GMV-algebras satisfy De Morgan and double negation laws, in the definition of the corresponding monadic algebras, it is impossible to use only one of the existential and universal quantifiers as initial; the other is then definable as the dual of the original one. However, in contrast to the monadic MV-algebras and GMV-algebras, monadic Heyting algebras, monadic BL-algebras, and other logical algebras require using both kinds of quantifiers simultaneously because these quantifiers are not mutually definable.

Originally, effect algebras were introduced as partial algebraic structure, and several attempts have been made to combine effect algebras with total algebraic structure [21–23]. However, previously, "monadic" was just introduced and studied on total algebraic structure. In our paper, we try to introduce the "monadic" into the effect algebra which is a typical partial algebraic structure. We believe that this can be helpful to study the effect algebras better.

The paper is organized as follows. In Section 2, we recall some basic definitions and properties in effect algebras which will be used in the remainder of the paper. In Section 3, we design the axiomatic system of existential quantifiers on effect algebras. Since effect algebras also satisfy the double negation laws, we just use existential quantifier to define monadic effect algebras. Also, the universal quantifier is definable as the dual of the existential quantifiers. Then, we give some nontrivial examples for monadic effect algebras and study some properties of existential quantifiers and universal quantifiers. Moreover, we introduce relatively complete subalgebras of effect algebras and study the relationship between the existential quantifiers and the relatively complete subalgebras. In Section 4, we introduce monadic ideals on monadic effect algebras. In particular, we focus on the relationship between Riesz monadic ideals and Riesz monadic congruences and the relationship between the lattice of all monadic ideals of $(E, \exists)$ and the lattice of all ideals of $\exists E$. Moreover, we consider in which case the quotient structure $(E/I, \exists_I)$ is also a monadic effect algebra. Finally, we introduce and characterize monadic simple and monadic subdirectly irreducible effect algebras.

2. Preliminaries

In this section, we summarize some definitions and results about effect algebras, which will be used in the following part, and we shall not cite them every time they are used.

Definition 1 (see [1]). An effect algebra (EA for short) is a structure $(E; \oplus, 0, 1)$ consisting of a set E, elements 0 and 1 in E are the zero and the unit, and a partially defined operation $\oplus$ on E is called the orthosummation, such that for all $x, y, z \in E$:

$(EA1)$: if $x \oplus y$ is defined, then $y \oplus x$ is defined and $x \oplus y = y \oplus x$

$(EA2)$: if $x \oplus y$ and $(x \oplus y) \oplus z$ are defined, then $y \oplus z$ and $x \oplus (y \oplus z)$ are defined and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

$(EA3)$: for each $x \in E$, there exists a unique element $x \in E$ called the orthosupplement of x, such that $x \oplus x = 1$

$(EA4)$: $x \oplus 1$ is defined only if $x = 0$

Remark 1 (see [1]). In an effect algebra E, a partial ordering is defined by $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It turns out that the element c, if it exists, is uniquely defined. If $a \oplus b$ is defined, we say that a and b are orthogonal and write $a \perp b$. Moreover, $a \perp b$ if and only if $a \leq b$. Moreover, if $a \vee b$ and $a \wedge b$ exist for all $a, b \in E$, then we say that E is a lattice ordered effect algebra.

The partial operation $\ominus$ by $b \ominus a := c$ iff $a \oplus c = b$. Clearly, z is defined iff $a \leq b$. With respect to this partial order, we have $0 \leq a \leq 1$ for all $a \in E$. In particular, $a = 1 \ominus a$. Moreover, if $a \perp b$, then $a \oplus b = (a \ominus b) = (b \ominus a)$. The other partial operation $\odot$ by $a \odot b := (a \oplus b)$. Clearly, $a \odot b$ is defined iff $a \leq b$ iff $b \leq a$.

Proposition 1 (see [1]). *Let E be an effect algebra and $a, b, c \in E$. Then,*

(1) If $a \leq b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$

(2) If $a \leq b \leq c$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$

(3) If $a \leq b \leq c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and also $(c \ominus b) \oplus (b \ominus a) = c \ominus a$

Definition 2 (see [1]). Let E be an effect algebra and $F \subseteq E$. F is called a subalgebra of E if (i) $0, 1 \in F$, (ii) for $p \in F$, $p \in F$, and (iii) for $p, q \in F$ with $p \perp q$, $p \oplus q \in F$.

Definition 3 (see [3]). A subset I of an effect algebra E is called an ideal, if for all $a, b \in E$:

$(I1)$: if $a, b \in I$ and $a \perp b$, then $a \oplus b \in I$

$(I2)$: if $a \in I$ and $b \leq a$, then $b \in I$

An ideal I of an effect algebra E is called a Riesz ideal, if for all $a, c, d \in E$, where $c \perp d$

$(I3)$: if $a \in I$ and $c \oplus d \geq a$, then there exists $h, k \in I$ with $h \leq c, k \leq d$ and $h \oplus d \geq a$

An ideal I of E is called a proper ideal if $I \neq E$. Also, a proper ideal I of E is called a maximal ideal if it is not contained in any proper ideal of E. We denote the set of all ideals of E by $I(E)$ and the set of all Riesz ideals of E by $RI(E)$. Moreover, for any $X \subseteq E$, there will be a smallest ideal $(X]$ with $X \subseteq (X]$. We call $(X]$ the ideal generated by X.

Lemma 1 (see [3]). *Let E be an effect algebra and $X \subseteq E$. Then, the ideal $(X]$ generated by X can be constructed in the following way: define $I_0 = X$, $I_{n+1} := \cup_{x \in I_n} E[0, x]$, and $I_{n+2} := \{x \oplus y\colon x, y \in I_{n+1}, x \perp y\}$, $n = 0, 1, \cdots$. Clearly, $I_n \subseteq I_{n+1} \subseteq I_{n+2}$, $n = 0, 1, \cdots$. Then, $(X] = \bigcup_{n=0}^{\infty} I_n$.*

Definition 4 (see [3]). A congruence on an effect algebra E is an equivalence relation $\sim$ such that for all $a, a_1, b, b_1 \in E$:

C1: if $a_1 \sim a$, $b_1 \sim b$, $a \perp b$, and $a_1 \perp b_1$, then $a_1 \oplus b_1 \sim a \oplus b$

C2: if $a_1 \sim a$ and $a \perp b$, then there exists $b_0 \in E$ with $a_1 \perp b_0$ and $b_0 \sim b$

A congruence $\sim$ on E is a Riesz congruence provided that the following holds for all $a, b \in E$

C3: if $a \sim b$, then there exists $c \in E$ such that $c \perp a$, $c \perp b$, and $a \oplus c \sim 1 \sim b \oplus c$

Given an ideal I of an effect algebra E, we may define a relation $\sim_I$ on E as follows: $a \sim_I b \Leftrightarrow$ there exist $i, j \in I$: $i \leq a$, $j \leq b$ $a \ominus i \leq b$, and $b \ominus j \leq a$. According to the relationship between ideals and congruences on effect algebras, we have the following theorem.

Theorem 1 (see [3]). *Let E be an effect algebra. Then, the following holds:*

(i) If I is a Riesz ideal, then $\sim_I$ *is a Riesz congruence and* $[0]_{\sim_I} = I$
(ii) If $\sim$ *is a Riesz congruence, then* $I = [0]_\sim$ *is a Riesz ideal and* $\sim_I = \sim$
(iii) The map which assigns to every Riesz congruence the equivalence class of 0 is an order isomorphism between the lattice of all Riesz congruence and the lattice of all Riesz ideal, whose inverse is the map $I \mapsto \sim_I$

Theorem 2 (see [3]). *For a congruence* $\sim$ *on an effect algebra E, the set of all congruence class is denoted by* $E/\sim$*, i.e.,* $E/\sim\ = \{[x] | x \in E\}$*. We define* $[a] \perp [b]$ *if there exists* $a_1, b_1 \in E$ *such that* $a_1 \sim a$*,* $b_1 \sim b$*, and* $a_1 \perp b_1$ *and put* $[a] \oplus [b] = [a_1 \oplus b_1]$*. Then,* $(E/\sim, \oplus, [0], [1])$ *is an effect algebra which is called a quotient effect algebra of E.*

Definition 5 (see [4]). Let $(E; \oplus, 0, 1)$ be an effect algebra. Then,

(1) E is called simple, if it has exactly two ideals: $\{0\}$ and E
(2) E is called subdirectly irreducible if among the nontrivial congruence relations of E, there exists the least one

Definition 6. (see [8]). Let $M = (M, \oplus, \odot, *, 0, 1)$ be an MV-algebra. A mapping $\exists$: $M \longrightarrow M$ is called an existential quantifier on M, if for all $x, y \in M$, the following conditions hold:

$(E1) x \leq \exists x$
$(E2) \exists (x \vee y) = \exists x \vee \exists y$
$(E3) \exists (\exists x)^* = (\exists x)^*$
$(E4) \exists (\exists x \oplus \exists y) = \exists x \oplus \exists y$
$(E5) \exists (x \odot y) = \exists x \odot \exists y$
$(E6) \exists (x \oplus y) = \exists x \oplus \exists y$

3. Monadic Effect Algebra

In this section, we use the existential quantifier to construct the axiomatic system of monadic effect algebras. Also, we introduced relatively complete subalgebras of effect algebras and prove that there exists a one-to-one correspondence between the set of the existential quantifiers and the set of relatively complete subalgebras.

