Research Article

Characterizations of Integral Type for Weighted Classes of Analytic Banach Function Spaces

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The aim of this study is to give some new definitions of Banach spaces of holomorphic functions. Some holomorphic characterizations of integral type for some classes of Banach spaces of holomorphic functions are established in the unit disc U .

1. Introduction and Preliminaries

Numerous studies of analytic function spaces by several classes of functions are introduced and intensively studied. The theory of function spaces provides interesting tools in many active branches of mathematics, especially in mathematical analysis such as in operator theory, measure theory, and differential equations. In the present study, we aim to give the definition of g -Bloch space of holomorphic functions. Using the new class of functions, some essential relations between it and some other known classes are investigated.

Next, we report the recent advancements of the concepts of specific-weighted classes of holomorphic function spaces. The choice of the appropriate functions gives the specific essential properties of the underlying weighted classes of functions that can have an important impact for the study.

Specific weighted classes of holomorphic function spaces and concepts are presented. Let $U = \{w: w \in \mathbb{C}, |w| < 1\}$, and $H(U)$ denote the class of all holomorphic functions that belonging to U .

The known analytic Bloch-type space [1–5] is defined by

$$\mathcal{B} = \left\{ f: f \in H(U) \text{ and } \sup_{w \in U} (1 - |w|^2) |f'(w)| < \infty \right\}. \quad (1)$$

The analytic little Bloch-type space $\mathcal{B}_0$ is symbolized by $\mathcal{B}_0$, for which

$$\mathcal{B}_0 = \left\{ f: f \in H(U) \text{ and } \lim_{|w| \rightarrow 1^-} (1 - |w|^2) |f'(w)| = 0 \right\}. \quad (2)$$

For numerous global studies on Bloch-type spaces, we refer to [6–13] and others.

The analytic Dirichlet-type space [1, 14] is given by

$$\mathcal{D} = \left\{ f: f \in H(U) \text{ and } \int_U |f'(w)|^2 d\sigma_w < \infty \right\}, \quad (3)$$

where $d\sigma_w = dx dy$.

Using Green's function $g(w, w_0) = \ln|(1 - \bar{w}_0 w)/(w_0 - w)| = \ln(1/|\phi_{w_0}(w)|)$ as a weight function, the analytic $\mathcal{Q}_p$ -spaces is defined in [1] by

$$\mathcal{Q}_p = \left\{ f: f \in H(U) \text{ and } \sup_{w_0 \in U} \int_U |f'(w)|^2 g^p(w, w_0) d\sigma_w < \infty \right\}. \quad (4)$$

By analytic $\mathcal{Q}_p$ -spaces, some interesting relationships between analytic Dirichlet-type space and the analytic Bloch-type space are obtained [1].

For intensive research on analytic $\mathcal{Q}_p$ spaces, we may refer to [1, 15–18] and others. Also, there are certain specific generalizations of these weighted classes of analytic functions in C^n [12, 19, 20]. On the other hand, there are some interesting extensions using quaternion-valued functions setting ([21–24]).

Hereafter, we set

$$\varphi_{w_0}(w) = \frac{w_0 - w}{1 - \bar{w}_0 w}, \quad w \neq w_0, \quad (5)$$

and set

$$\varphi_{w_0}(w) = C < 1, \quad \text{when } w = w_0. \quad (6)$$

The modified Green's function is introduced by

$$\mathbf{g}(w, w_0) = \ln \left| \frac{1 - \bar{w}_0 w}{w_0 - w} \right| = \ln \frac{1}{|\varphi_{w_0}(w)|}. \quad (7)$$

Motivated by the modified Green's function, the following definitions can be presented.

Definition 1. Let $0 < m < \infty$ and $0 < n < \infty$. For the function $f \in H(U)$, we define the analytic $\mathbf{g}$ -Bloch space $\mathcal{B}_g^{n,m}$ as follows:

$$\mathcal{B}_g^{n,m} = \left\{ f: \sup_{w, w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)| < \infty \right\}. \quad (8)$$

Furthermore, assume that

$$\mathcal{B}_g^{n,m}(f) = \sup_{w, w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)|. \quad (9)$$

Example 1. Let $m \in (0, \infty)$. Suppose that

$$f(w) = \int_{|w| < 1} \left(\log \frac{1}{|\varphi_a(w)|} \right)^m dw. \quad (10)$$

It is very obvious that the function f is a $\mathbf{g}$ -Bloch function.

