


Research Article

On Randic, Seidel, and Laplacian Energy of NEPS Graph

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Received 9 January 2022; Accepted 8 February 2022; Published 28 February 2022

Academic Editor: Gul Rahmat

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Let Z be the simple graph; then, we can obtain the energy $E(Z)$ of a graph Z by taking the absolute sum of the eigenvalues of the adjacency matrix of Z . In this research, we have computed different energy invariants of the noncompleted extended P-Sum (NEPS) of graph Z_i . In particular, we investigate the Randic, Seidel, and Laplacian energies of the NEPS of path graph P_n , with any base \mathcal{B} . Here, n denotes the number of vertices and i denotes the number of copies of path graph P_n . Some of the results depend on the number of zeroes in base elements, for which we use the notation j .

1. Introduction

The NEPS of the graphs is a graph Z whose vertex set is equal to the simple Cartesian product of the vertices' sets of the graphs [1]. If $Z_1, Z_2, Z_3, \dots, Z_K$ are K graphs having $v(Z_1), v(Z_2), v(Z_3), \dots, v(Z_K)$ vertices sets, respectively, then the vertex set of NEPS of graphs is defined as

$$V(Z) = v(Z_1) \times v(Z_2) \times v(Z_3) \times \dots \times v(Z_K). \quad (1)$$

The existence of the edges of NEPS graph is depending on the base elements:

$$(\beta_1, \beta_2, \beta_3, \dots, \beta_K) \in \overline{\mathcal{B}}, \quad (2)$$

whereas some $\beta_i = 0$ and some $\beta_i = 1$. When $\beta_i = 0$, it means there is no edge because in this case $u_i = v_i$. However, when $\beta_i = 1$, then there exists an edge between (u_1, u_2, \dots, u_K) and (v_1, v_2, \dots, v_K) iff Z_i has an edge between u_i and v_i .

The NEPS of the graph is abbreviated as noncomplete extended p-sum of the graph. It has many graph operations as special cases, and names are as follows: the product, the sum, and the strong product of graphs [1]. The energy of graph is the summation of $|\lambda_i|$ of adjacency matrix $A(G)$ of graph G [2, 3]. Energy of graphs, first introduced by Ivan

Gutman, has remarkable chemical application; see [4–6], for details.

This energy have applications in image analysis [7, 8], which is been used in investigation of medical fields like in brain activity. Randic energy of graph G is the summation of $|\rho_i|$, where ρ_i are the eigenvalues of the Randic matrix $R(G)$ [9, 10]. It has endless applications in chemistry and became an attractive field of research; for further details, see [11–13]. In [7], Haemers defined the Seidel energy; let the Seidel matrix of a graph G be represented as $SE(G)$, and θ_i are the eigenvalues of this matrix; then, the energy of graph is the summation of $|\theta_i|$, where $\theta - i$ is of the Seidel matrix. More results on this energy can be found in [14–16]. These energies have lot of work in calculating upper and lower bounds of different graphs. In Section 2, we first show NEPS of the cyclic graph and how the energy of NEPS with any other basis do not have any effect on energies, see [17–19]. Then, how this result is used in calculating Randic, Seidel, and Laplacian energy of this graph?

Example 1. Consider two graphs P_2 and C_3 ; then, the vertex set of NEPS of P_2 and C_3 is stated as

$$V(Z) = V(P_2) \times V(C_3). \quad (3)$$

Let $V(P_2) = (1, 2)$ and $V(C_3) = (1', 2', 3')$; then,

$$V(Z) = \{(1, 1'), (1, 2'), (1, 3'), (2, 1'), (2, 2'), (2, 3')\}. \quad (4)$$

The NEPS graph is dependent on the base elements, and the possible base elements are $\{(1, 1), (1, 0), (0, 1)\}$. If we take the base element $(1, 1)$, then the graphical view of $\text{NEPS}\{P_2 \times C_3; (1, 1)\}$ is expressed as in Figure 1.

If we take base element $(0, 1)$, then its graph is shown in Figure 2.