Definition 7. Let $E = (E, \oplus, 0, 1)$ be an effect algebra. A mapping $\exists$: $E \longrightarrow E$ is called an existential quantifier on E, if for all $x, y \in E$, the following conditions hold:

$(E1)$: $x \leq \exists x$
$(E2)$: if $x \leq y$, then $\exists x \leq \exists y$
$(E3) \exists (\exists x) = \exists x$
$(E4) \exists (\exists x \oplus \exists y) = \exists x \oplus \exists y$, whenever $\exists x \oplus \exists y$ is defined

Example 1. Let $E = (E, \oplus, 0, 1)$ is an effect algebra. Clearly, we have that id_E is an existential quantifier on E.

From the above example, we know that monadic effect algebra is a generalization of effect algebra.

Example 2. Let $E = [0, 1]$. For any $x, y \in E$, we define $x \oplus y = x + y$ if and only if $x + y \leq 1$ and $x = 1 - x$, then $(E, \oplus, 0, 1)$ is an effect algebra. For any $n \in N$, we define $\exists_n$ as follows:

$$\exists_n x = \begin{cases} 0, & x = 0, \\ \\ \dfrac{i}{n}, & x \in \left(\dfrac{i-1}{n}, \dfrac{i}{n}\right] i = 1, 2, \ldots, n. \end{cases} \tag{1}$$

We can check that $\exists_n$ is an existential quantifier on E.

Example 3. Let $E = \{0, a, b, c, d, 1\}$, where $0 \leq a, b \leq c, d \leq 1$. Define operation $\oplus$ and as follows (see Tables and 21):

Then, $(E, \oplus, \neg, 0, 1)$ is an effect algebra. We define $\exists$ as follows: $\exists 0 = 0, \exists a = \exists d = d, \exists b = b$, and $\exists c = \exists 1 = 1$. We can easily check that $\exists$ is an existential quantifier on E.

Remark 2. As is well known, we call a lattice ordered effect algebra which satisfies $a \wedge b = 0 \Rightarrow a \leq b$ as an MV-effect algebra, and MV-algebras and MV-effect algebras are in one-to-one correspondence; we always identify them. However, the existential quantifiers on MV-algebras and MV-effect algebras are different. We can check that an existential quantifier on MV-algebra is an existential quantifier on MV-effect algebra, but the converse not holds. Let $(E, \exists)$ be the monadic effect algebra given by Example 3; we can check that E is an MV-effect algebra. Since $a \oplus a = b$ and $\exists a = d$, $\exists a \oplus \exists a = d \oplus d$ is not defined, which implies that $\exists$ does not satisfy the E6 in the definition of monadic MV-algebras.

Proposition 2. *Let M be an MV-algebra and* $\exists$ *be an existential quantifier on E. Then, for all* $x, y \in E$*, the following conditions are valid:*

(1) $\exists 1 = 1$ *and* $\exists 0 = 0$
(2) $\exists \exists x = \exists x$
(3) $x \leq \exists y \Leftrightarrow \exists x \leq \exists y$
(4) $\exists x \leq \exists x$
(5) If $\exists x \oplus \exists y$ *is defined, then* $\exists (x \oplus y) \leq \exists x \oplus \exists y x$
(6) The set $\exists E$ *is an effect subalgebra,* $\exists E = \{x \in E: \exists x = x\}$*, and* $\exists$ *on* $\exists E$ *is the identity on* $\exists E$
(7) If $y \leq x$*, then* $\exists (\exists x \ominus \exists y) = \exists x \ominus \exists y$
(8) If $y \leq x$*, then* $\exists x \ominus \exists y \leq \exists (x \ominus y)$
(9) $\exists (\exists x \odot \exists y) = \exists x \odot \exists y$*, whenever* $\exists x \odot \exists y$ *is defined*
(10) If $x \leq y$*, then* $\exists (x \odot y) \leq \exists x \odot \exists y$
(11) $\exists (\exists x \wedge \exists y) = \exists x \wedge \exists y$*, whenever* $\exists x \wedge \exists y$ *exist*

Proof

(1) By $E1$, $1 \leq \exists 1$; thus, $\exists 1 = 1$ as 1 is the greatest element of E. By $E3$, $\exists 0 = \exists 1 = \exists (\exists 1) = \exists 1 = 1 = 0$.

Table 1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	1	—	—
b	b	c	—	—	1	—
c	c	1	—	—	—	—
d	d	—	1	—	—	—
1	1	—	—	—	—	—

Table 2

	$\neg$
0	1
a	c
b	d
c	a
d	b
1	0

(2) By (1) and $E4$, $\exists\exists x = \exists(\exists x \oplus 0) = \exists(\exists x \oplus \exists 0) = \exists x \oplus \exists 0 = \exists x \oplus 0 = \exists x$.

(3) If $x \leq \exists y$, by $E2$ and (2), we have $\exists x \leq \exists\exists y = \exists y$; conversely, if $\exists x \leq \exists y$, by (1), we have $x \leq \exists x \leq \exists y$.

(4) By (1), we have $x \leq \exists x$; then, $\exists x \leq x \leq \exists x$.

(5) If $\exists x \oplus \exists y$ is defined, then we have $\exists x \leq \exists y$, so $x \leq \exists x \leq \exists y \leq y$; then, we get $x \oplus y$ defined. Therefore, $\exists(x \oplus y) \leq \exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$.

(6) For any $x \in \exists E$, there exists $y \in E$, such that $x = \exists y$; then, we have $\exists x = \exists\exists y = \exists y = x$. Therefore, $\exists E = \{x \in E\colon \exists x = x\}$. For any $x, y \in \exists E$, we have $x = \exists x$ and $y = \exists y$; if $x \oplus y$ is defined, then $x \oplus y = \exists x \oplus \exists y = \exists(\exists x \oplus \exists y)$. Therefore, $x \oplus y \in \exists E$. For any $x \in \exists E$, then $x = \exists x$, so $\exists x = \exists\exists x = \exists x = x$. Therefore, $x \in \exists E$. By the above, we have $\exists E$ as an effect subalgebra. Finally, since $\exists\exists x = \exists x$, we have $\exists$ on $\exists E$ as the identity on $\exists(E)$.

(7) If $y \leq x$, then $\exists y \leq \exists x$, so $\exists x \ominus \exists y$ is defined. Now, we assume $\exists x \ominus \exists y = z$; then, $\exists x \ominus \exists y = (\exists x \oplus \exists y) = (\exists\exists x \oplus \exists y) = (\exists(\exists\exists x \oplus \exists y)) = \exists(\exists(\exists\exists x \oplus \exists y))$, so $z \in \exists E$. Then, we have $z = \exists z$. Therefore, $\exists(\exists x \ominus \exists y) = \exists z = z = \exists x \ominus \exists y$.

(8) If $y \leq x$, then there exists $z \in E$, such that $y \oplus z = x$, so $\exists x \ominus \exists y = \exists(y \oplus z) \ominus \exists y \leq (\exists y \oplus \exists z) \ominus \exists y = \exists z = \exists(x \ominus y)$.

(9) $\exists(\exists x \odot \exists y) = \exists(\exists x \oplus \exists y) = \exists(\exists\exists x \oplus \exists\exists y) = \exists\exists(\exists\exists x \oplus \exists\exists y) = \exists(\exists\exists x \oplus \exists\exists y) = \exists(\exists\exists x \oplus \exists\exists y) = (\exists x \oplus \exists y) = \exists x \odot \exists y$.