Remark 1. Using Definition 1, relationships between analytic Dirichlet-type functions and analytic Bloch functions can be characterized.

When $m = 0$ and $0 < n = \alpha < \infty$, then we will obtain α -Bloch space. The case $m \in (-\infty, 0)$ induces some other types of analytic function spaces with different behaviors and can be studied separately.

The little analytic $\mathbf{g}$ -Bloch space $\mathcal{B}_g^{n,m,0}$ can be considered as a subspace of the analytic $\mathbf{g}$ -Bloch space that includes all $f \in \mathcal{B}_g^{n,m}$, with

$$\lim_{|w| \rightarrow 1^-} \lim_{|w_0| \rightarrow 1^-} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |f'(w)| = 0. \quad (11)$$

Remark 2. The symbol $\mathcal{B}_g^{n,m}(f)$ stands for a seminorm, while the usual norm can be defined as

$$\|f\|_{\mathcal{B}_g^{n,m}} = |f(0)| + \mathcal{B}_g^{n,m}(f). \quad (12)$$

Applying the above norm, the space $\mathcal{B}_g^{n,m}$ is a complete normed space (Banach space).

The next lemma can be applied for some results in this study.

Lemma 1 (See [14]). Let $0 < s < \infty$ and assume that $|w_0| < 1$. Then,

$$\int_{\Gamma_w} \frac{1}{|1 - \bar{w}_0 w|^{2s}} d\Gamma_w \leq \frac{k}{(1 - |w_0|)^s}, \quad (13)$$

where $k > 0$, and Γ_w defines the boundary of U .

2. $\mathbf{g}$ -Bloch Space and Dirichlet Space

Some essential characterizations between the analytic Dirichlet-type space and the analytic $\mathbf{g}$ -Bloch space are given in this section.

Proposition 1. Let $f \in H(U)$ and assume that $|w_0| < 1$. Then,

$$\frac{(1 - |w|^2)^2}{\mathbf{g}^m(w, w_0)} |f'(w_0)|^2 \leq \frac{4}{\pi R^2} \int_U |f'(w_0)|^2 d\sigma_w, \quad (14)$$

where $0 < R < 1$.

Proof. Because the pseudohyperbolic disk can be symbolized by

$$D(w_0, R) = \left\{ w: \rho(w, w_0) = \left| \frac{w - w_0}{1 - \overline{w_0}w} \right| < R \right\}, \quad (15)$$

where w_0 is its center and R is its radius, then we infer that

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \int_{D(w_0, R)} |f'(w)|^2 d\sigma_w \\ &= \pi R^2 (1 - |w_0|^2)^2 |f'(w_0)|^2 \geq \pi R^2 (1 - |w_0|^2)^2 |f'(w_0)|^2. \end{aligned} \quad (16)$$

Using the definition of the modified Green's function as well as the inequalities,

$$\begin{aligned} 0 < \varphi_{w_0}(w_0) &= C < 1, \\ \left(1 - |\varphi_{w_0}(w)|^2\right) &\leq 2\mathbf{g}(w, w_0), \text{ (see [25])}. \end{aligned} \quad (17)$$

We can obtain

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \pi R^2 \frac{(1 - |w_0|^2)^2}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^2} |f'(w_0)|^2 \\ &> 2\pi R^2 \frac{(1 - |w_0|^2)^2}{C = \mathbf{g}^2(w_0, w_0)} |f'(w_0)|^2. \end{aligned} \quad (18)$$

The proof of Proposition 1 is therefore finished. $\square$

Corollary 1. In view of Proposition 1, for $|w_0| < 1$, we have the following interesting inclusion:

$$\mathcal{D} \subset \mathcal{B}_{\mathbf{g}}^{2,2}. \quad (19)$$

Proposition 2. Let $f \in H(U)$ and assume that $|w_0| < 1$. Also, we suppose that $n \in [1, \infty)$ and $m \in (0, \infty)$, then

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2 \leq \frac{2^m}{\pi R^2} \int_U |f'(w)|^2 d\sigma_w, \quad (20)$$

where $0 < R < 1$.

Proof. The proof of Proposition 2 can be obtained as in the proof of Proposition 1, with the following changes:

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\geq \pi R^2 \frac{(1 - |w_0|^2)^2}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^m} |f'(w_0)|^2 \\ &\geq \frac{\pi}{2^m} R^2 \frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2. \end{aligned} \quad (21)$$

$\square$

Corollary 2. Proposition 2 results that

$$\mathcal{D} \subset \mathcal{B}_{\mathbf{g}}^{2n,m}, \quad (22)$$

with $|w_0| < 1$, $n \in [1, \infty)$, and $m \in (0, \infty)$.