Now, for $(1, 0)$, we have Figure 3.

If we take three copies of P_2 and $(1, 1, 1)$ as the base element, then its NEPS graph is expressed as in Figure 4.

Now, for three copies of P_2 , with base $(1, 0, 1)$, the NEPS graph is shown in Figure 5.

The NEPS graph was first time introduced by D. Cvetkovic and R. Lucic [3], and after some time, it was redefined by S.C. Shee [20]. It has defined various graph operations; some of the results we explain as follows. Let Z_1 and Z_2 be any two graphs; if we take bases $\{(1, 0), (0, 1)\}$, then the resulting graph Z is the sum of Z_1 and Z_2 ; if the base element is $(1, 1)$, then the graph Z is the tensor product of Z_1 and Z_2 . Furthermore, the connectedness of the NEPS graph in [2] and NEPS operation on cordial graphs in [8] have been discussed.

The energy of the graph Z is defined as the absolute sum of the eigenvalues of the adjacency matrix of Z . It was first time introduced by Ivan Gutman [5]. The adjacency matrix of the simple graph Z is a matrix whose entries are zeroes and ones. If the vertices $v_i, v_j \in E$, then $A_{ij} = 1$; otherwise, $A_{ij} = 0$. It is denoted by $A(Z)$.

A Randic matrix $R(Z)$ is a matrix whose entries are denoted by r_{ij} and is defined as $R = [r_{ij}]$. If u_i and u_j are not adjacent or $u_i = u_j$, then $r_{ij} = 0$, but if u_i and u_j are adjacent, then $r_{ij} = 1/\sqrt{d_i d_j}$, where d_i and d_j are degrees of u_i and u_j , respectively. The Randic energy of a graph Z can be evaluated by taking the absolute sum of the eigenvalues of the Randic matrix. If the eigenvalues of the Randic matrix are denoted by μ_i , then we can express the Randic energy of Z as [21]

$$RE(Z) = \sum_{i=1}^n |\mu_i|. \quad (5)$$

Milan Randic is the first person who has introduced the Randic index [22] such as

$$R(Z) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}}. \quad (6)$$

Gutman et al. have explained that the Randic index is used to form the Randic matrix [5]. They also introduced the energy of the graphs [9, 10]; then, they extended this topic and defined the Randic energy [15]. Dasa et al. have discussed the upper and lower bounds on the Randic energy of the graphs [4].

Let Z be a graph having vertices $\{v_1, v_2, v_3, \dots, v_n\}$; then, the Seidel matrix of Z is a $n \times n$ matrix including the entries $-1, 0$, and 1 . The Seidel matrix is expressed as $S(Z) = [s_{ij}]$. If

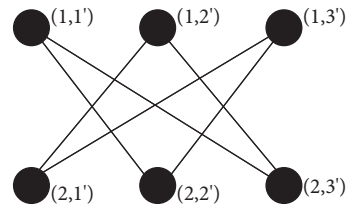


FIGURE 1: $\text{NEPS}\{P_2 \times C_3; (1, 1)\}$.

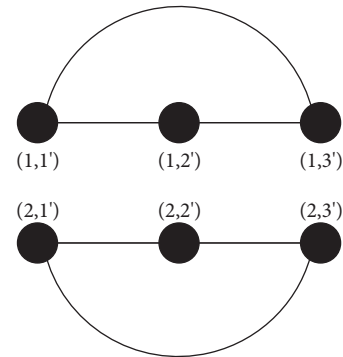


FIGURE 2: $\text{NEPS}\{P_2 \times C_3; (0, 1)\}$.

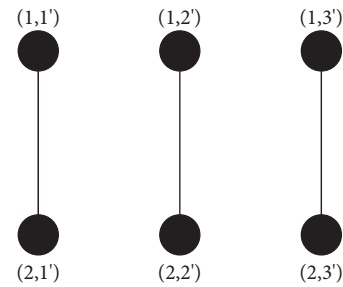


FIGURE 3: $\text{NEPS}\{P_2 \times C_3; (1, 0)\}$.

