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Research Article

Common Fixed Point Theorems for F-Kannan–Suzuki Type Mappings in TVS-Valued Cone Metric Space with Some Applications

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This research paper generalizes and extends various fixed-point results that demonstrate common fixed-point theorems for F-Kannan–Suzuki type mappings in TVS-valued cone metric spaces. The results are supported using interpretative exemplifications and applications that include nonlinear fractional as well as two-point periodic ordinary differential equations.

1. Introduction

The fixed-point theory is at the foundation of nonlinear analysis, which is a prominent research area of mathematics. Fixed point theory is, in fact, a simple, powerful, and useful tool for nonlinear analysis. It also has fruitful applications in mathematics and in various scientific domains, including physics, chemistry, computer science, etc. As a result, this theory has attracted a large number of researchers who are guiding the theory's growth in various areas.

In 1922, Banach [1] established a fixed point theorem in metric space which states that if $\mathcal{X}$ is a complete metric space and $G: \mathcal{X} \longrightarrow \mathcal{X}$ is a contraction map, i.e., $\varrho(G\sigma, G\varsigma) \leq \kappa\varrho(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathcal{X}$ and $\kappa \in [0, 1)$, then G has a unique fixed point or $G\sigma = \sigma$ has a unique solution. In addition to an acceptable contraction condition, the metrical common fixed-point theorems usually include constraints on commutativity, continuity, completeness, and appropriate containment of ranges of detailed maps. The goal of researchers in this field is to weaken one or more of these conditions. The use of weak conditions of commutativity is to improve common fixed point theorems in analysis. Connell [2] provided an example of a noncomplete metric space X, but every contraction on it has a fixed point. Kannan [3] proposed an alternative contractive condition that was not the same as the Banach contraction condition. Also, Subrahmanyam [4] proved the converse of Banach fixed-point theorem using Kannan mapping. Furthermore, to evaluate a fixed point for a stringent type Kannan contraction, the assumption of continuity of the mapping and the compactness requirement on metric space are necessary.

In 2007, Huang and Zhang [5] generalised the Banach fixed point theorem by introducing the structure of cone metric by substituting real numbers with an ordered Banach space and establishing a convergence criterion for sequences in a cone metric space. In normal cone metric space, Huang and Zhang [5] proved some fixed-point theorems for Kannan type contractive conditions; nevertheless, Rezapour and Hamlbarani [6] neglected this idea in some results by Huang. For normal and nonnormal cones in cone metric spaces, several authors have examined fixed point theorems and common fixed-point theorems for self-mappings. We refer to the reader [7–10] and the references therein. By relaxing the normalcy criteria set by Huang and Zhang [5], Beg et al. [11], investigated common fixed points for a pair of maps on topological vector space (TVS) valued cone metric spaces in 2009. They demonstrated that the class of TVS-valued cone metric spaces is larger than the class of cone metric spaces, used in [12–16] and the references therein. Recently, Hu and Gu [17] proved some fixed point theorems of λ-contractive mappings in Menger PSM-spaces. For a class of contractive mappings, Reich and Alexander [18] generalised fixed points and convergence results. In Hausdroff TVS, Ram and Lai [19] presented the existence results on generalised strong operator equilibrium problems. In TVS-Cone Metric Spaces, Dubey and Mishra [20] demonstrated some fixed-point results of single-valued mapping for -distance. Using some facts about topological vector space, Tas [21] constructed a new notion of a TVS cone S- metric space. Lee [22] introduces chain recurrent set, trapping region, attracting set and repelling set for a flow f on a TVS-cone metric space. By using generalised metric spaces, Ge and Yang [23] proved a common generalisation of TVS-cone metric spaces, partial metric spaces and b-metric spaces, and a unified approach is proposed for some fixed point results. Later, Suzuki [24] and Rida et al. [25] gave a generalisation of the Banach contraction principle that characterises metric completeness.

Wardowski [26] used a new sort of contraction called F-contraction to give an intriguing generalisation of the Banach fixed point theorem. Many scholars have used his method to build new fixed-point theorems since then. The associated results and references can be found in [27–30] and the references therein. Piri and Kumam [28], extended Wardowski's [26] results in 2014 by introducing the notion of F-Suzuki contraction and obtained some intriguing results utilising the Secelean [29] concept. In the complete b-metric spaces, Alsulami et al. [31] demonstrated fixed points of generalised F-Suzuki type Contractions. Budhia et al. [32] proved an extension of almost-F and F-Suzuki contractions with graph and demonstrated some applications to fractional calculus whereas Chandok et al. [33] formulated some fixed point results for the generalised F-Suzuki type contractions in b-metric spaces. Derouiche and Ramou [34] proved new fixed-point results for F-contractions of Suzuki Hardy-Rogers type in b-metric spaces and provided some applications. Beg et al. [11] proposed a fixed point of orthogonal F-Suzuki contraction mapping on 0-complete b-metric spaces with some applications. Mani et al. [35] introduced generalised orthogonal F-contraction and orthogonal F-Suzuki contraction mappings and proved some fixed point theorems for a self-mapping in orthogonal metric space. Vujakovic and Radenovic [36] introduced certain fixed point results for F-contraction of Piri-Kumam-Dung-type mappings in metric spaces.

In 2019, Goswami et al. [27] introduced F-contractive type mappings in b-metric spaces and proved some fixed point results with suitable examples. Recently, Batra et al. [37] noticed in their subsequent analysis that the definition introduced by Goswami et al. [27] is not meaningful in general. Therefore, they provided suitable examples to support their opinion on this definition. Also, due to these reasons, Batra et al. [37] presented F-contraction and Kannan mapping concepts for defining F-Kannan mappings, which is, in a true sense, a generalisation of Kannan mappings.

This paper aims to extend and generalise the results due to Batra et al. [37], Filipovic et al. [38], Morales and Rojas [9], Rahimi et al. [39] and Wangwe and Kumar [40] using a pair of two self-mappings in F-Kannan–Suzuki type mapping in TVS-valued cone metric space, where we consider a map to be sequentially convergent, one to one and continuous. By doing so, we will extend several other results of the same setting in the literature. Finally, we will provide some applications to the nonlinear Riemann–Liouville fractional differential equation and nonlinear Volterra-integral differential equation.

2. Preliminaries

The definitions, lemmas, and theorems will help us prove our main points in the upcoming sections.

In 1968, Kannan [3] developed a new contractive condition and proved the following theorem for self mappings in complete metric spaces as a result of a generalisation of the Banach fixed point theorem.

Theorem 1 (see [3]). *Let $G\colon \mathscr{X} \longrightarrow \mathscr{X}$ be a self mapping on a complete metric space $(\mathscr{X}, \varrho)$ such that*

$$\varrho(G\sigma, G\varsigma) \leq \kappa\{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)\}, \tag{1}$$

for all $\sigma, \varsigma \in \mathscr{X}$ and $0 \leq \kappa \leq (1/2)$. Then, G possesses a unique fixed point $\sigma^* \in \mathscr{X}$ and for any $\sigma \in \mathscr{X}$ the iterate sequence $\{G^n\sigma\}$ converges to σ^*.

Equation (1) is equivalent to

$$\varrho(G\sigma, G\varsigma) \leq \frac{\kappa}{2}\{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)\}, \tag{2}$$

for some $\kappa \in (0, 1)$.

Definition 1 (see [11]). Let $(\mathscr{E}, \tau)$ be always a topological space and $\mathscr{P}$ a subset of $\mathscr{E}$. Then, $\mathscr{P}$ is called a cone if the following hold:

(i) $\mathscr{P}$ is a nonempty, closed and $\mathscr{P} \neq \{0\}$;
(ii) $\lambda\sigma + \mu\varsigma \in \mathscr{P}$ for all $\sigma, \varsigma \in \mathscr{P}$ and nonnegative real number λ, μ;
(iii) $\mathscr{P} \cap (-\mathscr{P}) = \{0\}$.

For given cone $\mathscr{P} \subseteq \mathscr{E}$. If the interior of $\mathscr{P}(\text{int}\mathscr{P})$, is nonempty we say that $\mathscr{P}$ is solid. If $\mathscr{P}$ is solid cone, then $\mathscr{P}$ is a component of $\mathscr{P}$, and in this case we use the notation $\sigma \ll \varsigma$ to indicate that $\varsigma - \sigma \in \text{int}\mathscr{P}$. Note that if $\sigma \ll \varsigma$ and $\varsigma \leq \upsilon$, then $\sigma \ll \upsilon$ for all $\sigma, \varsigma, \upsilon \in \text{int}\mathscr{P}$.

The following axioms satisfy TVS-valued cone complete metric space.

Definition 2 (see [11]). Let $\mathscr{X}$ be a nonempty set and the mapping $\varrho\colon \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{E}$, satisfies the following:

(i) $0 \leq \varrho(\sigma, \varsigma)$, for all $\sigma, \varsigma \in \mathscr{X}$ and $\varrho(\sigma, \varsigma) = 0$ if and only if $\sigma = \varsigma$;
(ii) $\varrho(\sigma, \varsigma) = \varrho(\varsigma, \sigma)$, for all $\sigma, \varsigma \in \mathscr{X}$;
(iii) $\varrho(\sigma, \varsigma) \leq \varrho(\sigma, \upsilon) + \varrho(\upsilon, \varsigma)$, for all $\sigma, \varsigma, \upsilon \in \mathscr{X}$.