Remark 3. Corollaries 1 and 2 interpret that the analytic Dirichlet-type space can be considered as a subspace of the analytic $\mathbf{g}$ -Bloch space when $m = n = 2$ as well as for $n \in [1, \infty)$ and $m \in (0, \infty)$, respectively.

Proposition 3. Let $f \in H(U)$ and $f \in \mathcal{B}_{\mathbf{g}}^{n,m}$. Suppose that

$$I(m, n) = \int_0^1 \frac{(\ln(1/r))^m}{(1 - r^2)^{1-n}} r dr < \infty. \quad (23)$$

Then, for $0 < m < \infty$ and $0 < n < \infty$ with $m + n > 0$, we obtain that

$$\sup_{w \in U} \int_U |f'(w)|^2 d\sigma_w \leq (2)^{4-m} \pi I(m, n) (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2. \quad (24)$$

Proof. From the definition of $\mathbf{g}$ -Bloch space, we have

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w)| \leq \mathcal{B}_{\mathbf{g}}^{n,m}(f). \quad (25)$$

By a change of the variables technique, we infer that

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_U \frac{(\ln(1/|w_0|))^m}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^n} \\ &\quad \frac{(1 - |\varphi_{w_0}(w_0)|^2)^2}{(1 - |w_0|^2)^2} d\sigma_w. \end{aligned} \quad (26)$$

Since,

$$\left(1 - |\varphi_{w_0}(w_0)|^2\right) = \frac{(1 - |w_0|^2)(1 - |w|^2)}{|1 - \overline{w_0}w|^2}. \quad (27)$$

Therefore,

$$\begin{aligned} \int_U |f'(w)|^2 d\sigma_w &\leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_U \frac{(\ln(1/|w_0|))^m}{\left(1 - |\varphi_{w_0}(w_0)|^2\right)^n} d\sigma_w \\ &\leq \pi 2^{4-n} (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 \int_0^1 \frac{(\ln(1/r))^m}{(1 - r^2)^{1-n}} r dr, \end{aligned} \quad (28)$$

which implies that

$$\int_U |f'(w)|^2 d\sigma_w \leq (\mathcal{B}_{\mathbf{g}}^{n,m}(f))^2 k_1, \quad (29)$$

where $0 < k_1 < \infty$. The proof is therefore established.

Propositions 2 and 3 result in the following fundamental theorem. $\square$

Theorem 1. Let $f \in H(U)$; then, we have equivalence between the following statements:

- (i) $f \in \mathcal{B}_{\mathbf{g}}^{n,m}$ with $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n > 0$
- (ii) $f \in \mathcal{D}$.

Remark 4. The obtained results in Theorem 1 reflexed the major role of the newly definition of the analytic $\mathbf{g}$ -Bloch space which has used to get relations between analytic Dirichlet-type space $\mathcal{D}$ and $\mathcal{B}_{\mathbf{g}}^{n,m}$ space.

3. $\mathbf{g}$ -Bloch Functions and $\mathcal{Q}_p$ Functions

Proposition 4. For $0 \leq m < \infty$, $1 \leq n < \infty$, let $f \in H(U)$ and $|w_0| < 1$; hence,

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w_0, w_0)} |f'(w_0)|^2 \leq \frac{2^{2m+4}}{\rho^2(1 - \rho^2)^p} \int_U |f'(w)|^2 (1 - |\varphi_{w_0}(w)|^2)^p d\sigma_w, \quad (30)$$

where $0 < \rho < 1$ and $p \in (0, \infty)$.

Proof. From subharmonicity principle, the following inequality can be easily obtained:

$$|h(0)|^2 \leq \frac{1}{\rho^2} \int_{D(0, \rho)} |h(z)|^2 d\sigma_w, \quad (31)$$

Since $f \in H(U)$ is holomorphic on $D(w_0, \rho)$, then use (31) to the function

$$h = f' \circ \varphi_{w_0}. \quad (32)$$

Applying the technique of change of variables, the next inequality can be inferred:

$$\begin{aligned} |f'(w_0)|^2 &\leq \frac{1}{\rho^2} \int_{D(0, \rho)} |f'(\varphi_{w_0}(w))|^2 d\sigma_w \\ &= \frac{1}{\rho^2} \int_{D(w_0, \rho)} |f'(w)|^2 \left(\frac{1 - |\varphi_{w_0}(w)|^2}{1 - |w|^2} \right)^2 d\sigma_w. \end{aligned} \quad (33)$$