(10) If $x \leq y$ is defined, then $x \leq y$, so $\exists x \leq \exists y$, and so $\exists x \leq \exists x \leq \exists y$. Then, we have $\exists x \odot \exists y$ defined. Therefore, $\exists(x \odot y) \leq \exists(\exists x \odot \exists y) = \exists x \odot \exists y$.

(11) Assume that $d = \exists x \wedge \exists y$ is defined in E. We show that $\exists d = d$. Check, $\exists d \leq \exists\exists x = \exists x$ and $\exists d \leq \exists\exists y = \exists y$. This yields $\exists d \leq \exists x \wedge \exists y = d$, so $\exists d = d$. Then, $\exists(\exists x \wedge \exists y) = \exists x \wedge \exists y$.

In monadic MV-algebras, the existential quantifier $\exists$ and the universal quantifier $\forall$ can be mutually induced. Similarly, by means of $\exists$, we define $\forall$ on E by the rule

$$\forall x\colon = (\exists x)(R). \tag{2}$$

□

Proposition 3. *Let E be an effect algebra and $\exists$ be an existential quantifier on E. If $\forall$ is defined by (R), then the following conditions are satisfied:*

$(A1)$: $\forall x \leq x$

$(A2)$: if $x \leq y$, then $\forall x \leq \forall y$

$(A3)\forall(\forall x) = \forall x$

$(A4)$: $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$, whenever $\forall x \odot \forall y$ is defined

Proof

$(A1)$: since $x \leq \exists x$, then $x = x \geq (\exists x) = \forall x$

$(A2)$: if $x \leq y$, then $y \leq x$, so $\exists y \leq \exists x$, and then $\forall x = (\exists x) \leq (\exists y) = \forall y$

$(A3)\forall(\forall x) = \exists(\forall x) = \exists((\exists x)) = y(\exists x) = \forall x$

$(A4)\forall(\forall x \odot \forall y) = (\exists((\exists x) \odot (\exists y))) = \exists(\exists x \oplus \exists y) = (\exists x \oplus \exists y) = ((\exists x) \oplus (\exists y)) = \forall x \odot \forall y$

An unary operation $\forall\colon E \longrightarrow E$ on an effect algebra that satisfies $(A1)-(A4)$ will be called a universal quantifier. □

Proposition 4. *Let $E = (E, \oplus, 0, 1)$ be an effect algebra and $\forall$ be a universal quantifier on E. Define $\exists x = (\forall x)$; then, $\exists$ is an existential quantifier on E.*

Proof. It is similar to the above proposition.

From Propositions 2 and 3, we get that for an effect algebra, there exists a one-to-one correspondence between existential quantifiers and universal quantifiers. Therefore, alike monadic MV-algebras, we can choose only one between the existential quantifiers and the universal quantifiers to construct the axiomatic system of monadic effect algebras. In order to study the monadic ideals and the related theory on effect algebra and for an ideal I on an effect algebra, when $\exists x \in I$, we can get $\forall x \in I$, but the reverse side not holds. Therefore, we use the existential quantifiers to construct the axiomatic system of monadic effect algebras which is more natural and convenient. □

Definition 8. Let E be an effect algebra and $\exists$ be an existential quantifier on E. Then, the couple $(E, \exists)$ is called a monadic effect algebra.

Proposition 5. *Let $\forall$ be a universal quantifier on an effect algebra E. For all $x, y \in E$, the following conditions are valid:*

(1) $\forall 0 = 0$ and $\forall 1 = 1$

(2) $\forall\forall x = \forall x$

(3) $\forall x \leq y \Leftrightarrow \forall x \leq \forall y$

(4) $\forall x \leq \forall x$

(5) If $\forall x \odot \forall y$ is defined, then $\forall x \odot \forall y \leq \forall (x \odot y)$

(6) $\forall (\forall x \oplus \forall y) = \forall x \oplus \forall y$, whenever $\forall x \oplus \forall y$ is defined

(7) If $x \oplus y$ is defined, then $\forall x \oplus \forall y \leq \forall (x \oplus y)$

(8) The set $\forall E$ is an effect subalgebra, $\forall E = \{x \in E\colon \forall x = x\}$, and $\forall$ on $\forall E$ is the identity on $\forall E$

(9) If $y \leq x$, then $\forall (\forall x \ominus \forall y) = \forall x \ominus \forall y$

(10) If $y \leq x$, then $\forall (x \ominus y) \leq \forall x \ominus \forall y$

(11) $\forall (\forall x \vee \forall y) = \forall x \vee \forall y$, whenever $\forall x \vee \forall y$ exists

Proof

(1) Since $\forall 0 \leq 0$, we have $\forall 0 = 0$ as 0 is the least element of E. $\forall 1 = \forall 0 = \forall 0 = \exists 0 = 0 = 1$.

(2) $\forall \forall x = \forall (\forall x \odot 1) = \forall (\forall x \odot \forall 1) = \forall x \odot \forall 1 = \forall x \odot 1 = \forall x$.

(3) If $\forall x \leq y$, we have $\forall x \leq \forall \forall y = \forall y$; conversely, if $\forall x \leq \forall y$, we have $\forall x \leq \forall y \leq y$.

(4) Since $\forall x \leq x$, we have $\forall x \leq x \leq \forall x$.

(5) If $\forall x \odot \forall y$ is defined, then we have $\forall x \leq \forall y$, so $x \leq \forall x \leq \forall y \leq y$; therefore, $x \odot y$ is defined. So, $\forall x \odot \forall y = \forall (\forall x \odot \forall y) \leq \forall (x \odot y)$.

(6) $\forall (\forall x \oplus \forall y) = \forall (\forall x \odot \forall y) = \forall (\forall \forall x \odot \forall \forall y) = \forall \forall (\forall \forall x \odot \forall \forall y) = \forall (\forall \forall x \odot \forall \forall y) = (\forall \forall x \odot \forall \forall y) = (\forall x \odot \forall y) = \forall x \oplus \forall y$.

(7) If $x \oplus y$ is defined, then $x \leq y$; thus, $\forall x \leq x \leq y \leq \forall y$. Therefore, $\forall x \oplus \forall y$ is defined. So, $\forall x \oplus \forall y = \forall (\forall x \oplus \forall y) \leq \forall (x \oplus y)$.

(8 and 9) Similar to (6) and (7) of Proposition 2.

(10) If $y \leq x$, then there exists z such that $y \oplus z = x$ and $\forall y \leq \forall x$, so $\forall x \ominus \forall y = \forall (y \oplus z) \ominus \forall y \geq (\forall y \oplus \forall z) \ominus \forall y = \forall z = \forall (x \ominus y)$.

(11) Assume that $d = \forall x \vee \forall y$ is defined in E. We show that $\forall d = d$. Check $\forall d \geq \forall \forall x = \forall x$ and $\forall d \geq \forall \forall y = \forall y$. This yields $\forall d \geq \forall x \vee \forall y = d$, so $\forall d = d$. Then, $\forall (\forall x \vee \forall y) = \forall x \vee \forall y$. □

Proposition 6. *Let $(E, \exists)$ be a monadic effect algebra and $\forall$ be the universal quantifier defined by (R). For all $x, y \in E$, the following conditions are valid:*

(1) $\forall \exists x = \exists x$ and $\exists \forall x = \forall x$

(2) $\forall x = x \Leftrightarrow \exists x = x$

(3) $\exists x = \forall x$ and $\forall x = \exists x$

(4) $\forall (\exists x \odot \exists y) = \exists x \odot \exists y$ and $\exists (\forall x \odot \forall y) = \forall x \odot \forall y$, whenever $\exists x \odot \exists y$ and $\forall x \odot \forall y$ are defined

(5) $\forall (\exists x \oplus \exists y) = \exists x \oplus \exists y$ and $\exists (\forall x \oplus \forall y) = \forall x \oplus \forall y$, whenever $\exists x \odot \exists y$ and $\forall x \odot \forall y$ are defined

(6) If $y \leq x$, then $\exists (\forall x \ominus \forall y) = \forall x \ominus \forall y$ and $\forall (\exists x \ominus \exists y) = \exists x \ominus \exists y$

(7) $(\exists, \forall)$ establishes a Galois connection

Proof

(1) $\forall \exists x = \exists \exists x = \exists x = \exists x$ and $\exists \forall x = \forall \forall x = \forall x = \forall x$.

(2) If $\forall x = x$, then we have $x \in \forall (E)$, so there exists y, such that $\forall y = x$. Thus, $\exists x = \exists \forall y = \forall y = x$. Conversely, if $\exists x = x$, then we have $x \in \exists (E)$, so there exists y, such that $\exists y = x$. Thus, $\forall x = \forall \exists y = \exists y = x$.