Then, ϱ is called a cone metric on $\mathcal{X}$, and $(\mathcal{X}, \varrho)$ is called topological vector space valued cone metric space.

Example 1 (see [12, 9, 41]). Let $\mathcal{E} = (C_{[0,1]}, \mathbb{R}^2)$, $\mathcal{P} = \{(\sigma, \varsigma) \in \mathcal{E} | \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$, $\mathcal{X} = \mathbb{R}$ and $\varrho: \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{E}$ such that $\varrho(\sigma, \varsigma) = |\sigma - \varsigma| \psi(t)$, where $\psi(t) = e^t$. Then, $(\mathcal{X}, \Sigma)$ is a TVS-valued cone metric space.

The following definition is due to Beg et al. [11] in TVS-valued cone metric space.

Definition 3 (see [8]). Let $(\mathcal{X}, \varrho)$ be a topological vector space valued cone metric space, and let $x \in \mathcal{X}$ and $\{\sigma_n\}_{n \geq 1}$ be a sequence in $\mathcal{X}$. Then,

(i) $\{\sigma_n\}_{n \geq 1}$ converges to $\mathcal{X}$ whenever for every $c \in \mathcal{E}$ with $0 \ll c$ there is a natural number $\mathbb{N}$ such that $\varrho(\sigma_n, \sigma) \ll c$ for all $n \geq \mathbb{N}$. We denote this by

$$\lim_{n \longrightarrow \infty} \sigma_n = \sigma \Leftrightarrow \sigma_n \longrightarrow \sigma. \tag{3}$$

(ii) $\{\sigma_n\}_{n \geq 1}$ is Cauchy sequence whenever for every $c \in \mathcal{E}$ with $0 \ll c$, there is a natural number $\mathbb{N}$ such that $\varrho(\sigma_n, \sigma_m) \ll c$ for all $n, m \geq \mathbb{N}$.

(iii) $(\mathcal{X}, \varrho)$ is called topological vector space valued cone metric space if every Cauchy sequence is convergent.

Definition 4 (see [42]). Let $\mathcal{X}$ be a topological space. If (σ_n) is a sequence of points of $\mathcal{X}$, and if $n_1 < n_2 < \ldots < n_i < \ldots$ is an increasing sequence of positive integers, then the sequence (ς_i) defined by setting $\varsigma_i = \sigma_{n_i}$ is called a subsequence of the sequence (σ_n). The space $\mathcal{X}$ is said to be sequentially compact if every sequence of points of $\mathcal{X}$ has a convergent subsequence.

Definition 5 (see [43]). Let $(\mathcal{X}, d)$ be a metric space. A mapping $G: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be sequentially convergent if we have, for every sequence $\{\varsigma_n\}$, if $\{G\varsigma_n\}$ is convergence then $\{\varsigma_n\}$ also is convergence. G is said to be subsequentially convergent if we have, for every sequence $\{\varsigma_n\}$, if $\{G\varsigma_n\}$ is convergence then $\{\varsigma_n\}$ has a convergent subsequence.

The extended version of sequentially convergent mappings in TVS-valued cone metric space is given as follows.

Definition 6 (see [9]). Let $(\mathcal{X}, \varrho)$ be a cone metric space, $\mathcal{P}$ is a solid cone and $G: \mathcal{X} \longrightarrow \mathcal{X}$. Then

(i) G is said to be continuous if

$$\lim_{n \longrightarrow \infty} \sigma_n = \sigma \Rightarrow \lim_{n \longrightarrow \infty} G\sigma_n = G\sigma, \tag{4}$$

for all $\sigma_n \in \mathcal{X}$,

(ii) G is said to be sequentially convergent if we have, for every sequence (ς_n), if $G\varsigma_n$ is convergent, then ς_n also is convergent,

(iii) G is said to be subsequentially convergent if we have, for every sequence (ς_n) and $G\varsigma_n$ is convergent, implies ς_n has a convergent subsequence.

In 2011, Filipovic et al. [38] generalised Theorem 3.1 and Theorem 3.5 from [9] by using the sequentially convergent mappings in cone metric space and considered $\mathcal{P}$ to be a solid cone. They proved results on two self mappings as follows.

Definition 7 (see [38]). Let $(\mathcal{X}, \varrho)$ be a cone metric space and $T, f: \mathcal{X} \longrightarrow \mathcal{X}$ two mappings. A mapping f is said to be T- Hardy-Rogers contraction if there exists $a_i \geq 0, i = 1, \ldots, 5$ with $\sum_{i=1}^{5} a_i \leq 1$ such that for all $\sigma, \varsigma \in \mathcal{X}$.

$$\begin{aligned} \varrho(Tf\sigma, Tf\varsigma) \leq{}& a_1 \varrho(T\sigma, T\varsigma) + a_2 \varrho(T\sigma, Tf\sigma) + a_3 \varrho(T\varsigma, Tf\varsigma) \\ &+ a_4 \varrho(T\sigma, Tf\varsigma) + a_5 \varrho(T\varsigma, Tf\sigma). \end{aligned} \tag{5}$$

Theorem 2 (see [38]). *Let $(\mathcal{X}, \varrho)$ be a complete cone metric space and $\mathcal{P}$ a solid cone, in addition let $T: \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f: \mathcal{X} \longrightarrow \mathcal{X}$ a T-hardy-Rogers contraction. Then,*

(i) For every $\sigma_0 \in \mathcal{X}$ the sequence $Tf^n\sigma_0$ is Cauchy.

(ii) There is $v_{\sigma_0} \in \mathcal{X}$ such that $\lim_{n \longrightarrow \infty} Tf^n\sigma_0 = v_{\sigma_0}$.

(iii) T is sequentially convergent, then $(f^n\sigma_0)$ has a convergent, subsequence.

(iv) There is a unique $u_{\sigma_0} \in \mathcal{X}$ such that $fu_{\sigma_0} = u_{\sigma_0}$.

(v) If T is sequentially convergent, then for each $\sigma_0 \in \mathcal{X}$ the iterate sequence $(f^n\sigma_0)$ converges to u_{σ_0}.

Theorem 3 (see [38]). *Let $(\mathcal{X}, \varrho)$ be a complete cone metric space and $\mathcal{P}$ a solid cone, in addition let $T: \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f: \mathcal{X} \longrightarrow \mathcal{X}$ such that $F(f) \neq \varnothing$ and that*

$$\varrho\left(Tf\sigma, Tf^2\sigma\right) \preceq \lambda \varrho(T\sigma, Tf\sigma), \tag{6}$$

holds for some $\lambda \in (0,1)$ and for all $\sigma \in \mathcal{X}, \sigma \neq f\sigma$. Then f has property $\mathcal{P}$.

Remark 1 (see [44]). Let $F(T)$ denote the fixed point set of a map T. A map T has property $\mathcal{P}$ if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$. We shall say that a pair of maps T and f has property Q if $F(T) \cap F(f) = F(T^n) \cap F(f^n)$ for each $n \in \mathbb{N}$.

Secelean [29] proved the following lemma.

Lemma 1 (see [29]). *Let $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an increasing function and $\{x_n\}$ be a sequence of positive real numbers. Then the following holds:*

(a) If $\lim_{n \longrightarrow \infty} F(\alpha_n) = -\infty$, then $\lim_{n \longrightarrow \infty} \alpha_n = 0$,

(b) If $\inf F = -\infty$ and $\lim_{n \longrightarrow \infty} \alpha_n = 0$, then $\lim_{n \longrightarrow \infty} F(\alpha_n) = -\infty$,

Let $\mathfrak{F}$ be the set of all functions defined as $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$, which satisfies the following conditions:

(F1) F is strictly increasing i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$

(F2″) there is a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ or $\inf F = -\infty$

(F3″) F is continuous on $(0, \infty)$

The following function $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ belongs to $\mathfrak{F}$:

(i) $F_1(z) = \ln z$

(ii) $F_2(z) = -(1/z)$

(iii) $F_3(z) = -(1/z) + z$

Definition 8 (see [28]). Let $(\mathcal{X}, \varrho)$ be a metric space. A mapping $G: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be an F-Suzuki contraction if there exists $\tau > 0$, such that for all $\sigma, \varsigma \in \mathcal{X}$ with $G\sigma \neq G\varsigma$

$$\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma) \Rightarrow \tau + F(\varrho(G\sigma, G\varsigma)) \leq F(\varrho(\sigma, \varsigma)), \tag{7}$$

where $F \in \mathfrak{F}$.