From [5], we have

$$\left(\frac{1 - |\varphi_{w_0}(w)|^2}{1 - |w|^2} \right)^2 \leq \left(\frac{4}{(1 - |w_0|^2)} \right)^2. \quad (34)$$

This yields that

$$|f'(w_0)|^2 \leq \frac{(4/\rho)^2}{(1 - |w_0|^2)^2} \int_{D(w_0, \rho)} |f'(w)|^2 d\sigma_w. \quad (35)$$

Because,

$$\begin{aligned} (1 - |\varphi_{w_0}(w)|^2)^p &\geq (1 - \rho^2)^p, \\ \varphi_{w_0}(w_0) &= 0. \end{aligned} \quad (36)$$

Then,

$$\begin{aligned} \int_{D(w_0, \rho)} |f'(w)|^2 (1 - |\varphi_{w_0}(w)|^2)^p d\sigma_w &\geq \frac{\rho^2(1 - \rho^2)^p}{16} (1 - |w_0|^2)^2 |f'(w_0)|^2 \\ &\geq \frac{\rho^2(1 - \rho^2)^p}{16} \frac{(1 - |w_0|^2)^2}{(1 - |\varphi_{w_0}(w_0)|^2)^2} |f'(w_0)|^2 \\ &\geq \frac{\rho^2(1 - \rho^2)^p}{2^{2m+4}} \frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^m(w, w_0)} |f'(w_0)|^2. \end{aligned} \quad (37)$$

Using the inequality

$$\int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w \geq \int_{D(w_0, \rho)} |f'(w)|^2 \left(1 - |\varphi_{w_0}(w_0)|^2\right)^p d\sigma_w, \quad (38)$$

we can deduce that

$$\frac{(1 - |w_0|^2)^{2n}}{\mathbf{g}^{2m}(w_0, w)} |f'(w_0)|^2 \leq \frac{2^{2m+4}}{\rho^2 (1 - \rho^2)^p} \int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w. \quad (39)$$

The proof is therefore completely finished. $\square$

Corollary 3. By Proposition 4, for $p > 0$, $0 \leq m < \infty$, $1 \leq n < \infty$, and $|w_0| < 1$, the following inclusions can be easily proved:

$$\mathcal{B}_{\mathbf{g}}^{2n, 2m}(f) \leq \frac{2^{2m+4}}{\rho^2 (1 - \rho^2)^p} \mathcal{Q}_p^2(f), \quad (40)$$

where

$$\mathcal{Q}_p(f) = \sup_{w_0 \in U} \int_U |f'(w)|^2 \left(1 - |\varphi_{w_0}(w)|^2\right)^p d\sigma_w \approx \sup_{w_0 \in U} \int_U |f'(w)|^2 (\mathbf{g}(w, w_0))^p d\sigma_w < \infty. \quad (41)$$

Proposition 5. Let $f \in H(U)$. Assume that $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n + p > 0$. Then, we have that

$$I(m, n, p) = \lambda \int_0^1 \frac{(\ln(1/r))^{m+p}}{(1-r)^{1-n}} r dr, \quad (43)$$

and λ is an absolute positive constant.

$$\int_U |f'(w)|^2 (\mathbf{g}(w, w_0))^p d\sigma_w \leq I(m, n, p) (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f), \quad (42)$$

Proof. It is not hard to see that

where

$$\int_U |f'(w)|^2 \mathbf{g}^p(w_0, w) d\sigma_w \leq 4 (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f) \int_U \frac{(\ln(1/|w|))^{m+p}}{(1 - |w|^2)^{2-2n}} d\sigma_w. \quad (44)$$

Applying Lemma 1, we obtain

$$\begin{aligned} \int_U |f'(w)|^2 \mathbf{g}^p(w_0, w) d\sigma_w &\leq 8\pi (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f) \int_0^1 \frac{(\ln(1/r))^{m+p}}{(1-r)^{1-n}} r dr \\ &= I(m, n, p) (\mathcal{B}_{\mathbf{g}}^{m, n})^2(f). \end{aligned} \quad (45)$$

Combining Corollary 3 and Proposition 5, we have the following theorem: $\square$

Theorem 2. Let $h \in H(U)$. Assume that $0 \leq m < \infty$ and $0 < n < \infty$ with $m + n + p > 0$ as well as $0 < p < \infty$. Therefore, the next assertions can be equivalent:

$$(a) \ h \in \mathcal{B}_{\mathbf{g}}^{m, n}$$

$$(b) \ h \in \mathcal{Q}_p$$

$$(c) \ h \in \mathcal{Q}_p \text{ for some } 0 < p < \infty.$$

Proof. (a) $\Rightarrow$ (b) can be established using Proposition 5. (b) $\Rightarrow$ (c) is quite obvious. (c) $\Rightarrow$ (a) can be proved by Corollary 3. $\square$

Remark 5. Theorem 2 investigates relations between the new type of holomorphic g -Bloch functions and the weighted holomorphic Dirichlet-type functions.