(3) $x \exists x = \forall x = \forall x$ and $\forall x = \exists x = \exists x$.

(4) $\forall (\exists x \odot \exists y) = \forall (\forall \exists x \odot \forall \exists y) = \forall \exists x \odot \forall \exists y = \exists x \odot \exists y$. $\exists (\forall x \odot \forall y) = \exists (\exists \forall x \odot \exists \forall y) = \exists \forall x \odot \exists \forall y = \forall x \odot \forall y$.

(5) $\forall (\exists x \oplus \exists y) = \forall (\forall \exists x \oplus \forall \exists y) = \forall \exists x \oplus \forall \exists y = \exists x \oplus \exists y$. $\exists (\forall x \oplus \forall y) = \exists (\exists \forall x \oplus \exists \forall y) = \exists \forall x \oplus \exists \forall y = \forall x \oplus \forall y$.

(6) If $y \leq x1$, then $\forall x \ominus \forall y$ and $\exists x \ominus \exists y$ are defined. $\exists (\forall x \ominus \forall y) = \exists (\exists \forall x \ominus \exists \forall y) = \exists \forall x \ominus \exists \forall y = \forall x \ominus \forall y$ and $\forall (\exists x \ominus \exists y) = \forall (\forall \exists x \ominus \forall \exists y) = \forall \exists x \ominus \forall \exists y = \exists x \ominus \exists y$.

(7) $\exists \forall x = \forall x \leq x$ and $\forall \exists x = \exists x \geq x$, so $(\exists, \forall)$ establishes a Galois connection.

Let $\forall$ be a universal quantifier on E. We denote the kernel of $\forall$ by $\mathrm{Ker}(\forall) = \{x \in E\colon \forall x = 0\}$. Moreover, we call $\forall$ as faithful if $\mathrm{Ker}(\forall) = \{0\}$. □

Proposition 7. *Let $\forall$ be a universal quantifier on an effect algebra E. Then, the following conditions are valid:*

(1) If $\forall$ is faithful, then $x < y$ implies $\forall x < \forall y$

(2) $\forall$ is faithful $\Leftrightarrow \forall x = x$, for any $x \in E$

(3) $\forall$ is faithful $\Leftrightarrow \exists x = x$, for any $x \in E$

Proof

(1) Suppose that $x < y$, and $\forall x = \forall y$ and $\forall (y \ominus x) \leq \forall y \ominus \forall x = 0$ giving $y \ominus x = 0$ so that $x = y$; this is a contradiction.

(2) For any $x \in E$, $\forall (x \ominus \forall x) \leq \forall x \ominus \forall \forall x = 0$; since $\forall$ is faithful, we have $x \ominus \forall x = 0$; then, $x = \forall x$. The converse is clear.

(3) $\forall x = x \Leftrightarrow \exists x = x$.

Next, we study the relationship between the existential quantifiers and the relatively complete subalgebras on monadic effect algebras.

A subalgebra E_0 of an effect algebra E is said to be relatively complete if for every $a \in E$, the set $\{b \in E_0 | a \leq b\}$ has the least element in E_0, which is denoted by

$$\inf\{b \in E_0 | a \leq b\} \text{ or } \wedge_{a \leq b \in E_0} b. \tag{3}$$

□

Theorem 3. *Let $(E, \exists)$ be a monadic effect algebra. Then, $\exists E$ is a relatively complete subalgebra of E and $\exists a = \inf\{b \in \exists E | a \leq b\}$.*

Proof. By Proposition 2 (6), $\exists E$ is a subalgebra of E. Let $a \in E$ and $A = \{b \in \exists E | a \leq b\}$. Then, $\exists a \in \exists E$ and for any $b \in A$, we have $b = \exists b$. So, $\exists a \leq \exists b = b$; thus, $\exists a$ is the least element of A. Therefore, $\exists E$ is a relatively complete subalgebra of E. □

Remark 3. Let E be the effect algebra given in Example 2. Then, by Theorem 3, we know that, for any $n \in N$, the subset $\{0, 1/n, 2/n, \ldots, n-1/n, 1\}$ is a relatively complete subalgebra of E.

Theorem 4. *Let $E = (E; \oplus, 0, 1)$ be an effect algebra and E_0 be a relatively complete subalgebra of E. For any $a \in E$, define $\exists a = \inf\{b \in E_0 | a \leq b\}$. Then, $E = (E; \oplus, \exists, 0, 1)$ is a monadic effect algebra.*

Proof. We only need to prove $\exists$ defined above which is an existential quantifier on E.

($E1$): $a \leq b$, where $\{b \in E_0 | a \leq b\}$. Then, we get $a \leq \inf\{b \in E_0 | a \leq b\} = \exists a$.

($E2$): if $x \leq y$, then $\{b \in E_0 | y \leq b\} \subseteq \{b \in E_0 | x \leq b\}$. So, $\inf\{b \in E_0 | x \leq b\} \leq \inf\{b \in E_0 | y \leq b\}$, i.e., $\exists x \leq \exists y$.

For any $x \in E_0$, since $x \leq x$, we have $\inf\{b \in E_0 | x \leq b\} \leq x$, i.e., $\exists x \leq x$. Also, by the definition of $\exists$, we have $x \leq \exists x$. Therefore, for any $x \in E_0$, we have $x = \exists x$.

($E3$): for any $x \in E$, we have $\exists x \in E_0$; since E_0 is a subalgebra of E, we get $\exists x \in E_0$, and hence $\exists(\exists x) = \exists x$.

($E4$): $\exists x, \exists y \in E_0$, and E_0 is a subalgebra of E. Hence, if $\exists x \oplus \exists y$ is defined, then $\exists x \oplus \exists y \in E_0$, and so we have $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$.

Therefore, $E = (E; \oplus, \exists, 0, 1)$ is a monadic effect algebra.

Let A and B be two posets. Consider a function $h: A \longrightarrow B$; the function $k: B \longrightarrow A$ is called left adjoint to h, if $k(b) \leq a \Leftrightarrow b \leq h(a)$ for any $a \in A$ and $b \in B$. □

Theorem 5. *Let E be an effect algebra. Then, there exists a one-to-one correspondence between each pair of the following sets:*

(1) The set of all the existential quantifiers on E, denoted by $E(E)$

(2) The set of all the relatively complete subalgebras on E, denoted by $RCS(E)$

(3) The set of all the pairs (E, E_0), where E_0 is a subalgebra of E and the canonical embedding $h: E_0 \longrightarrow E$ has left adjoint function, denoted by $P(E, E_0)$

Proof

(i) Consider the mapping $f: E(E) \longrightarrow \mathrm{RCS}(E)$, i.e., $f(\exists) = \exists E$. For any $\exists \in E(E)$, according to Theorem 3, $\exists E$ is a relatively complete subalgebra of E. Also, consider the mapping $g: \mathrm{RCS}(E) \longrightarrow E(E)$, i.e., $g(E_0) = \exists'$. For any relatively complete subalgebra $E_0 \in \mathrm{RCS}(E)$, according to Theorem 4, $\exists' a = \inf\{b \in E_0 | a \leq b\}$ is an existential quantifier on E. So, for any $a \in E$, we have $\exists a = \inf\{b \in \exists E | a \leq b\} = \exists' a$. Therefore, $g^\circ f = id_{E(E)}$. On the other hand, for any $E_0 \in \mathrm{RCS}(E)$, suppose $g(E_0) = \exists'$; then, $f(\exists') = \exists' E$. For any $a \in E_0$, by the definition of $\exists'$, we have $a = \exists' a \in \exists' E$, so $E_0 \subseteq \exists' E$. Conversely, for any $a \in \exists' E$, there exists $b \in E$, which satisfies $\exists' b = a$. Since $\exists'$ is an existential quantifier, then $\exists' a = \exists' \exists' b = \exists' b = a$; therefore, $a = \exists' a = \inf\{b \in E_0 | a \leq b \in E_0\}$, i.e., $\exists' E \subseteq E_0$. Therefore, $\exists' E = E_0$, which implies that $f^\circ g = id_{\mathrm{RCS}(E)}$. According to the above, there exists a one-to-one correspondence between (1) and (2).