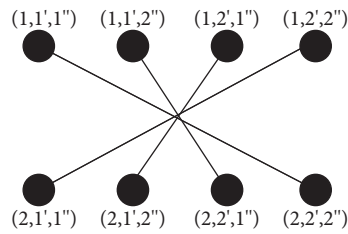


FIGURE 4: $\text{NEPS}\{P_2 \times P_2 \times P_2; (1, 1, 1)\}$.

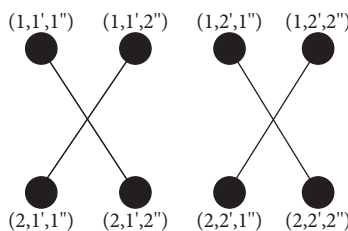


FIGURE 5: $\text{NEPS}\{P_2 \times P_2 \times P_2; (1, 0, 1)\}$.

$v_i = v_j$, then $s_{ij} = 0$; if $v_i \neq v_j$ and there exist an edge between v_i and v_j in Z , then $s_{ij} = -1$, but if $v_i \neq v_j$ and there does not exist an edge between v_i and v_j in Z , then $s_{ij} = 1$. Let the eigenvalues of the Seidel matrix be denoted by θ_i , then the Seidel energy is expressed as the absolute sum of eigenvalues and written as

$$SE(Z) = \sum_{i=1}^n |\theta_i|. \tag{7}$$

Liu Jian-ping and Liu Bo-lian have explained [14] the seidel energy. Seidel energy and its bounds have been calculated by a sharp method which is made by P. Nageswari and P. B. Sarasija [11].

Now, we discuss on Laplacian energy As $D(Z)$ is a diagonal matrix whose entries are the degree of the vertices of Z and $A(Z)$ is the adjacency matrix, then the Laplacian matrix is stated as $L(Z) = D(Z) - A(Z)$. If the eigenvalues of the Laplacian matrix is denoted by μ_i , then the Laplacian energy of the graph Z is expressed as

$$LE(Z) = \sum_{i=1}^n |\mu_i - d(Z)|, \tag{8}$$

where $d(Z) = 2|E(Z)|/|V(Z)|$. The Laplacian matrix has been introduced by Grone and Merris [23] such as $L(Z) = D(Z) - A(Z)$. Gutman and Zhou, in 2006 [6], defined a result that energy of graph cannot be exceeded from the Laplacian energy of that graph; also, they explained some properties of Laplacian energy in [24]. Zhou has been working on energy and Laplacian energy [25] and gave some useful results. Dragan Stevanovica, Ivan Stankovicb, and Marko Milosevicb have explained some positive and negative results between the relation of energy and Laplacian energy.

We discuss some important results that are made on the path graphs by defining its spectrum, whereas the spectrum of Z is the nonincreasing sequence of the distinct eigenvalues $\mu_1, \mu_2, \mu_3, \dots, \mu_n$ of the adjacency matrix $A(Z)$ of Z . In particular, we explained some results regarding Randic, seidel, and Laplacian energies of NEPS of path graphs. Some results of the Randic, seidel, and Laplacian energies depend on j and i , for $i > j$. Whereas, j and i denote the number of zeroes in base elements $(\beta_1, \beta_2, \beta_3, \dots, \beta_K) \in \mathcal{B}$ and number of copies of the path graph, respectively. Also, the dimensions of the base elements are based on i .

Such as, if we take $j = 1$ and $i = 3$, then, in this case, possible base elements are

$\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. One more important point is that we will discuss on all base elements but will not include the ones such as $(1, 1, \dots, 1)$.

2. Main Results

Our main focus is to compute Randic, seidel, and Laplacian energy. The broad generation of energies that is consisted on different graph matrices was the first to categorize the Laplacian energy. This is defined in the form of spectrum. The spectrum of the Laplacian matrix consists of the eigenvalues.