In 2014, Piri and Kumam [28] established a generalisation of Banach contraction principle, which is as follows:

Theorem 4 (see [28]). *Let $(\mathcal{X}, \varrho)$ be a complete metric space and $G: \mathcal{X} \longrightarrow \mathcal{X}$ be a F-Suzuki contraction. Then G has a unique fixed point $\sigma^* \in \mathcal{X}$ and for every $\sigma_0 \in \mathcal{X}$ a sequence $\{G^n\sigma_0\}_{n\in\mathbb{N}}$ is convergent to σ^*.*

Remark 2 (see [28]). We denote by $\mathfrak{F}$ the set of all functions satisfying F-suzuki type contraction condition due to [28, 29] and let denote by $\mathcal{F}$ the set of all functions satisfying F-contraction condition by Wardowski [26], then

(i) $\mathcal{F} \nsubseteq \mathfrak{F}$

(ii) $\mathfrak{F} \nsubseteq \mathcal{F}$

(iii) $\mathcal{F} \cap \mathfrak{F} \neq \varnothing$

For more details on F-Suzuki contraction mapping, one can see [31–33] and the references therein.

Motivated by Batra et al. [37], we use the following notations: Let $\mathcal{X}$ be a nonempty set and $(\mathcal{X}, \varrho)$ denotes the metric space with metric ϱ. Let the cardinality of a set A is denoted by card$\{A\}$ and Fix G is set of all fixed points of a mapping G.

Batra et al. [37] gave a new generalisation family of contraction called F-Kannan mapping and introduced the following definition:

Definition 9 (see [37]). Let F be a mapping satisfying $(F1) - (F3)$. A mapping $G: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be an F-Kannan mapping if the following holds:

(K1)

$$G\sigma \neq G\varsigma \Rightarrow G\sigma \neq \sigma \text{ or } G\varsigma \neq \varsigma. \tag{8}$$

(K2) $\exists \Upsilon > 0$ such that

$$\Upsilon + F(\varrho(G\sigma, G\varsigma)) \leq F\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \tag{9}$$

for all $\sigma, \varsigma \in \mathcal{X}$, with $G\sigma \neq G\varsigma$.

The remark presented below is due to Batra et al. [37].

Remark 3 (see [37]). By properties of F, it follows that every F-Kannan mapping T on a metric space $(\mathcal{X}, \varrho)$, satisfies following condition:

$$\varrho(G\sigma, G\varsigma) \leq \frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}, \tag{10}$$

for every $\sigma, \varsigma \in \mathcal{X}$.

Furthermore, it is concluded that Card$\{\text{Fix}\, G\} \leq 1$. Let G be a self map of a metric space $(\mathcal{X}, \varrho)$. G is said to be a Picard operator (PO) if G has unique fixed point σ^* and $\lim_{n\longrightarrow\infty} G^n\sigma = \sigma^*$ for all $\sigma \in \mathcal{X}$.

Then the family of all functions $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying the condition $(F1) - (F3)$ is denoted by $\mathcal{F}$.

We recall the following examples from Batra et al. [37] of such functions $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ which satisfies $(F1) - (F3)$:

Example 2 (see [37]). Let $F_1: \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1) - (F3)$ are satisfied by $F_1(z)$. In fact $(F3)$ holds for every $k \in (0, 1)$

$$\varrho(G\sigma, G\varsigma) \leq e^{-\Upsilon}\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \tag{11}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $G\sigma \neq G\varsigma$.

Thus, if $G: \mathcal{X} \longrightarrow \mathcal{X}$ is a Kannan mapping with constant $\kappa \in (0, 1)$ satisfying

$$\varrho(G\sigma, G\varsigma) \leq \kappa\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \tag{12}$$

for every $\sigma, \varsigma \in \mathcal{X}$, then it also satisfies (8) and (11) with $\Upsilon = \ln(1/\kappa)$. In fact, whenever $G\sigma \neq G\varsigma$, then from (12), we get $G\sigma \neq \sigma$ or $G\varsigma \neq \varsigma$.

The following lemma introduced by Batra et al. [37].

Lemma 2 (see [37]). *Let $(\mathcal{X}, \varrho)$ be a metric space and $G: \mathcal{X} \longrightarrow \mathcal{X}$ be a F-Kannan mapping. Then, $\varrho(G^n\sigma, G^{n+1}\sigma) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathcal{X}$.*

Batra et al. [37] introduced a F-Kannan mapping using the properties by Subrahmanyam [4] which is an extension of Goswami et al. [27] and Wardowski [26] results. They proved the following result.

Theorem 5 (see [37]). *Let $(\mathcal{X}, \varrho)$ be a complete metric space and suppose $G: \mathcal{X} \longrightarrow \mathcal{X}$ is a F-Kannan mapping, then G is a Picard operator (PO).*

Using the following definitions, we introduce some fundamental properties for a fixed point and common fixed point theorems.

Definition 10 (see [45]). Let (G, f) be a pair of self-mappings on a metric space $(\mathcal{X}, \varrho)$. Then coincidence point of the pair (G, f) is a point $\sigma \in \mathcal{X}$ such that $(G\sigma) = (f\sigma) = \sigma^*$, then σ^* is called coincidence point of the pair (G, f). If $\sigma^* = \sigma$, then σ is said to be a common fixed point of f and G.

Definition 11 (see [46]). Let G, f be self-mappings of a nonempty set $\mathcal{X}$. A point $\sigma \in \mathcal{X}$ is coincidence point of G and f if $t = G\sigma = f\sigma$. The set of coincidence point of G and f is denoted by $C(G, f)$.

Definition 12 (see [46, 47]). Let (T, f) be a pair of self-mappings on a metric space $(\mathcal{X}, \varrho)$. Then, the pair (T, f) is said to be as follows:

(i) Commuting if, for all $\sigma \in \mathcal{X}$, $G(f\sigma) = f(G\sigma)$,

(ii) Weakly commuting if, for all $\varrho(G(f\sigma), f(G\sigma)) \leq \varrho(G\sigma, f\sigma)$,

(iii) Compatible if $\lim_{n\longrightarrow\infty}\varrho(Gf\sigma_n, fG\sigma_n) = 0$, whenever σ_n is a sequence in $\mathcal{X}$ such that $\lim_{n\longrightarrow\infty}G\sigma_n = \lim_{n\longrightarrow\infty}f\sigma_n = t$,

(iv) Weakly compatible if, for all $G(f\sigma) = f(G\sigma)$, for every coincidence point $\sigma \in \mathcal{X}$.

3. Main Results

To prove this section's main results, we commerce by obtaining a more general version of Definition 8 and 9 using a pair of two self mappings in F-Kannan–Suzuki type mapping setting. We denotes $(\mathcal{X}, \varrho)$ as a TVS-valued cone metric space.

Definition 13. Let F be a mapping satisfying $(F1) - (F3)$. A pair of two self mapping $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ is said to be an F-Kannan–Suzuki type mapping if the following holds:

(FKS1)

$$Gf\sigma \neq Gf\varsigma \Rightarrow Gf\sigma \neq \sigma \text{ or } Gf\varsigma \neq \varsigma. \tag{13}$$

(FKS2) there exists $\vartheta > 0$ such that

$$\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \sigma)$$
$$\Rightarrow \vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right], \tag{14}$$

for all $\sigma, \varsigma \in \mathcal{X}$, with $Gf\sigma \neq Gf\varsigma$ and $F \in \mathfrak{F}$.

Following remark is motivated by the work of Batra et al. [37] given as follows.

Remark 4. By properties of F, it follows that every F-Kannan–Suzuki type mapping G on a TVS-valued cone metric space $(\mathcal{X}, \varrho)$, satisfies the following condition:

$$\varrho(Gf\sigma, Gf\varsigma)) \leq \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}, \tag{15}$$

for every $\sigma, \varsigma \in \mathcal{X}$.

We give the following examples in the context of a pair of two self mappings:

Example 3. Let $F_1\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1) - (F3)$ are satisfied by $F_1(z)$. In fact $(F3)$ holds for every $\kappa \in (0, 1)$. Moreover, condition (14) takes the form:

$$\varrho(Gf\sigma, Gf\varsigma)) \leq e^{-\vartheta}\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right], \tag{16}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

Thus, if $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ is a Kannan mapping with constant $\kappa \in (0, 1)$ satisfying

$$\varrho(Gf\sigma, Gf\varsigma)) \leq \kappa\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right]. \tag{17}$$

for every $\sigma, \varsigma \in \mathcal{X}$. Then it also satisfies (16) and (14) with $\vartheta = \ln(1/\kappa)$.

Example 4. Let $F_2\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_2(z) = -(1/z), z > 0$. Then, $(F1) - (F3)$ are satisfied by $F_2(z)$. Condition (14) takes the form:

$$\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)}{1 - \vartheta\varrho(Gf\sigma, Gf\varsigma)}, \tag{18}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

Example 5. Let $F_3\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_3(z) = -(1/z), z > 0$. Then, $(F1) - (F3)$ are satisfied by $F_3(z)$. Condition (14) takes the form:

$$\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)\left([(\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma))/2]^2 - 1\right)}{\varrho(Gf\sigma, Gf\varsigma) + \vartheta\left(\varrho(Gf\sigma, Gf\varsigma)^2 - 1\right)}, \tag{19}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

We prove the following lemma which is an extension of Lemma 2.

Lemma 3. *Let $(\mathcal{X}, \varrho)$ be a complete TVS-valued cone metric space and $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ be an F-Kannan–Suzuki type mapping. Then,*

$$\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{20}$$

for all $\sigma \in \mathcal{X}$.

Proof. Suppose that σ_0 is an arbitrary point in $\mathcal{X}$. If $Gf^n x_0 = Gf^{n+1}\sigma_0$ for some $n \in \mathbb{N}$, then sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{X}$, and hence the sequence $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathcal{X}$.