(ii) Let E_0 be a relatively complete subalgebra of E. We define $\exists_h(a) = \inf\{b \in E_0 | a \leq b\}$, and $h: E_0 \longrightarrow E$ is the canonical embedding. If $\exists_h(a) \leq b$, where $a \in E$ and $b \in E_0$, then $b \in \{c \in E_0 | a \leq c\}$; hence, $a \leq b = h(b)$. Conversely, if $a \leq h(b)$, then $a \leq b$, so $b \in \{c \in E_0 | a \leq c\}$, and so $\exists_h(a) \leq b$. Hence, $\exists_h(a) \leq b \Leftrightarrow a \leq h(b)$. Therefore, the canonical embedding $h: E_0 \longrightarrow E$ has left adjoint function, and the left adjoint function of h is $\exists_h$. On the other hand, for a given pair (E, E_0), where E_0 is a subalgebra of E, the canonical embedding $h: E_0 \longrightarrow E$ has left adjoint function. Now, we prove $\inf\{b \in E_0 | a \leq b\}$ exists. For any $a \in E$, $\exists_h(a) \in E_0$. For any $x \in \{b \in E_0 | a \leq b\}$, we have $a \leq x = h(x)$, so $\exists_h(a) \leq x$. Also, $\exists_h(a) \leq \exists_h(a)$; then, $a \leq h\exists_h(a) = \exists_h(a)$, so we have $a \leq \exists_h(a)$. Hence, $\exists_h(a)$ is the least element in $\{b \in E_0 | a \leq b\}$. Therefore, $\inf\{b \in E_0 | a \leq b\} = \exists_h(a)$. And so, there exists a one-to-one correspondence between (2) and (3).

(iii) By (i) and (ii), we know that there exists a one-to-one correspondence between (1) and (3). □

4. Monadic Ideals on Monadic Effect Algebras

In this section, we introduce and investigate monadic ideals on monadic effect algebras. Also, in order to get some more and better results, we introduce strong existential quantifier. Moreover, we introduce and characterize simple and subdirectly irreducible monadic effect algebras.

Definition 9. Let $E = (E; \oplus, \exists, 0, 1)$ be a monadic effect algebra and I be an ideal of E. Then, I is called a monadic ideal if for any $i \in I$, we have $\exists i \in I$.

When I is a Riesz ideal and I is also a monadic ideal, we say I is a Riesz monadic ideal. We denote the sets of all the monadic ideals and the Riesz monadic ideals of $(E, \exists)$ by $\mathrm{MI}(E)$ and $\mathrm{RMI}(E)$, respectively.

Example 4. For any $n \in N$, let $(E, \exists_n)$ be the monadic effect algebras given by Example 2. Then, $\{0\}$ is the only monadic ideal in each $(E, \exists_n)$.

Example 5. Let $(E, \exists)$ be the monadic effect algebra given in Example 3. Take $I_1 = \{0, a, d\}$ and $I_2 = \{0, b\}$; we can check that I_1 and I_2 are monadic ideal of $(E, \exists)$. $I_3 = \{0, a\}$ is an ideal but not a monadic ideal. $a \in I_3$, but $\exists a = d \in I_3$.

Example 6. Let $(E, \exists)$ be the monadic effect algebra given in Example 3. Take $I_1 = \{0, a, d\}$, $I_2 = \{0, b\}$, $I_3 = \{0, a\}$, and $I_4 = \{0, c\}$. We can check that I_2 is a Riesz monadic ideal. I_1 is not a Riesz monadic ideal as I_1 is not a Riesz ideal. Since $d \leq a \oplus c$, $h, k \in I_1$ does not exist such that $d \leq h \oplus k$. I_3 is not a Riesz monadic ideal as I_3 is not a monadic ideal. I_4 is not a Riesz monadic ideal as I_4 is neither a monadic ideal nor a Riesz ideal.

Proposition 8. *Let $(E, \exists)$ be a monadic effect algebra and $X \subseteq E$. The monadic ideal generated by X is denoted by $(X]_{\exists}$; then, $(X]_{\exists}$ can be constructed in the following way: define $I_0 = X$, $I_{n+1} := \cup_{x \in \exists I_n} E[0, x]$, and $I_{n+2} := \{x \oplus y: x, y \in I_{n+1}, x \perp y\}$, $n = 0, 1, \cdots$. Clearly, $I_n \subseteq I_{n+1} \subseteq I_{n+2}$, $n = 0, 1, \cdots$. Then, $(X]_{\exists} = \cup_{n=0}^{\infty} I_n$.*

Proof. Clearly, $(X]_{\exists}$ is an ideal of $(E, \exists)$. For any $t \in (X]_{\exists}$, there exists $n \in N$ such that $t \in I_n$. If $I_n = \cup_{x \in \exists I_{n-1}} E[0, x]$, then there exists $y \in \exists I_{n-1}$ such that $t \leq y$; then, $\exists t \leq \exists y = y$, which implies $\exists t \in I_n \subseteq (X]_{\exists}$. If $I_n = \{x \oplus y: x, y \in I_{n-1}, x \perp y\}$, then there exist $a, b \in I_{n-1}$ such that $t = a \oplus b$; then, $\exists t = \exists (a \oplus b)$ and $\exists (a \oplus b) \in I_{n+1}$, which implies $\exists t \in I_{n+1} \subseteq (X]_{\exists}$. Therefore, $(X]_{\exists}$ is a monadic ideal of $(E, \exists)$. □

Definition 10. Let $(E, \exists)$ be a monadic effect algebra. If for any $x, y \in E$, $x \perp y$ and $x \oplus y \neq 1$ imply $\exists x \perp \exists y$; then, we say the existential quantifier $\exists$ is a strong and $(E, \exists)$ is a strong monadic effect algebra.

Example 7. Let $E = (E, \oplus, 0, 1)$ be an effect algebra. We have that id_E is a strong existential quantifier on E.

Example 8. Let $(E, \exists)$ be the monadic effect algebra given by Example 3. Then, we have $a \perp a$ on E and $\exists a = d$, but $d \perp d$. Hence, the existential quantifier $\exists$ is not strong.

Example 9. Let E={0, a, b, c, 1}. Define operation $\oplus$ and $\neg$ as follows:(see Tables 3 and 4):

Then, $(E, \oplus, \neg, 0, 1)$ is an effect algebra. We define $\exists$ as follows: $\exists 0 = 0$, $\exists a = \exists b = b$, and $\exists c = \exists 1 = 1$. We can easily check that $\exists$ is a strong existential quantifier on E.

Table 3

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	b	c	1	—
b	b	c	1	—	—
c	c	1	—	—	—
1	1	—	—	—	—

Table 4

	$\neg$
0	1
a	c
b	b
c	a
1	0

Proposition 9. *Let $(E, \exists)$ be a strong monadic effect algebra $a \in \exists E$ and I be a monadic ideal. If $J = (\{a\} \cup I]$, then J is also a monadic ideal.*

Proof. For any $x \in J$, since the $\oplus$ is commutative and associative, then there exist $n \in N$ and $i \in I$ such that $x \leq na \oplus i$. If $na \oplus i = 1$, then $\exists x \leq \exists (na \oplus i) = \exists 1 = 1$. Then, $\exists x \in J$. If $na \oplus i \neq 1$, then $\exists x \leq \exists (na \oplus i) \leq \exists (na) \oplus \exists i \leq n \exists a \oplus \exists i = na \oplus \exists i$ as $\exists$ is strong. Since $\exists i \in I$, then $na \oplus \exists i \in J$; it follows that $\exists x \in J$. Therefore, J is a monadic ideal of $(E, \exists)$. □

Theorem 6. *Let $(E, \exists)$ be a strong monadic effect algebra and I be an ideal of E. Then,*

(1) I is a monadic ideal of $(E, \exists)$ if and only if $I = (I \cap \exists E]$

(2) There exists a lattice isomorphism between the lattice of all monadic ideals of $(E, \exists)$ and the lattice of all ideals of $\exists E$

Proof

(1) Let $(E, \exists)$ be a monadic effect algebra. $I \subseteq E$ is a monadic ideal. Let $i \in I$, then $\exists i \in I$ and $\exists i \in I \cap \exists E$; since $i \leq \exists i$, we get $i \in (I \cap \exists E]$, so $I \subseteq (I \cap \exists E]$. Moreover, since $I \cap \exists E \subseteq I$, then we have $(I \cap \exists E] \subseteq (I] \subseteq I$. Therefore, we get $I = (I \cap \exists E]$.