Theorem 1. Let n and m be the number of vertices and edges of $NEPS\{Z_i = P_{2_i}; \mathcal{B}\}$, respectively, for $i = 2, 3, \dots, K$, where i denotes the number of copies of P_2 . Then,

$$RE[NEPS\{Z_i = P_{2_i}; \mathcal{B}\}] = 2m. \tag{9}$$

Proof. Let $Z = NEPS\{Z_i = P_{2_i}; \mathcal{B}\}$, for $i = 2, 3, \dots, K$, be a graph. Here, i denotes the number of copies of P_2 (which is a path graph heaving 2 vertices and 1 edge) and \mathcal{B} denotes the base element which depend on the number of copies of P_2 .

In that case, the spectrum of the Randic matrix of Z is defined as

$$\left\{ \frac{2m}{n} \left(\frac{n}{2} \text{time} \right), \frac{-2m}{n} \left(\frac{n}{2} \text{time} \right) \right\}, \tag{10}$$

$$RE(Z) = \frac{n}{2} \times \left| \frac{2m}{n} \right| + \frac{n}{2} \times \left| \frac{-2m}{n} \right| = m + m = 2m.$$

Thus, we conclude the proof:

$$RE[NEPS\{Z_i = P_{2_i}; \mathcal{B}\}] = 2m. \tag{11} \quad \square$$

Theorem 2. Let n and m be the number of vertices and edges of $NEPS\{Z_i = P_{3_i}; \mathcal{B}\}$, respectively, where number of copies of P_3 and number of zeroes in the base element are designated by i and j , respectively. Then, for any j and $i > j$, we have

$$RE[NEPS\{Z_i = P_{3_i}; \mathcal{B}\}] = 3^j 2^{i-j}. \tag{12}$$

Proof. Let $Z = NEPS\{Z_i = P_{3_i}; \mathcal{B}\}$, for $i = 2, 3, \dots, K$, be a graph. For any j and $i > j$, we have the spectrum of the Randic matrix of Z :

$$\{1(3^j 2^{i-j-1} \text{time}), 0(n - 3^j 2^{i-j} \text{time}), -1(3^j 2^{i-j-1} \text{time})\}. \tag{13}$$

Thus, we follow the result:

$$RE(Z) = (3^j 2^{i-j-1})|1| + (n - 3^j 2^{i-j})|0| + (3^j 2^{i-j-1})|-1| \\ = 3^j 2^{i-j-1} + 3^j 2^{i-j-1},$$

$$RE(Z) = RE[NEPS\{Z_i = P_{3_i}; \mathcal{B}\}] = 3^j 2^{i-j}. \tag{14} \quad \square$$

Theorem 3. Let $n = 4^i$ and m be the number of vertices and edges of $NEPS\{Z_i = P_{4_i}; \mathcal{B}\}$, respectively. Then, for any j and $i > j$, we have

$$RE[NEPS\{Z_i = P_{4_i}; \mathcal{B}\}] = \sum_{t=0}^{i-j} {}^{i-j}C_t \times 2^{i+j-t}. \tag{15}$$

Proof. Let $Z = NEPS\{Z_i = P_{4_i}; \mathcal{B}\}$ be a graph. Then, for any $j \geq 1$ and $i > j$, we have the spectrum of the Randic matrix as given below:

$$\left\{ \frac{-1}{2^t} \left(i-j C_t \times \frac{n}{2^{i-j+1}} \text{time} \right), \frac{1}{2^t} \left(i-j C_t \times \frac{n}{2^{i-j+1}} \text{time} \right) \right\}, \quad (16)$$

where $0 \leq t \leq i - j$,

$$\begin{aligned} RE(Z) &= \left(\sum_{t=0}^{i-j} i-j C_t \times \frac{n}{2^{i-j+1}} \right) \times \left| \frac{-1}{2^t} \right| + \left(\sum_{t=0}^{i-j} i-j C_t \times \frac{n}{2^{i-j+1}} \right) \\ &\quad \times \left| \frac{1}{2^t} \right| \\ &= 2 \times \left(\sum_{t=0}^{i-j} i-j C_t \times \frac{n}{2^{i-j+1}} \right) \times \left(\frac{1}{2^t} \right). \end{aligned} \quad (17)$$