4. A Specific Criteria

The following symbol stands for a non-Euclidean distance of hyperbolic-type between the points w_0 and w in U .

$$d(w_0, w) = \ln \sqrt{\frac{|1 - \overline{w_0}w| + |w_0 - w|}{|1 - \overline{w_0}w| - |w_0 - w|}} \quad (\text{see [26]}). \quad (46)$$

Now, for $0 < R < \infty$, set

$$\begin{aligned} T(w, R) &= \{w_0 \in U: d(w_0, w) < R\}, \\ T_1(w, R) &= \{w_0 \in U: d(w_0, w) = R\}. \end{aligned} \quad (47)$$

Let $h \in H(U)$ be the nonconstant analytic function, and let $S_h(w, R)$ define the concerned area of the Riemannian image $P(w, R)$ of $T(w, R)$ by the function h , and assume that $\mathcal{S}_h^*(w, R)$ defines the area of the image $P(w, R)$ of $T(w, R)$ using the function h . It should be noted that $P(w, R)$ defines the projection to C . The length of the Riemannian image of

$T_1(w, R)$ by h is denoted by $M_h(z, \rho)$, and the symbol $\mathcal{M}_h(z, \rho)$ defines the length of the outer boundary of $P(w, R)$. If Γ defines a specific bounded domain in C , the concept of outer boundary means that we are working in the boundary of $C \setminus B$, where B may be defined as the unbounded component for the complement $C \setminus \Gamma$.

Now, we clearly have the following inequalities:

$$\begin{aligned} S_h(w, R) &\geq \mathcal{S}_h^*(w, R), \\ M_h(w, R) &\geq \mathcal{M}_h(w, R), \end{aligned} \quad (48)$$

where $0 < R < \infty$ and each $w \in U$.

Proposition 6. Let $h \in H(U)$ be a nonconstant analytic function. Suppose that $1 \leq n < \infty$ and $0 \leq m < \infty$, then the following inequalities can be deduced:

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{1}{\theta(R)} \sqrt{\frac{2^m \mathcal{S}_h^*(w, R)}{\pi(1 - \rho^2)^m}}. \quad (49)$$

Also,

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{2^m (1 - |w|^2)^n}{(1 - |\varphi_{w_0}(w)|^2)^m} |h'(w)| \leq \frac{2^{m-1} \mathcal{M}_h(w, R)}{(1 - \rho^2)^m \pi \theta(R)}, \quad (50)$$

where $\theta(R) = ((e^{2R} - 1)/(e^{2R} + 1))$.

Proof. First, we suppose that $h'(w) \neq 0$. Now, we suppose that

$$f(w_0) = h\left(\frac{w_0 + w}{1 + \overline{w_0}w}\right) = b_0 + b_1 w_0 + b_2 w_0^2 + b_3 w_0^3 + \dots, \quad (51)$$

with $|w_0| < 1$, where $b_0, b_1, b_2, b_3, \dots$ are the positive constants. Now,

$$(1 - |w|^2)^n |h'(w)| \leq b_1 = (1 - |w|^2) |h'(w)| \neq 0, \quad (52)$$

which implies that

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{(1 - |w|^2)}{\mathbf{g}^m(w, w_0)} |h'(w)| \neq 0 < \infty. \quad (53)$$

Using Theorems 1 and 2 in [27], the following inequalities can be deduced:

$$\begin{aligned} \pi \theta^2(R) |b_1|^2 &\leq \mathcal{S}_h^*(w, R), \\ 2\pi \theta(R) |b_1| &\leq \mathcal{M}_h(w, R). \end{aligned} \quad (54)$$

Since, $0 < |\varphi_{w_0}(w)| < \rho < 1$, then

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{1}{\theta(R)} \sqrt{\frac{2^m \mathcal{S}_h(w, R)}{\pi(1 - \rho^2)^m}}. \quad (55)$$

Also,

$$\frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} |h'(w)| \leq \frac{2^m (1 - |w|^2)^n}{(1 - |\varphi_{w_0}(w)|^2)^m} |h'(w)| \leq \frac{2^{m-1} \mathcal{M}_h(w, R)}{(1 - \rho^2)^m \pi \theta(R)}. \quad (56)$$

Remark 6. In the proof of Proposition 6, positive coefficients are considered to keep the convergence of Taylor or Fourier power series in its region.