Conversely, let $I = (I \cap \exists E]$; if $I = E$, clearly, I is a monadic ideal. If $I \neq E$, for I_0 in $(I \cap \exists E]$, we can easily check that $\exists I_0 \subseteq (I \cap \exists E]$. For I_1 in $(I \cap \exists E]$, for any $x \in I_1$, then there exists $a \in I_0$ such that $x \leq a$; then, we have $\exists x \leq \exists a = a \in I_0$, so $\exists x \in I_1$; hence, $\exists I_1 \in (I \cap \exists E]$. For I_2 in $(I \cap \exists E]$, for any $x \in I_2$, there exists $a, b \in I_1$ such that $a \perp b$ and $x = a \oplus b$; then, $\exists x = \exists (a \oplus b) \leq \exists a \oplus \exists b$ as $\exists$ is strong and $a \oplus b \neq 1$. Since $\exists a, \exists b \in I_1$, then $\exists a \oplus \exists b \in I_2$. We have $\exists x \in I_3$. Therefore, we get $\exists I_2 \subseteq (I \cap \exists E]$. The rest may be deduced by analogy. Therefore, for any I_n in $(I \cap \exists E]$, we have $\exists I_n \subseteq (I \cap \exists E]$. Then, we have $\exists (I \cap \exists E] \subseteq (I \cap \exists E]$, so $(I \cap \exists E]$ is a monadic ideal of $(E, \exists)$. Therefore, we get that if $I = (I \cap \exists E]$, then I is a monadic ideal of $(E, \exists)$.

(2) First, we prove that, for any ideal I of $\exists E$, we have $I = (I] \cap \exists E$.

If $I = \exists E$, clearly, we have $I = (I] \cap \exists E$. If I is proper, it is clear that $I \subseteq (I] \cap \exists E$. Conversely, for I_0 in $(I]$, we have $I_0 = I$; then, $\exists I_0 = I$. For I_1 in $(I]$, for any $x \in I_1$, there exists $y \in I_0$, such that $x \leq y$; then, $\exists x \leq \exists y$. Since $\exists I_0 = I$ and I is an ideal, we have $\exists x \in I$; then, $\exists I_1 \subseteq I$. For I_2 in $(I]$, for any $x \in I_2$, there exists $a, b \in I_1$ such that $a \perp b$ and $x = a \oplus b$. If $a \in I_1$, then $\exists a \in \exists I_1 \subseteq I$, and so $\exists a \in I$, Since $a \in I_1$, then $\exists a \in \exists I_1 \subseteq I$, so $\exists a \oplus \exists a = 1 \in I$, which is a contradictory to I is proper, so $a \notin I_1$. Therefore, $a \oplus b \neq 1$. Then, $\exists x = \exists (a \oplus b) \leq \exists a \oplus \exists b$ as $\exists$ is strong and $a \oplus b \neq 1$. Since I is an ideal, then we have $\exists a \oplus \exists b \in I$; it follows that $\exists x \in I$ and so $\exists I_2 \subseteq I$. The rest may be deduced by analogy. We have that, for any I_n in $(I]$, we have $\exists I_n \subseteq I$. Also, for any I_n in $(I]$, for any $x \in I_n$, $\exists x \in \exists I_n \subseteq I$. If $x \in I_n$, $\exists x \in \exists I_n \subseteq I$, so $\exists x \in I$, and so $\exists x \oplus \exists x = 1 \in I$, which is a contradictory to I that is proper, so $a \notin I_n$. It follows that if I is proper, then we have $(I]$ that is proper. For I_0 in $(I]$, $I_0 \cap \exists E = I \subseteq I$. For I_1 in $(I]$, for any $x \in I_1 \cap \exists E$, there exists $a \in I_0$ such that $x \leq a$; then, $x = \exists x \leq \exists a \in I$, so $x \in I$. Therefore, $I_1 \cap \exists E \subseteq I$. For I_2 in $(I]$, for any $x \in I_2 \cap \exists E$, there exists $a, b \in I_1$ such that $x = a \oplus b$, so $x = \exists x = \exists (a \oplus b) \leq \exists a \oplus \exists b$ as $\exists$ is strong and $a \oplus b \neq 1$. Since $a, b \in I_1$, we have $\exists a, \exists b \in I$, and so $x \leq \exists a \oplus \exists b \in I$. Therefore, $I_2 \cap \exists E \subseteq I$. The rest may be deduced by analogy. For any I_n in $(I]$, we have $I_n \cap \exists E \subseteq I$. Therefore, for any ideal I of $\exists E$, we have $I = (I] \cap \exists E$.

Next, we prove that there exists a lattice isomorphism between the lattice of all monadic ideals of $(E, \exists)$ and the lattice of all ideals of $\exists E$. We have already known that $\exists E$ is an effect algebra. Then, correspondence $I \longrightarrow (I]$ maps any ideal I of $\exists E$ to the monadic ideal $(I]$ of $(E, \exists)$. Moreover, $I = (I] \cap \exists E$. On the other hand, let J be a monadic ideal of $(E, \exists)$, then $J = (J \cap \exists E]$, where $J \cap \exists E$ is an ideal of $\exists E$. So, we have found the required $1-1$ correspondence between the set of monadic ideals of $(E, \exists)$ and the set of ideals of $\exists E$. Moreover, it is clear that $I_1 \subseteq I_2 \Rightarrow (I_1] \subseteq (I_2]$ and $J_1 \subseteq J_2 \Rightarrow J_1 \cap \exists E \subseteq J_2 \cap \exists E$. Therefore, there exists a lattice isomorphism between the lattice of all monadic ideals of E and the lattice of all ideals of E_0. ☐

Definition 11. Let $E = (E; \oplus, \exists, 0, 1)$ be a monadic effect algebra and $\sim$ be a congruence relation on E. $\sim$ is called a monadic congruence relation if $a \sim b$ implies that $\exists a \sim \exists b$, for any $a, b \in E$.

Proposition 10. *If I is a Riesz monadic ideal of a monadic effect algebra $(E, \exists)$, we define a relation $\sim_I$ on $(E, \exists)$ as follows: $a \sim_I b \Leftrightarrow$ there exist $i, j \in I$: $i \leq a, j \leq b, a \ominus i \leq b$, and $b \ominus j \leq a$; then, $\sim_I$ is a monadic congruence of $(E, \exists)$.*

Proof. By Theorem 1, we have already known that if I is a Riesz ideal, then $\sim_I$ is a congruence of E. Now, we prove $\sim_I$ is a monadic congruence of E. If $a \sim_I b$, then there exists $i, j \in I$ such that $i \leq a, j \leq b, a \ominus i \leq b$, and $b \ominus j \leq a$. Then, $\exists a \ominus \exists i \leq \exists (a \ominus i) \leq \exists b$ and $\exists b \ominus \exists j \leq \exists (b \ominus j) \leq \exists a$; since I is a Riesz monadic ideal, we have $\exists i, \exists j \in I$. Then, $\exists a \sim_I \exists b$. Therefore, $\sim_I$ is a monadic congruence of $(E, \exists)$. ☐

Theorem 7. *Let $(E, \exists)$ be a monadic effect algebra. Then, the following holds:*

(i) *If I is a Riesz monadic ideal, then $\sim_I$ is a Riesz monadic congruence and $[0]_{\sim_I} = I$*

(ii) *If $\sim$ is a Riesz monadic congruence, then $I = [0]_\sim$ is a Riesz monadic ideal and $\sim_I = \sim$*

(iii) *There exists an order isomorphism between the lattice of all Riesz monadic congruence and the lattice of all Riesz monadic ideal*

Proof

(a) From Theorem 1 and Proposition 10, the result is stragightforward.

(b) By Theorem 1, we get $I = [0]_\sim$ which is a Riesz ideal. So, we just prove $I = [0]_\sim$ is a Riesz monadic ideal. For any $x \in [0]_\sim$, we have $x \sim 0$. Since $\sim$ is a Riesz monadic congruence, then $\exists x \sim \exists 0 = 0$, so $\exists x \in [0]_\sim$. Therefore, $I = [0]_\sim$ is a Riesz monadic ideal and $\sim_I = \sim$.