For $n = 4^i$, we have

$$RE(Z) = RE[\text{NEPS}\{Z_i = P_{4_i}; \mathcal{B}\}] = \sum_{t=0}^{i-j} i-j C_t \times 2^{i+j-t}. \quad (18)$$

Hence, it is done. □

Theorem 4. For n number of vertices and $m = n/2$ number of edges, we have

$$SE[\text{NEPS}\{G_i = P_{2_i}; \mathcal{B}\}] = a_{\log_2^n} + (2n - 3), \quad (19)$$

where $a_{\log_2^n} = a_{\log_2^{n-2}} + 2^{\log_2^{n-1}}$, for $a_1 = 1$.

Proof. Let $Z = \text{NEPS}\{Z_i = P_{2_i}; \mathcal{B}\}$, for $i = 2, 3, \dots, K$, be a graph. Then, for n vertices of Z , we have $m = n/2$ edges in Z ; we have the spectrum of the seidel matrix as

$$\left\{ a_{\log_2^n}, \frac{2m}{n} \left(\frac{n}{2} \text{time} \right), \frac{-6m}{n} \left(\frac{n}{2} - 1 \text{time} \right) \right\}, \quad (20)$$

where $a_{\log_2^n} = a_{\log_2^{n-2}} + 2^{\log_2^{n-1}}$, for $a_1 = 1$.

Thus, we have

$$\begin{aligned} SE(Z) &= a_{\log_2^n} + \frac{n}{2} \left| \frac{2m}{n} \right| + \left(\frac{n}{2} - 1 \right) \left| \frac{-6m}{n} \right| \\ &= a_{\log_2^n} + 4m - \frac{6m}{n} \\ &= a_{\log_2^n} + 2m \left(2 - \frac{3}{n} \right), \end{aligned} \quad (21)$$

for $m = n/2$; we have

$$= a_{\log_2^n} + (2n - 3). \quad (22) \quad \square$$

Theorem 5. Let $n = 3^i$ be the number of vertices; we have $m = 8n/9$ as edges of $\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}$, where i is the number of copies of P_3 . Furthermore, suppose that j is the number of zeroes in the base element; then, for any $j \geq 1$ and $i = j + 2$, we have

$$SE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = 8a_{j+1} + \frac{14n - 18}{9}, \quad (23)$$

where $a_1 = 1$ and $a_{j+1} = 3a_j + 2$.

Proof. Let $Z = \text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}$ be the graph. Then, for $j \geq 1$ and $i = j + 2$, we have the spectrum of the seidel matrix of Z as given below:

$$\left\{ \frac{-45m}{8n} (a_{j+1} = 3a_j + 2 \text{time}), \frac{-9m}{8n} \left(\frac{15n}{27} \text{time} \right), \frac{27m}{8n} (a_{j+1} = 3a_j + 2 \text{time}), n - 2 \text{ (sum of two eigenvalues)} \right\}, \quad (24)$$

where $a_1 = 1$ and $a_{j+1} = 3a_j + 2$, for any $j \geq 1$.

By using this spectrum, we have

$$\begin{aligned} SE(Z) &= (a_{j+1}) \left| \frac{-45m}{8n} \right| + \left(\frac{15n}{27} \right) \left| \frac{-9m}{8n} \right| \\ &\quad + (a_{j+1}) \left| \frac{27m}{8n} \right| + (n - 2) \\ &= (a_{j+1}) \left(\frac{45m}{8n} \right) + \left(\frac{15n}{27} \right) \left(\frac{-9m}{8n} \right) \\ &\quad + (a_{j+1}) \left(\frac{27m}{8n} \right) + (n - 2) \\ &= \left(\frac{9m}{n} \right) a_{j+1} + \frac{5m}{8} + n - 2. \end{aligned} \quad (25)$$

For $m = 8n/9$, we have

$$SE(Z) = SE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = 8(a_{j+1}) + \frac{14n - 18}{9}. \quad (26)$$

□

Theorem 6. Let n and $m = n/2$ be the number of vertices and edges of the graph $Z = \text{NEPS}\{Z_i = P_{2_i}; \mathcal{B}\}$, respectively, for $i = 2, 3, \dots, K$. Then,

$$LE[\text{NEPS}\{Z_i = P_{2_i}; \mathcal{B}\}] = n. \quad (27)$$

Proof. We have n and $m = n/2$ be the number of vertices and edges of the graph $Z = \text{NEPS}\{Z_i = P_{2_i}; \mathcal{B}\}$, respectively, for $i = 2, 3, \dots, K$. Then, the average degree of Z is $d(Z) = 2m/n$.