Assume that $Gf^n\sigma_0 \neq Gf^{n+1}\sigma_0$ for any $n \in \mathbb{N}$. Then, by (14) with $\vartheta > 0$, we get

$$\begin{aligned}\frac{1}{2}\varrho(\sigma_0, G\sigma_0) &< \varrho(\sigma_0, G\sigma_0)\\ &\Rightarrow \vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right)\\ &\leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right].\end{aligned} \tag{21}$$

By Remark 4, we obtain

$$\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) \leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}. \tag{22}$$

Using (22) in (21), as results yields to

$$\vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right). \tag{23}$$

Letting $n \longrightarrow \infty$ in (23), we get

$$\begin{aligned}\vartheta + 0 &\leq 0,\\ \vartheta &\leq 0,\end{aligned} \tag{24}$$

which is a contradiction. Hence, $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$. □

Motivated by Batra et al. [37] and Filipovic et al. [38], we give a proof of an extended version of Theorem 2, 4, and 5 in F-Kannan–Suzuki type mappings with a pair of two self-mappings in complete TVS-valued cone metric space.

Theorem 6. *Let $(\mathcal{X}, \varrho)$ be a complete TVS-valued cone metric space and $\mathcal{P}$ a solid cone, in addition let $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ a G-F-Kannan–Suzuki type contraction. Then,*

(i) For every $\sigma_0 \in \mathcal{X}$ the sequence $Gf^n\sigma_0$ is convergent

(ii) There is $v^* \in X$ such that $\lim_{n\longrightarrow\infty} Gf^n\sigma_0 = v^*$

(iii) G is sequentially convergent, then $(f^n\sigma_0)$ has a convergent, subsequence

(iv) There is a unique $u^* \in X$ such that $fu^* = u^*$

(v) If G is sequentially convergent, then for each $\sigma_0 \in X$ the iterate sequence $(f^n\sigma_0)$ converges to u^*

Proof. By (i), we prove that $\{Gf^n\sigma_0\}$ is a Cauchy sequence. Let $\sigma_0 \in \mathcal{X}$ be arbitrary. If $Gf^n\sigma_0 = Gf^{n+1}\sigma_0$ for some $n \in \mathbb{N}$, then sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{X}$ and hence the sequence $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathcal{X}$. Suppose that $Gf^n\sigma_0 \neq Gf^{n+1}\sigma_0$ for any $n \in \mathbb{N}$. Then, by (14), Lemma 3 with $\vartheta > 0$, we get

$$\begin{aligned}\frac{1}{2}\varrho(\sigma_n, G\sigma_n) &< \varrho(\sigma_n, G\sigma_n) \Rightarrow\\ \vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) &\leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right].\end{aligned} \tag{25}$$

From Remark 4, we have

$$\begin{aligned}\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n x_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2},\\ 2\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right),\\ \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right).\end{aligned} \tag{26}$$

Using (26) in (25), as results yields to

$$\begin{aligned}\vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) &\leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right],\\ \vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) &\leq F\left[\frac{2\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right)}{2}\right],\\ \vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) &\leq F\left[\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right)\right].\end{aligned} \tag{27}$$

Letting $n \longrightarrow \infty$ in (27), we get

$$\begin{aligned}\vartheta + 0 &\leq 0,\\ \vartheta &\leq 0,\end{aligned} \tag{28}$$

which is a contradiction. Hence, $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\{Gf^n\sigma_0\}$ converges.

Since G is sequentially convergent, using (v), we prove that the iterate of a sequence $f^n\sigma_0$ converge to a fixed $u \in \mathcal{X}$. To see this, suppose $\sigma_0 \in \mathcal{X}$ be an arbitrary point in $\mathcal{X}$. Let the sequence $\{\sigma_n\}_{n\geq 1}$ be defined by $\sigma_{n+1} = f\sigma_n = f^{n+1}\sigma_0 = ff^n\sigma_0$ and $\sigma_n = f\sigma_{n-1} = f^n\sigma_0 = ff^{n-1}\sigma_0$, for $n \geq 1 \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\varrho(\sigma_n, \sigma_{n+1}) \leq \varrho(f\sigma_{n-1}, f\sigma_n) &= \varrho\left(f^n\sigma_0, f^{n+1}\sigma_0\right)\\ &= \varrho\left(ff^{n-1}\sigma_0, ff^n\sigma_0\right).\end{aligned} \tag{29}$$

equivalent to

$$\begin{aligned}\varrho(G\sigma_n, G\sigma_{n+1}) &\leq \varrho(Gf\sigma_{n-1}, Gf\sigma_n)\\ &= \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\\ &= \varrho\left(Gff^{n-1}\sigma_0, Tff^n\sigma_0\right).\end{aligned} \tag{30}$$

Let $\sigma = f^{n-1}\sigma_0$ and $\varsigma = f^n\sigma_0$, using inequality (14), we obtain

$$\begin{aligned}&\frac{1}{2}\varrho(\sigma_n, G\sigma_n) < \varrho(\sigma_n, G\sigma_n)\\ &\Rightarrow \vartheta + F\left(\varrho\left(Gff^{n-1}\sigma_0, Gff^n\sigma_0\right)\right) \leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gff^{n-1}\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gff^n\sigma_0\right)}{2}\right],\\ &F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left[\frac{\varrho(G\sigma_{n-1}, G\sigma_n) + \varrho(G\sigma_n, G\sigma_{n+1})}{2}\right] - \vartheta.\end{aligned} \tag{31}$$

Using Remark 4 and the increasing property of F, we deduce

$$\begin{aligned}\varrho\left(Gff^{n-1}\sigma_0, Gff^n\sigma_0\right) &\leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gff^{n-1}\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gff^n\sigma_0\right)}{2},\\ \varrho(G\sigma_n, G\sigma_{n+1}) &< \frac{\varrho(G\sigma_{n-1}, G\sigma_n) + \varrho(G\sigma_n, G\sigma_{n+1})}{2},\end{aligned} \tag{32}$$

and hence,

$$\begin{aligned}2\varrho(G\sigma_n, G\sigma_{n+1}) - \varrho(G\sigma_n, G\sigma_{n+1}) &< \varrho(G\sigma_{n-1}, G\sigma_n),\\ \varrho(G\sigma_n, G\sigma_{n+1}) &< \varrho(G\sigma_{n-1}, G\sigma_n).\end{aligned} \tag{33}$$

By $(F1)$, this implies that

$$F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) < F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right). \tag{34}$$

Consequently, we get

$$\vartheta + F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right), \tag{35}$$

so

$$F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - \vartheta. \tag{36}$$

By induction and (36), we deduce

$$\begin{aligned}&F\left(\varrho(G\sigma_{n+1}, G\sigma_{n+2})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - 2\vartheta,\\ &F\left(\varrho(G\sigma_{n+2}, G\sigma_{n+3})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - 3\vartheta,\\ &\Rightarrow F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - n\vartheta.\end{aligned} \tag{37}$$

Letting $n \longrightarrow \infty$ in (37), we find that

$$\lim_{n\longrightarrow\infty} F\left(\varrho\left(G\sigma_n, G\sigma_{n+1}\right)\right) = -\infty. \tag{38}$$

Consequently, using Lemma 1 and property $(F2'')$ of F results in

$$\lim_{n\longrightarrow\infty} \varrho\left(G\sigma_n, G\sigma_{n+1}\right) = 0. \tag{39}$$

Thus, there exists $n \in \mathbb{N}$ such that

$$\varrho\left(G\sigma_n, G\sigma_{n+1}\right) < \varrho\left(G\sigma_n, G^2\sigma_n\right) < c\varrho\left(\sigma_n, G\sigma_n\right) < \varrho\left(\sigma_n, G\sigma_n\right), \tag{40}$$

which is a contradiction. Hence, we have

$$\lim_{n\longrightarrow\infty} \varrho\left(\sigma_n, G\sigma_n\right) = 0. \tag{41}$$

Therefore, we have $\varrho\left(G\sigma_n, G\sigma_{n+1}\right) \longrightarrow 0$ as $n \longrightarrow \infty$. Denote $\alpha_n = \varrho\left(G\sigma_n, G\sigma_{n+1}\right) = 0$, for all $n \geq 1$ and $n \in \mathbb{N}$, for F-Kannan–Suzuki type mappings.