(c) From (a) and (b), we know that the mapping $f \colon I \mapsto \sim_I$ is a bijection, whose inverse is the map $g \colon \sim \mapsto [0]_\sim$. Now, we prove the mapping f and g are isotone. Let I and J be two Riesz monadic ideals of $(E, \exists)$ such that $I \subseteq J$. If $a \sim_I b$, then there exists $i, j \in I$ such that $a \ominus i \leq b$ and $b \ominus j \leq a$. Since $i, j \in I \subseteq J$, then $a \sim_J b$, which means $\sim_I \subseteq \sim_J$. On the other hand, let $\sim$ and $\approx$ be two Riesz monadic congruence of $(E, \exists)$ such that $\sim \subseteq \approx$. For any $i \in [0]_\sim$, we have $i \sim 0$; then, $i \approx 0$ as $\sim \subseteq \approx$. It follows that $i \in [0]_\approx$, which means that $[0]_\sim \subseteq [0]_\approx$. Hence, f and g are isotone. Therefore, f is the order isomorphism between the lattice of all Riesz monadic congruence and the lattice of all Riesz monadic ideal.

Let $(E, \exists)$ be a monadic effect algebra and I be a Riesz monadic ideal. We define the mapping $\exists_I \colon E/I \longrightarrow E/I$ by $\exists_I([x]) = [\exists x]$ for any $x \in E$. ☐

Proposition 11. *Let $(E, \exists)$ be a strong monadic effect algebra and I be a Riesz monadic ideal of $(E, \exists)$. Then, $(E/I; \exists_I)$ is also a strong monadic effect algebra.*

Proof. Since I is a Riesz monadic ideal, then by Theorem 1, $\sim_I$ is a monadic congruence. We denote $(E/\sim_I) = (E/I)$. By Theorem 2, we have already known that E/I is an effect algebra. Now, we prove $((E/I); \exists_I)$ is a strong monadic effect algebra.

$(E1)$: since $x \leq \exists x$, then there exists y such that $x \perp y$ and $x \oplus y = \exists x$. It follows that $[x] \perp [y]$ and $[x] \oplus [y] = [x \oplus y] = [\exists x]$. Therefore, $[x] \leq \exists_I [x]$.

$(E2)$: if $[x] \leq [y]$, then there exists $[a]$ such that $[x] \perp [a]$ and $[x] \oplus [a] = [y]$. It follows that there exists $x_1, a_1, y_1 \in E$ such that $x_1 \sim_I x, a_1 \sim_I a, y_1 \sim_I y$, and $x_1 \oplus a_1 = y_1$, so $x_1 \leq y_1$. Then, $\exists x_1 \leq \exists y_1$. There exists $b \in E$ such that $\exists x_1 \oplus b = \exists y_1$. Therefore, $\exists_I [x] \oplus [b] = [\exists x] \oplus [b] = [\exists x_1 \oplus b] = [\exists y_1] = \exists_I [y]$. Hence, $\exists_I [x] \leq \exists_I [y]$.

$(E3)\exists_I\exists_I[x] = [\exists\exists x] = [\exists x] = \exists_I[x]$.

$(E4)$: let $\exists_I[x]\oplus\exists_I[y]$ be defined. If $\exists_I[x]\oplus\exists_I[y] = [1]$, clearly, $\exists_I(\exists_I[x]\oplus\exists_I[y]) = \exists_I[x]\oplus\exists_I[y]$. Otherwise, since $\exists_I[x]\perp\exists_I[y]$, then there exists $a, b \in E$ such that $\exists x \sim_I a$, $\exists y \sim_I b$, $a\perp b$, and $\exists_I[x]\oplus\exists_I[y] = [a\oplus b]$. As $\exists$ is strong and $a\oplus b \neq 1$, we have $\exists a\perp\exists b$. Since $\exists x \sim_I a$, then $\exists x \sim_I \exists a$; it follows that $a \sim_I \exists a$. Similarly, we get $b \sim_I \exists b$. As $\sim_I$ is a monadic congruence, we have $a\oplus b \sim_I \exists a\oplus\exists b$. Therefore, $\exists_I(\exists_I[x]\oplus\exists_I[y]) = \exists_I([\exists x]\oplus[\exists y]) = \exists_I[a\oplus b] = \exists_I[\exists a\oplus\exists b] = [\exists(\exists a\oplus\exists b)] = [\exists a\oplus\exists b] = \exists_I[a]\oplus\exists_I[b] = \exists_I[x]\oplus\exists_I[y]$.

Therefore, $((E/I);\exists_I)$ is a monadic effect algebra. Moreover, if $[x]\perp[y]$ and $[x]\oplus[y] \neq [1]$, then there exists $a, b \in E$ such that $x \sim_I a$, $y \sim_I b$, and $a\perp b$. Then, $\exists x \sim_I \exists a$, $\exists y \sim_I \exists b$, and $\exists a\perp\exists b$ as $\exists$ is strong and $a\oplus b \neq 1$. So, $[\exists x]\perp[\exists y]$, which implies that $\exists_I[x]\perp\exists_I[y]$. Therefore, $((E/I);\exists_I)$ is a strong monadic effect algebra. □

Definition 12. Let $(E, \exists)$ be a monadic effect algebra. A proper monadic ideal I of $(E, \exists)$ is called a maximal monadic ideal if it is not strictly contained in any proper monadic ideal of $(E, \exists)$.

Example 10. Let $E=\{0, a, b, c, d, e, f, 1\}$, where $0 \leq a$, $b \leq c$, $d \leq 1$. Define operation $\oplus$ and $\neg$ as follows (see Tables 5 and 6):

Then, $(E, \oplus, \neg, 0, 1)$ is an effect algebra. We define $\exists$ as follows: $\exists 0 = 0, \exists a = \exists d = d, \exists b = b$, and $\exists c = \exists e = \exists f = \exists 1 = 1$. We can easily check that $\exists$ is an existential quantifier on E. Define $I_1 = \{0, a, d, e\}$; I_1 is a maximal ideal, but not a maximal monadic ideal because $\exists e = 1 \notin I_1$. Define $I_2 = \{0, b\}$, and we can check if I_2 is a maximal monadic ideal of $(E, \exists)$.

Table 5

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	d	c	1	—	—	—	—
b	b	c	—	—	1	—	—	—
c	c	1	—	—	—	—	—	—
d	d	—	1	—	—	—	—	—
e	e	—	—	—	—	—	1	—
f	f	—	—	—	—	1	—	—
1	1	—	—	—	—	—	—	—

Table 6

	$\neg$
0	1
a	c
b	d
c	a
d	b
e	f
f	e
1	0

Proposition 12. *Let I be a monadic ideal of a monadic effect algebra $(E, \exists)$. If for any $x \in E$, we have $\exists x \in I$ or $\exists x \in I$; then, I is a maximal monadic ideal of $(E, \exists)$.*

Proof. Suppose for any $x \in E$, we have $\exists x \in I$ or $\exists x \in I$. Assume I is not a maximal monadic ideal of $(E, \exists)$. Then, there exists a proper monadic ideal I strictly containing I, so there also exists $a \in J$, but $a \in I$. Hence, $\exists a \in I$; according to our assumption, $\exists a \in I$. Therefore, $\exists a \in I$. On the other hand, as $a \in J$ and J is a monadic ideal, we have $\exists a \in J$, so $\exists a\oplus\exists a = 1 \in J$. This contradicts the fact that J is a proper monadic ideal. Therefore, I is a maximal monadic ideal of $(E, \exists)$. □

Definition 13. A monadic effect algebra $(E, \exists)$ is called monadic simple if it has exactly two monadic ideals: $\{0\}$ and E.

Example 11. Let $(E, \exists)$ be the monadic effect algebra given by Example 9. We can easily check that $(E, \exists)$ is monadic simple.

Example 12. For any $n \in N$, let $(E, \exists_n)$ be the monadic effect algebra given by Example 2. We can check $(E, \exists_n)$ is a simple monadic effect algebra.

Proof. $\{0\}$ is the only monadic ideal in each $(E, \exists_n)$. □

Theorem 8. *Let $(E, \exists)$ be a strong monadic effect algebra. Then, the following conditions are equivalent:*

(1) $(E, \exists)$ is monadic simple

(2) $\exists E$ is simple

Proof

(1)$\Rightarrow$(2): let I be an ideal of $\exists E$ and $I \neq \{0\}$. We will show that $I = \exists E$. Consider $J = \{x \in E | x \leq i$, for some $i \in I\}$. If $x, y \in J$, then there exists $i, j \in I$ such that $x\oplus y \leq i\oplus j \in I$, so $x\oplus y \in J$. If $y \leq x$ and $x \in J$, then there exists $i \in I$ such that $y \leq x \leq i$, so $y \in J$. Moreover, if $x \in J$, then there exists $i \in I$ such that $x \leq i$, so $\exists x \leq \exists i = i$ and $\exists x \in J$. Therefore, J is a monadic ideal of $(E, \exists)$. Since $I \subseteq J$, $I \neq \{0\}$, and $(E, \exists)$ is monadic simple, we have $J = E$, so $1 \in J$; by the definition of J, we have $1 \in I$. Therefore, we have $I = \exists E$ which implies $\exists E$ is simple.