Now, the spectrum of the Laplacian matrix can be expressed as

$$\left\{ 2 - \frac{2m}{n} \left(\frac{n}{2} \text{time} \right), 0 - \frac{2m}{n} \left(\frac{n}{2} \text{time} \right) \right\}. \quad (28)$$

By using this spectrum, we get the required result:

$$LE(Z) = \frac{n}{2} \times \left| 2 - \frac{2m}{n} \right| + \frac{n}{2} \times \left| 0 - \frac{2m}{n} \right| = n - m + m = n. \quad (29) \quad \square$$

Theorem 7. Let n and m be the number of vertices and edges of $\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}$, respectively, where number of copies

of P_3 and number of zeroes in the base element are designated by i and j , respectively. Then, for $j \geq 1$, we have the following.

Case I: for $i = j + 1$, we have

$$LE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = \frac{10n}{9}. \quad (30)$$

Case II: for $i = j + 2$, we have

$$LE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = \frac{106n}{81}. \quad (31)$$

Case III: for $i = j + 3$, we have

$$LE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = \frac{1210n}{729}. \quad (32)$$

Proof. Let n and m be the number of vertices and edges of $Z = \text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}$, respectively, whose average degree is $2m/n$, where the number of copies of P_3 and number of zeroes in the base element are designated by i and j , respectively.

Case I: for any j and $i = j + 1$, then the spectrum of the Laplacian matrix is

$$\begin{aligned} & \left\{ 3 - \frac{2m}{n} \left(\frac{n}{3} \text{ time} \right), 1 - \frac{2m}{n} \left(\frac{n}{3} \text{ time} \right), 0 - \frac{2m}{n} \left(\frac{n}{3} \text{ time} \right) \right\}, \\ LE(Z) &= \frac{n}{3} \times \left| 3 - \frac{2m}{n} \right| + \frac{n}{3} \times \left| 1 - \frac{2m}{n} \right| + \frac{n}{3} \times \left| 0 - \frac{2m}{n} \right| \\ &= \frac{n}{3} \times \left(3 - \frac{2m}{n} \right) + \frac{n}{3} \times \left(\frac{2m}{n} - 1 \right) + \frac{n}{3} \times \left(\frac{2m}{n} \right) \\ &= \frac{2}{3} (n + m). \end{aligned} \quad (33)$$

In this case, for $n = 3^i$, number of vertices, we have $m = 2n/3$ edges. Using this condition, we follow the result:

$$LE(Z) = LE[\text{NEPS}\{Z_i = P_{3_i}; B\}] = \frac{10n}{9}. \quad (34)$$

Case II: for any j and $i = j + 2$, we have the spectrum of the Laplacian matrix as given below:

$$\left\{ 5 - \frac{2m}{n} \left(\frac{n}{9} \text{ time} \right), 4 - \frac{2m}{n} \left(\frac{n}{9} \text{ time} \right), 2 - \frac{2m}{n} \left(\frac{2n}{9} \text{ time} \right), 1 - \frac{2m}{n} \left(\frac{n}{3} \text{ time} \right), 0 - \frac{2m}{n} \left(\frac{2n}{9} \text{ time} \right) \right\}. \quad (35)$$