By (39), we prove that $\{G\sigma_n\}$ is a Cauchy sequence since $(\mathcal{X}, \varrho)$ is complete. Consider $n, m \in N$ such that $m > n$. Assume on the contrary that there exists $c > 0$ and sequences $\{p(n)\}_{n\geq 1}^{\infty}$ and $\{q(n)\}_{n\geq 1}^{\infty}$ such that

$$\begin{aligned} p(n) > q(n) > n, \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) \\ \geq c, \varrho\left(G\sigma_{p(n)-1}, G\sigma_{q(n)}\right) \leq c, \forall n \in \mathbb{N}. \end{aligned} \tag{42}$$

Using (iii) of Definition 2, we get

$$\begin{aligned} \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) &\leq \varrho\left(G\sigma_{p(n)}, G\sigma_{p(n)-1}\right) \\ &\quad + \varrho\left(G\sigma_{p(n)-1}, G\sigma_{q(n)}\right) \\ &\leq \varrho\left(G\sigma_{p(n)}, G\sigma_{p(n)-1}\right) + c. \end{aligned} \tag{43}$$

From (39) and the above inequality, we have

$$\lim_{n\longrightarrow\infty} \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) = c. \tag{44}$$

From $(F3'')$, (44), and (14), we get

$$\begin{aligned} &\vartheta + F\left(\varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right)\right) \\ &\quad \leq F\left[\frac{\varrho\left(G\sigma_{p(n)-1}, G\sigma_{p(n)}\right) + \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right)}{2}\right]. \end{aligned} \tag{45}$$

Equivalently,

$$\begin{aligned} \vartheta + F(c) &\leq F(c), \\ \vartheta &\leq 0, \end{aligned} \tag{46}$$

which is a contradiction. So, $G\sigma_n = G\sigma_m$ for every $m \geq n$ in $\mathcal{X}$. Hence, $\{G\sigma_n\}$ is a Cauchy sequence in $\mathcal{X}$. The completeness of $\mathcal{X}$ ensures the existence of $u^* \in \mathcal{X}$ such that

$$\begin{aligned} \varrho\left(Gu^*, u^*\right) &= \lim_{n,m\longrightarrow\infty} \varrho\left(G\sigma_n, G\sigma_m\right) = 0 \\ &= \lim_{n\longrightarrow\infty} \varrho\left(G\sigma_n, u^*\right) = 0. \end{aligned} \tag{47}$$

By (47) and Definition 6, it follows that $G\sigma_{n+1} \longrightarrow u^*$ as $n \longrightarrow \infty$. By sequential continuity of f and G, we have

$$\begin{aligned} u^* &= \lim_{n\longrightarrow\infty} f^n\sigma_0 = \lim_{n\longrightarrow\infty} \sigma_n = \lim_{n\longrightarrow\infty} \sigma_{n+1} \\ &= \lim_{n\longrightarrow\infty} f\sigma_n = fu^*.u^* \\ &= \lim_{n\longrightarrow\infty} Gf^n\sigma_0 = \lim_{n\longrightarrow\infty} G\sigma_n = \lim_{n\longrightarrow\infty} G\sigma_{n+1} \\ &= \lim_{u\longrightarrow\infty} G^2\sigma_n = Gu^*. \end{aligned} \tag{48}$$

Since $\mathcal{X}$ is a complete metric space, there exists $u^* \in \mathcal{X}$ such that

$$\lim_{n\longrightarrow\infty} G\sigma_n = Gu^* = u^*. \tag{49}$$

Now, we prove that u^* is a fixed point of G. Thus, by (iii) of Definition 2 and $\varrho(u^*, Gu^*) \geq 0$, we have

$$\varrho\left(u^*, Gu^*\right) \leq \varrho\left(u^*, G\sigma_{n+1}\right) + \varrho\left(G\sigma_{n+1}, Gu^*\right). \tag{50}$$

By Remark 4, it implies that

$$\varrho\left(G\sigma_{n+1}, Gu^*\right) \leq \frac{\varrho\left(G\sigma_n, G\sigma_{n+1}\right) + \varrho\left(G\sigma_{n+1}, Gu^*\right)}{2}. \tag{51}$$

Applying (51) in (50), we obtain

$$\varrho\left(u^*, Gu^*\right) \leq \varrho\left(u^*, G\sigma_{n+1}\right) + \frac{\varrho\left(G\sigma_n, G\sigma_{n+1}\right) + \varrho\left(G\sigma_{n+1}, Gu^*\right)}{2}. \tag{52}$$

Letting $n \longrightarrow \infty$ and using in above inequality, we get

$$\begin{aligned} \varrho\left(u^*, Gu^*\right) &\leq \varrho\left(u^*, Gu^*\right) \\ &\quad + \frac{\varrho\left(u^*, Gu^*\right) + \varrho\left(Gu^*, Gu^*\right)}{2}, \\ \varrho\left(u^*, Gu^*\right) &\leq \varrho\left(u^*, Gu^*\right) + \frac{\varrho\left(u^*, Gu^*\right)}{2}, \\ \varrho\left(u^*, Gu^*\right) &\leq \frac{2\varrho\left(u^*, Gu^*\right) + \varrho\left(u^*, Gu^*\right)}{2}, \\ 2\varrho\left(u^*, Gu^*\right) &\leq 2\varrho\left(u^*, Gu^*\right) + \varrho\left(u^*, Gu^*\right), \\ 2\varrho\left(u^*, Gu^*\right) - 2\varrho\left(u^*, Gu^*\right) &\leq \varrho\left(u^*, Gu^*\right), \\ 0 &\leq \varrho\left(u^*, Gu^*\right), \end{aligned} \tag{53}$$

which is a contradiction. Hence, $Gu^* = u^*$.

Next, we prove that u^* is a unique fixed point of G. Assume on contrary that there exists $v^* \in \operatorname{int}(\mathcal{P})$ such that $u^* \neq v^*$ or $Gu^* \neq Gv^*$. Let $G\sigma_n \longrightarrow v^*$ and v^* is a fixed point of G. Using Remark 4 and (14), it follows that $u^* = v^*$ or $Gu^* = Gv^*$ which is a contradiction. Thus, u^* is a unique fixed point of G.

Moreover, G is a subsequentially convergent, $\{f^n\sigma_0\}$ has a convergent subsequence, there exists $\sigma^* \in X$ and $\{f^{n_k}x_0\}_{k=1}^{\infty}$ such that

$$\lim_{k\longrightarrow\infty} f^{n_k}x_0 = v^*. \tag{54}$$

Due to the continuity of G, it implies that

$$\lim_{k\longrightarrow\infty} Gf^{n_k}\sigma_0 = Gv^*. \tag{55}$$

By (49), we conclude that

$$Gv^* = u^*. \quad (56)$$

Using Remark 4 and $\lambda = (1/2)$, we get

$$\begin{aligned}
\varrho\left(Gff^{n_{k-1}}\sigma_0, Tff^{n_k}\sigma_0\right) &\leq \lambda\left(\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \varrho\left(Gf^{n_k}\sigma_0, Gff^{n_k}\sigma_0\right),\right.\\
&\leq \lambda\left(\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right),\right.\\
&\leq \lambda\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \lambda\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right),\\
(1-\lambda)\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right) &\leq \lambda\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right),\\
\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right) &\leq \frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right).
\end{aligned} \quad (57)$$

Thus, using (iii) of Definition 2, we have

$$\varrho\left(Gfv^*, Gv^*\right) \leq \varrho\left(Gfv^*, Gf^{n_k+1}\sigma_0\right) + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right). \quad (58)$$

By Remark 4,

$$\begin{aligned}
\varrho\left(Gfv^*, Gf^{n_k+1}\sigma_0\right) &= \varrho\left(Gfv^*, Gff^{n_k}\sigma_0\right)\\
&\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right)\right].
\end{aligned} \quad (59)$$

Using (57) and (59) in (58), we obtain

$$\begin{aligned}
\varrho\left(Gfv^*, Gv^*\right) &\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right)\right] + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right)\right] + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right)\\
&\leq \lambda\varrho\left(Gv^*, Gfv^*\right) + \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n_k-1}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right) + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \frac{\lambda}{1-\lambda}\left(\frac{\lambda}{1-\lambda}\right)^{n_k-1}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \frac{1}{1-\lambda}\varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \left(\frac{\lambda}{1-\lambda}\right)^{n_k}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \frac{1}{1-\lambda}\varrho\left(Gf^{n_k+1}\sigma_0, Gf^{n_k}\sigma_0\right).
\end{aligned} \quad (60)$$

Suppose that

$$\left(\frac{\lambda}{1-\lambda}\right)^{n_k}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) = \frac{c}{2}. \quad (61)$$

$$\frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) = \frac{c}{2}. \quad (62)$$

Letting $k \longrightarrow \infty$ and using Definition 3, (61), and (62) in (60), we obtain

$$\varrho\left(Gfv^*, Gv^*\right) \leq \frac{c}{2} + \frac{c}{2}, \quad (63)$$

which follows

$$\varrho\left(Gfv^*, Gv^*\right) \leq c. \quad (64)$$

Since G is one to one and continuous, $fv^* = v^*$. So, f has a fixed point. As $Gf^n\sigma_0$ is sequentially convergent, we conclude that $\{Gf^n\sigma_0\}$ converges to the fixed point of f. □

Next, we prove our second main results by extending Theorem 3 using an F-Kannan–Suzuki type mapping in TVS-valued cone metric space.

Theorem 7. *Let $(\mathcal{X}, \varrho)$ be a complete TVS-valued cone metric space and $\mathcal{P}$ a solid cone. In addition, let $G: \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous and sequentially mappings and $f: \mathcal{X} \longrightarrow \mathcal{X}$ such that $F(f) \neq \emptyset$, $\vartheta > 0$ and that*

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma)\\
&\Rightarrow \vartheta + F\left(\varrho\left(Gf\sigma, Gf^2\sigma\right)\right) \leq F\left(\varrho\left(G\sigma, Gf\sigma\right)\right),
\end{aligned} \quad (65)$$

holds for some $\lambda \in (0,1)$ and for all $\sigma \in \mathcal{X}, \sigma \neq f\sigma$. Then f has property Q.