Theorem 3. Let $h \in H(U)$ be the nonconstant analytic function. Suppose that $1 \leq n < \infty$ and $0 \leq m < \infty$; therefore, the equivalence between next assertions can be mutually obtained:

□

- (1) The function $h \in \mathcal{B}_g^{n,m}$
 (2) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} S_h(w, R) < \infty. \quad (57)$$

- (3) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} \mathcal{S}_h^*(w, R) < \infty. \quad (58)$$

- (4) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} M_h(w, R) < \infty. \quad (59)$$

- (5) The constant $0 < R < \infty$ can be obtained, for which

$$\sup_{w_0 \in U} \mathcal{M}_h(w, R) < \infty. \quad (60)$$

Proof. The assertions (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) can be deduced clearly. Additionally, (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are obvious. Next, suppose that (1) with

$$\sup_{w_0 \in U} \frac{(1 - |w|^2)^n}{\mathbf{g}^m(w, w_0)} h'(w) = \mathcal{B}_g^{n,m}(h) < \infty. \quad (61)$$

Hence,

$$\begin{aligned} S_h(w, R) &= \int_{T(w, R)} |h'(w_0)|^2 d\sigma_{w_0} \\ &\leq (\mathcal{B}_g^{n,m}(h))^2 \int_{|w_0| < \theta(R)} \frac{\mathbf{g}^m(w, w_0)}{(1 - |w|^2)^n} d\sigma_{w_0} \\ &= \eta_1(R) (\mathcal{B}_g^{n,m}(h)), \end{aligned} \quad (62)$$

where $\eta_1(R)$ is a positive constant depending on R . In addition, using the same way, we deduce that

$$\begin{aligned} \frac{(\mathcal{B}_g^{n,m}(h))^2 (\theta(R))^{2n}}{(1 - (\theta(R))^2)^{2m}} &= M_h(w, R) = \eta(R) \int_{T_1(w, R)} |f'(w_0)| d\sigma_{w_0} \\ &\leq \mathcal{B}_g^{n,m}(h) \int_{T_1(w, R)} \frac{\mathbf{g}^m(w, w_0)}{(1 - |w|^2)^n} d\sigma_{w_0} \\ &= \eta_2(R) \mathcal{B}_g^{n,m}(h), \end{aligned} \quad (63)$$

where $\eta(R)$ and $\eta_2(R)$ are the positive constants depending on R .

Additionally, the assertion (3) $\Rightarrow$ (1) as well as (5) $\Rightarrow$ (1) can be proved in view of Proposition 6. $\square$

Remark 7. Theorem 3 gives an interesting and global criterion for analytic $\mathbf{g}$ -Bloch-type function by the help of the concrete area as well as the concerned length of the images of both non-Euclidean unified discs and unified circles,

respectively. The obtained results in this section extended and improved some specific results in the study by Yamashita [26].

Remark 8. Quite recently, a new study of bicomplex functions was introduced in [28].

An interesting question can be formulated for the newly defined $\mathbf{g}$ -Bloch functions as follows. Can we discuss and study the new defined class of analytic $\mathbf{g}$ -Bloch type functions in the case of bicomplex functions?

5. Conclusions

Function spaces theory is developed, extended, and generalized to spaces of several complex variables ([8–13, 29]) also using quaternion-valued functions ([21–24, 30–33]). The intention of this study is to introduce a new type of analytic function spaces, which plays an interesting and global rule of studying complex function spaces. It should be emphasized that both the worked plane of the study (i.e., U) and the considered holomorphic functions of Bloch-type as well as specific properties of Green's function are extremely needed for the new classes.

The holomorphic classes of $\mathbf{g}$ -Bloch functions are defined and deeply considered using a modified Green's function. By the new function classes, some specific relations and inclusions for the holomorphic Dirichlet-type space as well as holomorphic $\mathcal{Q}_p$ spaces are obtained. On the other hand, an extension of Yamashita's result [26] is presented using holomorphic $\mathbf{g}$ -Bloch functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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