By following this spectrum, we have calculated the Laplacian energy:

$$\begin{aligned} LE(Z) &= \frac{n}{9} \times \left| 5 - \frac{2m}{n} \right| + \frac{n}{9} \times \left| 4 - \frac{2m}{n} \right| + \frac{2n}{9} \times \left| 2 - \frac{2m}{n} \right| + \frac{n}{3} \times \left| 1 - \frac{2m}{n} \right| \\ &+ \frac{2n}{9} \times \left| 0 - \frac{2m}{n} \right| \\ &= \frac{n}{9} \times \left(5 - \frac{2m}{n} \right) + \frac{n}{9} \times \left(4 - \frac{2m}{n} \right) + \frac{2n}{9} \times \left(2 - \frac{2m}{n} \right) + \frac{n}{3} \times \left(\frac{2m}{n} - 1 \right) + \frac{2n}{9} \times \left(\frac{2m}{n} \right). \end{aligned} \quad (36)$$

After some simplification, we obtain

$$LE(Z) = \frac{2}{9} (5n + m). \quad (37)$$

In this case, for $n = 3^i$, number of vertices, we have $m = 8n/9$ edges. The condition has carried out this case:

$$LE(Z) = LE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = \frac{106n}{81}. \tag{38}$$

Case III: for any j and $i = j + 3$, we have the spectrum of the Laplacian matrix as given below:

$$\left\{ 9 - \frac{2m}{n} \left(\frac{n}{27} \text{ time} \right), 6 - \frac{2m}{n} \left(\frac{n}{9} \text{ time} \right), 4 - \frac{2m}{n} \left(\frac{n}{9} \text{ time} \right), \right. \\ \left. 2 - \frac{2m}{n} \left(\frac{n}{3} \text{ time} \right), 1 - \frac{2m}{n} \left(\frac{7n}{27} \text{ time} \right), 0 - \frac{2m}{n} \left(\frac{4n}{27} \text{ time} \right) \right\}. \tag{39}$$

Using this spectrum, we have

$$LE(Z) = \frac{n}{27} \times \left| 9 - \frac{2m}{n} \right| + \frac{n}{9} \times \left| 6 - \frac{2m}{n} \right| + \frac{n}{9} \times \left| 4 - \frac{2m}{n} \right| + \frac{n}{3} \times \left| 2 - \frac{2m}{n} \right| \\ + \frac{7n}{27} \times \left| 1 - \frac{2m}{n} \right| + \frac{4n}{27} \times \left| 0 - \frac{2m}{n} \right| \\ = \frac{n}{27} \times \left(9 - \frac{2m}{n} \right) + \frac{n}{9} \times \left(6 - \frac{2m}{n} \right) + \frac{n}{9} \times \left(4 - \frac{2m}{n} \right) + \frac{n}{3} \times \left(\frac{2m}{n} - 2 \right) \\ + \frac{7n}{27} \times \left(\frac{2m}{n} - 1 \right) + \frac{4n}{27} \times \left(\frac{2m}{n} \right). \tag{40}$$

After some calculations, we obtain

$$LE(Z) = \frac{2}{27} (7n + 13m). \tag{41}$$

In this case, for $n = 3^i$, number of vertices, we have $m = 32n/27$ edges. By this condition, we have

$$LE(Z) = LE[\text{NEPS}\{Z_i = P_{3_i}; \mathcal{B}\}] = \frac{1210n}{729}. \tag{42}$$

Theorem 8. Let $n = 4^i$ and m are the number of vertices and edges of $\text{NEPS}\{Z_i = P_{4_i}; \mathcal{B}\}$, respectively; then, for $j \geq 1$, we have the following.

Case I: for $i = j + 1$, we have

$$LE[\text{NEPS}\{Z_i = P_{4_i}; \mathcal{B}\}] = n \times 12071 \times 10^{-4}. \tag{43}$$

Case II: for $i = j + 2$, we have

$$LE[\text{NEPS}\{Z_i = P_{4_i}; \mathcal{B}\}] = n \times 15731 \times 10^{-4}. \tag{44}$$

Proof. Let $Z = \text{NEPS}\{Z_i = P_{4_i}; \mathcal{B}\}$ be a graph that depends on j and i . We discuss the cases for different conditions of i and j .