(2)$\Rightarrow$(1): let I be a monadic ideal of $(E, \exists)$. Then, we have $I \cap \exists E$ as an ideal of $\exists E$; since $\exists E$ is simple, we get $I1 \cap \exists E = \{0\}$, or $I \cap \exists E = \exists E$. If $I \cap \exists E = \exists E$, then $\exists E \subseteq I$, so $1 \in I$. Then, we deduce that $I = E$. If $I \cap \exists E = \{0\}$, take $x \in I$. $\exists x \in I \cap \exists E = \{0\}$, so for any $x \in I$, we have $\exists x = 0$, but $x \leq \exists x$, so for any $x \in I$, we have $x = 0$, so $I = \{0\}$. Therefore, $(E, \exists)$ is monadic simple. □

Remark 4. When $(E, \exists)$ is a strong monadic effect algebra, there exists a more concise proof for the above theorem. Since $\exists$ is strong, by Theorem 6, we know there exists a lattice isomorphism between the lattice of all monadic ideals of $(E, \exists)$ and the lattice of all ideals of $\exists E$, which implies that the number of the elements in the sets $MI(E)$ and $I(\exists E)$ is equal. Therefore, when $\exists$ is strong, $(E, \exists)$ is monadic simple if and only if $\exists E$ is simple.

The result of Theorem 8 shows that we can use the partial structure $\exists E$ to characterize the total structure E. Also, $\exists E$ has the same structure as $(E, \exists)$, which reveals the significance of the image set $\exists E$.

Definition 14. A monadic effect algebra $(E, \exists)$ is said to be monadic subdirectly irreducible if for all of the nontrivial monadic congruence relations of $(E, \exists)$, there exists the least one.

Let $(E, \exists)$ be monadic subdirectly irreducible effect algebra and $\sim$ be the least nontrivial monadic congruence of $(E, \exists)$. Then, by Theorem 6, there is a Riesz monadic ideal I of $(E, \exists)$ such that $\sim_I = \sim$, which means I is the least Riesz monadic ideal of $(E, \exists)$ such that $I \neq \{0\}$. Thus, we can conclude that $(E, \exists)$ is monadic subdirectly irreducible if among the nontrivial Riesz monadic ideals of $(E, \exists)$, there exists the least one.

Proposition 13. *Let $(E, \exists)$ be a subdirectly irreducible monadic effect algebra and $I_1, I_2 \in RMI(E)$. If $I_1 \cap I_2 = \{0\}$, then $I_1 = \{0\}$ or $I_2 = \{0\}$.*

Proof. Suppose $I_1 \neq \{0\}$ and $I_2 \neq \{0\}$, then I_1 and $I_2 \in \{I \in \mathrm{RMI}(E) | I \neq \{0\}\}$, so $\cap\{I \in \mathrm{RMI}(E) | I \neq \{0\}\} \subseteq I_1 \cap I_2 = \{0\}$. Therefore, $\cap\{I \in \mathrm{RMI}(E) | I \neq \{0\}\} = \{0\}$, which contradicts to the fact that $(E, \exists)$ is subdirectly irreducible. Therefore, $I_1 = \{0\}$ or $I_2 = \{0\}$. □

Proposition 14. *Let $(E, \exists)$ be a monadic effect algebra. Then, the following conditions are equivalent:*

(1) $(E, \exists)$ is a subdirectly irreducible monadic effect algebra

(2) $\cap\{(x]_\exists | x \in E, x > 0\} \neq \{0\}$, where $(x]_\exists$ is the monadic ideal generated by x

Proof

(1)$\Rightarrow$(2): let $(E, \exists)$ be a subdirectly irreducible monadic effect algebra. We have $\cap\{I \in \mathrm{RMI}(E) | I \neq \{0\} \neq \{0\}$; then, $\cap\{(x]_\exists | x \in E,\ x > 0\} \neq \{0\}$.

(2)$\Rightarrow$(1): let $\cap\{(x]_\exists | x \in E, x > 0\} \neq \{0\}$, then there exists $a \in \cap\{(x]_\exists | x \in E, x > 0\}$. Now, we prove that, for any $I \in \mathrm{RMI}(E)$, if $I \neq \{0\}$, we have $a \in I$. In fact, if $I \neq \{0\}$, then there exist $x \in I$ and $x > 0$; by assuming, we have $a \in (x]_\exists$, so $a \in I$. Hence, $a \in \cap\ \{I \in \mathrm{RMI}(E) | I \neq \{0\}\}$. Therefore, $\cap\{I \in \mathrm{RMI}(E) | I \neq\ \{0\}\} \neq \{0\}$, which implies $(E, \exists)$ is a subdirectly irreducible monadic effect algebra. □

Theorem 9. *Let $(E, \exists)$ be a monadic effect algebra and $\forall$ be faithful. Then, the following conditions are equivalent:*

(1) $(E, \exists)$ is monadic subdirectly irreducible

(2) $\exists E$ is a subdirectly irreducible subalgebra of E

Proof

(1)$\Rightarrow$(2): let $(E, \exists)$be a subdirectly irreducible monadic effect algebra. Then, $\mathrm{RMI}(E) - \{0\}$ has a minimal element I. We have already known $\exists E$ is a subalgebra of E. Now, we will prove that $I \cap \exists E$ is the minimal ideal of $\exists E$ such that $I \cap \exists E \neq \{0\}$. First, $I \cap \exists E = \{0\}$; since $\forall I \subseteq I \cap \exists E = \{0\}$, then we have $\forall x = 0$ for any $x \in I$. Therefore, $I \subseteq \ker(\forall) = \{0\}$, and so $I = \{0\}$, which is a contradiction. Hence, $I \cap \exists E \neq \{0\}$. Next, we show that $I \cap \exists E$ is the minimal ideal of $\exists E$. Assume that J is an ideal of $\exists E$, so, clearly, $(J]$ is a monadic ideal of $(E, \exists)$ generated by J. By the minimality of I, we know that $I \subseteq (J]$, so $I \cap \exists E \subseteq (J] \cap \exists E = J$. Therefore, $I \cap \exists E$ is the minimal ideal of $\exists E$ such that $I \cap \exists E \neq \{0\}$. Therefore, $\exists E$ is a subdirectly irreducible subalgebra of E.

(2)$\Rightarrow$(1): let $\exists E$ be a subdirectly irreducible subalgebra of E. Then, there is a minimal ideal I of $\exists E$ such that $I \neq \{0\}$. So, $(I]$ is a monadic ideal of $(E, \exists)$ such that $(I] \neq \{0\}$. Now, we prove that $(I]$ is the minimal monadic ideal of $(E, \exists)$. Assume J is another nontrivial monadic ideal of $(E, \exists)$; then, we have $J \cap \exists E$ which is an ideal of $\exists E$. By the minimality of I, we have $I \subseteq J \cap \exists E$, so $(I] \subseteq (J \cap \exists E] = J$. Hence, $(I]$ is the minimal monadic ideal of $(E, \exists)$. Therefore, $(E, \exists)$ is a subdirectly irreducible monadic effect algebra. □

5. Conclusions

As effect algebras are partial algebraic structures, from $a \perp b$, we cannot get $\exists a \perp \exists b$ (naturally holds in total algebraic structure), which caused some difficulties for us to prove some propositions on monadic effect algebras. So, we introduce strong existential quantifiers as follows: if for any $x, y \in E$, $x \perp y$, and $x \oplus y \neq 1$, imply $\exists x \perp \exists y$. Also, we require $x \oplus y \neq 1$ because if for any $x, y \in E$, $x \perp y$ implies $\exists x \perp \exists y$, we can prove $\exists = id_E$. Another difference between monadic effect algebras and some monadic total algebras is that the class of monadic effect algebras just forms a quasivariety, but the class of monadic MV-algebras, monadic BL-algebras, and monadic residuated lattices forms a variety, respectively. Moreover, effect algebra is a model of unsharp measurement in quantum mechanical system. Therefore, using the existential quantifiers and the universal quantifiers to study the unsharp measurement in quantum mechanical system is our next work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the Ningxia Key Research and Development Program (special talents) (2018BEB04030), the Natural Science Foundation of Ningxia (2019AAC03073), and the Scientific Research Project of Ningxia Colleges (ZR1701).

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