Proof. By Remark 1, let $u \in F(G^n) \cap F(f^n)$ for some $n \in \mathbb{N}$. If $u = fu$, that is u is a unique fixed point of G and f. Hence, the proof completed. On contrary, we suppose $u \neq fu$. Let $\sigma = u = f^{n-1}u$ and $\varsigma = fu = ff^{n-1}u$ such that $f^{n-1} \neq ff^{n-1}$ and using (65), we get

$$\begin{aligned}
&\frac{1}{2}\varrho(u, Gu) < \varrho(u, fu),\\
&\varrho(u, Gu) < 2\varrho(u, fu),\\
&\Rightarrow \vartheta + F\left[\varrho\left(Gff^{n-1}u, Gf^2f^{n-1}u\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, Gff^{n-1}u\right)\right],\\
&\vartheta + F\left[\varrho\left(Gff^{n-1}u, Gff^{n}u\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, TGf^{n}u\right)\right].
\end{aligned} \tag{66}$$

Consequently, we get

$$F\left[\varrho\left(Gff^{n-1}u, Gff^{n}u\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, Gf^{n}u\right)\right] - \vartheta. \tag{67}$$

Repeating the same argument several times, we finally obtain

$$F\left[\varrho\left(Gff^{n-1}u, Gff^{n}u\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, Gf^{n}u\right)\right] - n\vartheta. \tag{68}$$

By following similar procedure as the proof of Theorem 6, we can conclude that $\varrho(Gu, Gfu) = c$, i.e., $Gu = Gfu$. Since G is one to one and sequentially convergent, then $u = fu$, which is a contradiction. Hence, $u \in F(G^n) \cap F(f^n)$. □

We give an example where generalised Kannan mapping will not be applicable. However, F- Kannan Suzuki type mapping is applicable.

Example 6. Consider the sequence $\mathcal{X} = \{0,1\} \cup \{(1/2), (1/3), (1/4), \ldots\}$ and d be an Euclidean metric on $\mathcal{X}$. Then $(\mathcal{X}, \varrho)$ is a TVS-valued cone complete metric space. Let the mapping $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ be determined as follows:

$$\begin{aligned}
f(0) &= 0,\\
f(1/i) &= \frac{1}{i+1},
\end{aligned} \tag{69}$$

for $n \geq 2$. Let there exist $\lambda \in [0, (1/2))$, so that, for all $\sigma, \varsigma \in X$ condition (1) is satisfied although is not true for every $\lambda > 0$. That is a contradiction; hence, Kannan's theorem cannot be applicable.

The mapping $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ be determined as

$$\begin{aligned}
G(0) &= 0,\\
G(1/i) &= \frac{1}{2^i}.
\end{aligned} \tag{70}$$

For all $i \geq 2$, G is continuous, one to one, and subsequentially convergent.

We consider a sequence $\{\sigma_i\}$ in $\mathcal{X}$ and assume that $\mathcal{X}$ is sequentially compact in complete TVS-valued cone metric space. By assumption $\mathcal{X}$ is sequentially compact with $\epsilon = 1$ we can cover the space $\mathcal{X}$ with finitely many balls of radius 1; then one of them contains many $\{\sigma_i\}$ for $i \geq 2$; i.e., there is a ball B_1 of radius 1 so that there is a subsequence of $\{\sigma_i\}$ whose members all belongs to B_1. We denote this subsequence by $\{\sigma_i\}$; thus, all $\{\sigma_i\}$ belongs to B_1.

Similar by sequentially compactness conditions with $\epsilon = (1/2)$, we can find a subsequence $\{\sigma_{i_2}\}$ of $\{\sigma_{i_1}\}$ and a ball B_2 of radius 1/2 so that all $\{\sigma_{i_2}\}$ belongs to B_2. Continuing this way, we obtain for any $k \geq 2$ a subsequence $\{\sigma_{i_k}\}$ of $\{\sigma_{i_k-1}\}$ and a ball B_k of radius 2^{-k} so that all $\{\sigma_{i_k}\}$ belongs to B_k.

Now, let $i, j \in \mathbb{N}$, $j > i$. Then, we show that (f, G) is a F-Kannan–Suzuki type mapping in TVS-valued cone metric space with respect to $F_2(z) = -(1/z)$ and $\vartheta \geq 0$. By using (FKS2) and $F_2(z)$, we have

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma)\\
&\Rightarrow \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)}{1 - \vartheta\varrho(Gf\sigma, Gf\varsigma)}.
\end{aligned} \tag{71}$$

To see this, we now calculate $\varrho(f\sigma, G\varsigma)$ for $\sigma = 1/i, \varsigma = 1/j, i \geq 1$.

$$\begin{aligned}
\varrho(G\sigma, Gf\sigma) &= \varrho(T(1/i), Gf(1/i))\\
&\leq \left|\frac{1}{2^i} - \frac{1}{2^{1/(i+1)}}\right| e^t.
\end{aligned} \tag{72}$$

$$\begin{aligned}
\varrho(Gf\sigma, Gf\varsigma) &= \varrho(Gf(1/i), Gf(1/j))\\
&\leq \left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t.
\end{aligned} \tag{73}$$

$$\begin{aligned}
\varrho(G\varsigma, Gf\varsigma) &= \varrho(G(1/j), Gf(1/j))\\
&\leq \left|2^{1/j} - 2^{1/(j+1)}\right| e^t.
\end{aligned} \tag{74}$$

Applying (72), (73), and (74) in (71) becomes

$$\begin{gathered}\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma),\\ \frac{1}{2}\varrho(1/i, G(1/i)) < \varrho(1/i, 1/j),\\ \varrho(1/i, G(1/j)) < 2\varrho(1/i, 1/j),\\ \left|\frac{1}{i} - \frac{1}{2^j}\right| e^t < 2\left|\frac{1}{i} - \frac{1}{j}\right| e^t,\\ \left|\frac{2^i - i}{2^i . i}\right| e^t < 2\left|\frac{j-i}{ij}\right| e^t,\\ \Rightarrow \frac{\varrho(G(1/i), Gf(1/i)) + \varrho(G(1/j), Gf(1/j))}{2} \leq \frac{\varrho(Gf(1/i), Gf(1/j))}{1 - \vartheta\varrho(Gf(1/i), Gf(1/j))},\\ \Rightarrow \frac{\left|\left(1/2^i\right) - \left(1/2^{1/(i+1)}\right)\right| e^t + \left|2^{1/j} - 2^{1/(j+1)}\right| e^t}{2} \leq \frac{\left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t}{1 - \vartheta\left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t}.\end{gathered} \tag{75}$$

Thus the inequality (71) and all conditions imposed in Theorem 6 are satisfied. Hence, G and f has unique fixed point that is $\nu^* = 0$ in $\{\mathscr{P} \subseteq \mathscr{E}\} \in \mathscr{X}$, where $\mathscr{P}$ is a solid cone.

4. Some Applications

Two applications of the theorem stated in the previous part will be presented in this section.

4.1. Existence of a Solution for Nonlinear Riemann–Liouville Type Fractional Differential Equation. As a convolution mapping, the nonlinear fractional differential equation is equally and identically utilized in several applications in the domains of science, engineering, and mathematics.

(i) In image processing: convolutional filtering is used in many essential algorithms in digital image processing, such as edge detection and related procedures. An out-of-focus photograph is created by convolutioning a crisp image with a lens function in optics. This is referred to as bokeh in photography. For example, applying blurring to a picture in image processing software.

(ii) In digital data processing: Savitzky–Golay smoothing filters are used for analyzing spectroscopic data in analytical chemistry. This can boost the signal-to-noise ratio while reducing spectral distortion along with a convolution in statistics that is weighted in moving average.

(iii) In acoustics: reverberation is the convolution of the original sound with echoes from objects surrounding the sound source. Convolution is a technique for mapping the impulse response of a physical room to a digital audio stream in digital signal processing. The imposition of a spectral or rhythmic structure on a sound is known as convolution in electronic music. This envelope or structure is frequently derived from a different sound. Filtering one signal via the other is called convolution of two signals.

(iv) In electrical engineering: the output of a linear time-invariant (LTI) system is obtained by the convolution of one function (the input signal) with a second function (the impulse response). At any one time, the output is the sum of all previous input function values, with the most recent values often having the most influence (expressed as a multiplicative factor). This component is provided by the impulse response function as a function of the time since each input value happened.

(v) In physics: a convolution operation can be found whenever there is a linear system with a "superposition principle." For example, in spectroscopy, line widening owing to the Doppler effect produces a Gaussian spectral line form on its own, whereas collision broadening produces a Lorentzian line shape. The Line form is a convolution of Gaussian and Lorentzian, which is a Voigt function, when both effects are active. The measured fluorescence in time-resolved fluorescence spectroscopy is a sum of exponential decays from each delta pulse, and the excitation signal may be considered as a chain of delta pulses.

(vi) In computational fluid dynamics: the convolution process is used in the large eddy simulation (LES) turbulence model to reduce the range of length scales required in computing, lowering the computational cost.