Case I: for $j \geq 1$ and $i = j + 1$, we have the spectrum for the Laplacian energy as

$$\left\{ 0 - \frac{2m}{n} \left(\frac{n}{4} \text{ time} \right), 0.5858 - \frac{2m}{n} \left(\frac{n}{4} \text{ time} \right), 2 - \frac{2m}{n} \left(\frac{n}{4} \text{ time} \right), 3.4142 - \frac{2m}{n} \left(\frac{n}{4} \text{ time} \right) \right\}, \tag{45}$$

where $d(Z) = 2m/n$ is the degree of the graph, and in this case, we have $m = 3n/4$ edges in graph Z :

$$\begin{aligned}
 LE(Z) &= \frac{n}{4} \times \left| 0 - \frac{2m}{n} \right| + \frac{n}{4} \times \left| 0.5858 - \frac{2m}{n} \right| + \frac{n}{4} \times \left| 2 - \frac{2m}{n} \right| + \frac{n}{4} \times \left| 3.4142 - \frac{2m}{n} \right| \\
 &= \frac{n}{4} \times \left(\frac{2m}{n} \right) + \frac{n}{4} \times \left(\frac{2m}{n} - 0.5858 \right) + \frac{n}{4} \times \left(2 - \frac{2m}{n} \right) + \frac{n}{4} \times \left(3.4142 - \frac{2m}{n} \right).
 \end{aligned}
 \tag{46}$$

After the simplification, we obtain

$$LE(Z) = LE[\text{NEPS}\{Z_i = P_4; \mathcal{B}\}] = n \times 12071 \times 10^{-4}.
 \tag{47}$$

Case II: for $j \geq 1$ and $i = j + 2$, we have the spectrum for the Laplacian energy as

$$\begin{aligned}
 &\left\{ 0 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right), 0.7076 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right), 1 - \frac{2m}{n} \left(\frac{n}{4} \right), \right. \\
 &2.3973 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right), 3 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right), 4 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right), \\
 &\left. 5.8951 - \frac{2m}{n} \left(\frac{n}{8} \text{time} \right) \right\}.
 \end{aligned}
 \tag{48}$$

Now, in this case, we have $m = 9n/8$ edges in graph Z :

$$\begin{aligned}
 LE(Z) &= \frac{n}{8} \times \left| 0 - \frac{2m}{n} \right| + \frac{n}{8} \times \left| 0.7076 - \frac{2m}{n} \right| + \frac{n}{4} \times \left| 1 - \frac{2m}{n} \right| + \frac{n}{8} \times \left| 2.3973 - \frac{2m}{n} \right| \\
 &\quad + \frac{n}{8} \times \left| 3 - \frac{2m}{n} \right| + \frac{n}{8} \times \left| 4 - \frac{2m}{n} \right| + \frac{n}{8} \times \left| 5.8951 - \frac{2m}{n} \right| \\
 &= \frac{n}{8} \times \left(0 - \frac{2m}{n} \right) + \frac{n}{8} \times \left(0.7076 - \frac{2m}{n} \right) + \frac{n}{4} \times \left(1 - \frac{2m}{n} \right) + \frac{n}{8} \times \left(2.3973 - \frac{2m}{n} \right) \\
 &\quad + \frac{n}{8} \times \left(3 - \frac{2m}{n} \right) + \frac{n}{8} \times \left(4 - \frac{2m}{n} \right) + \frac{n}{8} \times \left(5.8951 - \frac{2m}{n} \right).
 \end{aligned}
 \tag{49}$$

After some calculations, we get the result

$$LE(Z) = LE[\text{NEPS}\{Z_i = P_4; \mathcal{B}\}] = n \times 15731 \times 10^{-4}.
 \tag{50}$$

□

3. Conclusion

In this study, we investigated the Randic, seidel, and Laplacian energies of the NEPS of path graph P_n , with any base \mathcal{B} . Here, n denotes the number of vertices and i denotes the number of copies of path graph P_n . Some of the results depend on the number of zeroes in base elements, for which we use the notation j .

Data Availability

The data used to support the work are cited within the article as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work.

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