(vii) In probability theory: the convolution of the distributions of two independent random variables is the probability distribution of the sum of their distributions.

(viii) In kernel density estimation: a distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian.

(ix) In radiotherapy: in the handling of planning systems, the convolution-superposition algorithm is used in the majority of recent computation codes.

The above applications of a convolution show that the fractional derivative as convolution has multiple purposes. It can represent memory, like in the instance of elasticity theory. It may be understood as a filter, with the Caputo and Caputo–Fabrizio types in particular being viewed as a filter of the local derivative with power and exponent functions (one can see in [48]).

The purpose of this section is to provide an application of Theorem 6 to find a common solution of a nonlinear fractional differential equation, where we can apply F-Kannan–Suzuki type mappings in complete TVS-valued cone metric spaces.

Here, we investigates the Riemann–Liouville derivative fractional integral of order $\alpha > 0$. This form of fractional derivative for a continuous function g: $[0,\infty) \longrightarrow \mathbb{R}$ denoted by $D_a^{\alpha} f$, is given by

$$
\begin{aligned}
(D_{0+}^{\alpha})g(t) &= \left(\frac{d}{dt}\right)^{n-1} (I_{0+}^{\alpha})g(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1} g(s)\mathrm{d}s,
\end{aligned}
\tag{76}
$$

where $[\alpha]$ denotes the integer part of the real number α and $n = [\alpha]+1$, provided that the right hand side is pointwise defined on $(0,\infty)$. (see [49–54]). Also, the Riemann–Liouville fractional integral of order α is given by

$$
(I_0^{\alpha})g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g(s)\mathrm{d}s, \tag{77}
$$

for $\alpha > 0$. The notation $[\alpha]$ stands for largest integer not greater than α. If $\alpha = m \in \mathbb{N}$, then $(D_{0+}^{m})g(t) = g^{m}(t)$, for $t > 0$ and if $\alpha = 0$, then $(D_{0+}^{0})g(t) = g(t)$ for $t > 0$.

The following nonlinear fractional differential equation with integral boundary valued conditions is inspired by Kilbas et al. [55], Cabada and Hamdi [56], and Cabada and Wang [50].

$$
\begin{cases}
D_{0+}^{\alpha}\sigma(t) + g(t,\sigma(t)) = 0, & 0 < t < 1, \\
\sigma(0) = \sigma'(0), & \\
\sigma'(1) = \lambda \int_0^1 \sigma(s)\mathrm{d}s, & 0 < \lambda < 1,
\end{cases}
\tag{78}
$$

where D_{0+}^{α} denotes the Riemann–Liouville fractional derivative of order α and g: $[0,1] \longrightarrow \mathcal{X}$ is a continuous function.

We recall the following lemmas from Bai, and Lü [57].

Lemma 4. *Let $\alpha > 0$. If we assume $\sigma \in C(0,1) \cap L(0,1)$, then the fractional differential equation:*

$$
D_{0+}^{\alpha}\sigma(t) = 0, \tag{79}
$$

has

$$
\sigma(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{80}
$$

$C_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, as unique solution.

Since $D_{0+}^{\alpha} I_{0+}^{\alpha}\sigma(t) = \sigma$ for all $\sigma \in C(0,1) \cap L(0,1)$. From Lemma 4, we deduce the following lemma.

Lemma 5. *Assume that $\sigma \in C(0,1) \cap L(0,1)$, with fractional derivative of order $\alpha > 0$ that belongs to $\sigma \in C(0,1) \cap L(0,1)$. Then,*

$$
I_{0+}^{\alpha} D_{0+}^{\alpha}\sigma(t) = \sigma(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{81}
$$

for some $C_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, as unique solution.

The unique solution of (78) is given by

$$
\sigma(t) = \int_a^t G(t,s) g(s,u(s))\mathrm{d}s. \tag{82}
$$

Recall that the Green function related to the problem (78) is given by

$$
G_f(t,s) = \begin{cases}
\dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\[2ex]
\dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \le t \le s \le 1.
\end{cases}
\tag{83}
$$

Consider the space $\mathcal{X} = (C[0,1],\mathbb{R}^n)$, $\mathcal{E} = C[0,1]$ be endowed with the ordering $\sigma \le \varsigma$ if $\sigma(t) \le \varsigma(t)$ for all $t \in C[0,1]$ and define $\mathcal{P} \in \mathcal{E}$ by $\mathcal{P} = \{(\sigma,\varsigma) \in \mathcal{E}$: $\sigma(t), \varsigma(t) \ge 0\} \subset \mathbb{R}^2$, $\mathcal{X} = \mathbb{R}$.

This space defines the metric ϱ: $\mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{E}$ such that

$$
\varrho(\sigma,\varsigma) = \sup_{t\in[0,1]} \{|\sigma(t) - \varsigma(t)|\}\psi(t), \tag{84}
$$

$\forall \sigma,\varsigma \in \mathcal{X}$ and $\psi(t) = e^t$. Then, $(\mathcal{X},\varrho)$ is a TVS-valued cone metric space. A function $\sigma \in C([0,1],\mathcal{X})$ is a unique solution of the fractional differential integral equation (82) if and only if $\sigma = u^*$ is a solution of the nonlinear fractional differential equation (78).

Now, we prove the following theorem.

Theorem 8. *Suppose the following condition hold:*

(i) $G_f(t,s) \in C([0,1]\times[0,1], X) \geq 0$ *for all* $t, s \in [0,1]$

(ii) $\int_0^1 G_f(t,s) \leq \gamma(s)$ *for all* $t, s \in [0,1]$

(iii) $g \in C([0,1]\times\mathcal{X}, \mathcal{X})$ *is sequentially continuous*

(iv) *there exists a continuous function* $g\colon [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}_+$, *such that*

$$|g(t,\sigma(s)) - g(t,\varsigma(s))| \leq e^{-\vartheta}\gamma(s)|\sigma(s)-\varsigma(s)|, \tag{85}$$

for all $t \in [0,1]$ *and* $\vartheta > 0$, *such that*

$$\gamma(s) = \frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^{\alpha} + \lambda t^{\alpha}]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}. \tag{86}$$

Then, the fractional differential Equation 4.1 *has a common solution as a fixed point* $\sigma^* \in C([0,1],\mathcal{X})$.

Proof. Let us define a map $G, f\colon \mathcal{P} \longrightarrow \mathcal{E}$ by

$$Gf\sigma(t) = \int_0^1 G_f(t,s)g(s,\sigma(s))\mathrm{d}s, \tag{87}$$

for $t \in [0,1]$, then $Gf^n\sigma_0$ is sequentially continuous. This implies that $f \in Gf^n\sigma_0$ and $f^n\sigma_0$ possess a fixed point $u^* \in Gf$. To prove the existence of fixed point of Gf, we prove that Gf is sequentially and contraction. To show Gf is sequentially continuous, let $Gf\sigma \neq Gf\varsigma$, for all $\sigma, \varsigma \in [0,1]$. By condition (iv), we have

$$\begin{aligned}
|Gf\sigma - Gf\varsigma| &= \left|\int_0^1 G_f(t,s)g(s,\sigma(s))\mathrm{d}s - \int_0^1 G_f(t,s)g(s,\varsigma(s))\mathrm{d}s\right|,\\
&\leq \int_0^1 G_f(t,s)|g(s,\sigma(s)) - g(s,\varsigma(s))|\mathrm{d}s,\\
&\leq \left[\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\mathrm{d}s + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}\mathrm{d}s\right]e^{-\vartheta}|\sigma(s)-\varsigma(s)|e^{t},\\
&\leq \left[\frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^{\alpha} + \lambda t^{\alpha}]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}\right]e^{-\vartheta}|\sigma(s)-\varsigma(s)|.
\end{aligned} \tag{88}$$

This implies that

$$|Gf\sigma, Gf\varsigma| \leq e^{-\vartheta}\gamma(s)|\sigma-\varsigma|e^{t}. \tag{89}$$

Since $\gamma(s) < 1$, we have

$$|Gf\sigma, Gf\varsigma| \leq e^{-\vartheta}|\sigma-\varsigma|e^{t}. \tag{90}$$

Thus, for each $\sigma, \varsigma \in \mathcal{X}$, we have

$$\varrho(Gf\sigma, Gf\varsigma) \leq e^{-\vartheta}\mathbb{M}(\sigma,\varsigma). \tag{91}$$

Taking logarithms on both sides of (91) using $F_1(z) = \ln(z)$ and the property of F, we get

$$\ln(\varrho(Gf\sigma, Gf\varsigma)) \leq \ln\left(e^{-\vartheta}\mathbb{M}(\sigma,\varsigma)\right) \tag{92}$$

equivalent to

$$\vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F(\mathbb{M}(\sigma,\varsigma)), \tag{93}$$

where

$$\mathbb{M}(\sigma,\varsigma) = \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}. \tag{94}$$

Using (94) in (93) and applying F-Kannan–Suzuki type conditions leads to

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma,\varsigma)\\
&\Rightarrow \vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right].
\end{aligned} \tag{95}$$

For $\gamma \in [0,1)$, $\vartheta > 0$ satisfies F-Kannan–Suzuki type mapping. Therefore, Gf is a contraction mapping on X. Since all the conditions of Theorem 8 are satisfied. Therefore, there exists $u^* \in C([0,1])$ a common fixed point of G and f, that is, u^* is a solution to fractional nonlinear differential equation (78). □

4.2. The Existence of Coincidence Solution for Nonlinear Volterra-Integral Equations. This section investigates the coincidence solution for nonlinear Volterra-integral equations in the setting of TVS-valued cone metric spaces. Nieto [58] initiated the study of the existing solution of an ordinary differential equation. Since then, several authors utilized his ideas to find the solution of ordinary differential equations. In detail, one can see the literature in [55, 59–62] and the references therein.

Integral equation methods help solve many problems of the applied fields like mathematical economics and optimal control theory because this problem is often reduced to integral equations.

Integral equations appear in several forms. However, in this section, we are interested with the integral equation, namely, Volterra integral-differential equation which is of the form

$$u^n(t,\sigma) = f(t,\sigma) + \int_a^\sigma K(\sigma,t,u(t))\mathrm{d}t, \quad \text{where } u^n = \frac{d^n u}{d\sigma^n}. \tag{96}$$

The following integral equation inspired by [12, 63–66].

$$\begin{aligned} u(\sigma,\varsigma) &= l(\sigma,\varsigma) + \int_0^\sigma g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{97}$$

where l, g, h are given functions and u is unknown function to be found.

Let $C(G,f)$ be the class of continuous functions from the set G to the set f. We denote $\mathscr{E} = \mathbb{R}^+\times\mathbb{R}^+$, $\mathscr{E}_1 = \{l(\sigma,\varsigma,s)\colon 0\le s\le\sigma\le\infty, \varsigma\in\mathbb{R}^+\}$ and $\mathscr{E}_2 = \{l(\sigma,\varsigma,s,t)\colon 0\le s\le\sigma\le\infty, 0\le t\le\varsigma\le\infty\}$. We denote that $l\in C(\mathscr{E},\mathbb{R})$, $g\in C(\mathscr{E}_1\times\mathbb{R},\mathbb{R})$ and $h\in C(\mathscr{E}_2\times\mathbb{R},\mathbb{R})$

Denote $\mathscr{X}$ be the space of functions $z\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ satisfying

$$|z(\sigma,t)| = O\left(e^{\lambda(\sigma+\varsigma)}\right), \tag{98}$$

where λ is a positive constant, that is,

$$|z(\sigma,\varsigma)| \le M_0\left(e^{\lambda(\sigma+\varsigma)}\right), \tag{99}$$

for constant $M_0>0$. Let $(\mathscr{X},\|.\|)$ be a Banach space. Define a norm in the space X by

$$|z|_{\mathscr{X}} = \sup_{(\sigma,\varsigma)\in\mathscr{E}}\left[|z(\sigma,\varsigma)|e^{(-\lambda(\sigma+\varsigma))}\right]. \tag{100}$$

Define the mapping $G,f\colon \mathscr{X}\times\mathscr{X}\longrightarrow[0,\infty)$ by

$$\begin{aligned} Gf^n u(\sigma,\varsigma) &= l(\sigma,\varsigma) + \int_0^\sigma g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{101}$$

and

$$\begin{aligned} Gf^n v(\sigma,\varsigma) &= l(\sigma,\varsigma) + \int_0^\sigma g(\sigma,\varsigma,\varepsilon,v(\varepsilon,y))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h\sigma,\varsigma,\nu,\tau,v(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{102}$$

for $u,v\in\mathscr{X}$. The coincidence fixed point of $Gf^n u$ and $Gf^n v$ is also a solution of the Volterra integral-differential equation (97).

Now we prove the results by establishing the existence solution of a coincidence fixed point for a pair of self mappings:

Theorem 9. *Suppose the following conditions holds:*

(i) *For the continuous functions $g,h\in\mathscr{X}$, we have*

$$\begin{aligned} |g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma)) - g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))| &\le \gamma_1(\sigma,\varsigma,\varepsilon)|u-v|, \\ |h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau)) - h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))| &\le \gamma_2(\sigma,\varsigma,\nu,\tau)|u-v|, \end{aligned} \tag{103}$$

where $\gamma_1\in C(\mathscr{E}_1,[0,\infty))$ and $\gamma_2\in C(\mathscr{E}_2,[0,\infty))$.

(ii) *There exists a nonnegative constant δ such that $\delta<1$ and*

$$\int_0^\sigma \gamma_1(\sigma,\varsigma,\varepsilon)e^{\lambda(\varepsilon+\varsigma)}\mathrm{d}\varepsilon + \int_0^\sigma\int_0^\varsigma \gamma_2(\sigma,\varsigma,\nu,\tau)e^{\lambda(\nu+\tau)}\mathrm{d}\tau\mathrm{d}\nu \le \delta e^{\lambda(\sigma+\varsigma)-\vartheta}, \tag{104}$$

for all $\sigma,\varsigma,\varepsilon,\nu,\tau\in\mathscr{E}_1\cup\mathscr{E}_2$.

Then, the nonlinear Volterra-integral equation (97) has a unique solution in $\mathscr{E}_1\cup\mathscr{E}_2$ which is the coincidence fixed point of equations (101) and (102).

Proof. Let $G,f\colon\mathscr{X}\longrightarrow\mathscr{X}$ be two operators such that $Gf^\nu\in\mathscr{X}$ and $Gf^n v\in\mathscr{X}$. Now we verify that the two operators are contractive maps in $\mathscr{X}$. Let $u,v\in\mathscr{X}$. On contrary we claim that G and f are not contractive maps in $\mathscr{X}$. From equations (101) and (102), using condition (i) and (ii) of Theorem 9, we have

$$\begin{aligned}
\left|Gf^{n}u - Gf^{n}v\right|_{\mathcal{X}} &= l(\sigma,\varsigma) + \int_0^{\sigma} g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon + \int_0^{\sigma}\int_0^{\varsigma} h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\tau\mathrm{d}\nu \\
&\quad - l(\sigma,\varsigma) - \int_0^{\sigma} g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))\mathrm{d}\varepsilon - \int_0^{\sigma}\int_0^{\varsigma} h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))\mathrm{d}\tau\mathrm{d}\nu, \\
&\leq \int_0^{\sigma} |g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma)) - g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))|\mathrm{d}\varepsilon \\
&\quad + \int_0^{\sigma}\int_0^{\varsigma} |h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau)) - h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))|\mathrm{d}\tau\mathrm{d}\nu, \\
&\leq \left[\int_0^{\sigma} \gamma_1(\sigma,\varsigma,\varepsilon)e^{\lambda(\sigma+\varsigma)}\mathrm{d}\varepsilon + \int_0^{\sigma}\int_0^{\varsigma} \gamma_2(\sigma,\varsigma,\nu,\tau)e^{\lambda(\nu+\tau)}\mathrm{d}\tau\mathrm{d}\nu\right]|u-v|_{\mathcal{X}}, \\
&\leq \delta e^{\lambda(\sigma+\varsigma)-\vartheta}|u-v|_{\mathcal{X}}, \\
&\leq \delta e^{\lambda(\sigma+\varsigma)-\vartheta}|u-v|_{\mathcal{X}}, \\
\left|Gf^{n}u - Gf^{n}v\right|_{\mathcal{X}} &\leq \delta e^{-\vartheta}|u-v|_{\mathcal{X}}e^{\lambda(\sigma+\varsigma)}, \\
\varrho(Gfu,Gfv) &\leq \delta e^{-\vartheta}M(u,v),
\end{aligned} \tag{105}$$

which is a contradiction. Hence u is a common fixed of G and f, also a solution to integral (97).

From (105), since $\delta < 1$ and using FKS2 of Definition 13, where

$$M(u,v) = \frac{\varrho(Gu,Gfu) + \varrho(Gv,Gfv)}{2}, \tag{106}$$

we have

$$\varrho(Gfu,Gfv) \leq e^{-\vartheta}M(u,v). \tag{107}$$

Using $F_1(z) = \ln z$ by taking natural logarithms in both sides of (107), we get

$$\vartheta + \varrho(Gfu,Gfv) \leq M(u,v). \tag{108}$$

By (106), we obtain a F-Kannan–Suzuki contraction as defined in Definition 13. Thus, all conditions imposed in Theorem 6 and Theorem 9 are satisfied. Hence, u^* is a common fixed point of G and f in X. □

5. Conclusion

The novelty of this study to fixed point theory is the fixed point result given in Theorem 6. This theorem provides the common fixed points conditions for a pair of two self mappings in TVS-valued cone metric spaces. This paper extended and generalised the results due to Batra et al. [37], Filipovic et al. [38], Morales and Rojas [9], Rahimi et al. [39], and Wangwe and Kumar [40] using a pair of two self-mappings in F-Kannan–Suzuki type mapping in TVS-valued cone metric space, where we consider a map to be sequentially convergent, one to one and continuous. By doing so, we extended several other results of the same setting in the literature. These results have some applications in many areas of applied mathematics, especially in nonlinear Riemann–Liouville fractional differential equation and nonlinear Volterra-integral differential equation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. In addition, all authors read and approved the final manuscript.

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