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Research Article

Bivariate Chebyshev Polynomials to Solve Time-Fractional Linear and Nonlinear KdV Equations

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This work concerns the numerical solutions of a category of nonlinear and linear time-fractional partial differential equations (TFPDEs) that are called time-fractional inhomogeneous KdV and nonlinear time-fractional KdV equations, respectively. The fractional derivative operators are of the Caputo type. Two-variable second-kind Chebyshev wavelets (SKCWs) are constructed using one-variable ones; then, utilizing corresponding integral operational matrices leads to an approximate solution to the problem under study. Also, it is found that the perturbation term tends to zero even if a finite number of the basis functions is adopted. To exhibit the applicability and efficiency of the proposed scheme, two models of the KdV equations are given.

1. Introduction

Many scientists and researchers are interested in fractional integral and derivative operators as mathematical tools for modeling diverse physical, chemical, and biological phenomena [1–5]. Fractional operators have the memory property, and this characteristic converts them into a powerful tool for studying real-world problems [6–8]. Different fractional derivative operators have been introduced by researchers for successfully and effectively modeling scientific phenomena. For example, the fractional pseudohyperbolic telegraph partial differential equation employing the Caputo fractional derivative was solved in [9] utilizing the explicit finite difference method. Generalized Caputo and Caputo fractional derivatives were studied in [10], and the nonlinear heat equation in the sense of the generalized Caputo derivative was solved by fractional Green's functions, the generalized Laplace transform, and generalized Mellin transform. A type of the fractional diffusion equation in the sense of the Grunwald–Letnikov derivative was solved by Gorenflo and Abdel-Rehim in [11] using a difference scheme. The $(2+1)$-dimensional fractional Ablowitz–Kanup–Newell–Segur equation in the sense of the conformable derivative was studied to extract general analytical wave solutions in [12] implementing the $\exp(-\phi(\xi))$-expansion method. A modified definition of the conformable fractional derivative was presented in [13], and then, the exact solutions of linear and nonlinear time- and space-fractional mixed partial differential equations involving a new fractional derivative were obtained applying the invariant subspace method. Abu-Shady and Kaabar proposed the generalized fractional derivative (GFD) and showed that this operator coincides with the Caputo and Riemann–Liouville fractional derivatives [14, 15]. Therefore, this computational tool can be used to model different scientific phenomena. Nonlinear fractional partial differential equations (FPDEs) have attracted wide attention for describing many phenomena in engineering, physics, material science, and acoustics [7, 16–19]. Korteweg and de Vries introduced a class of nonlinear evolution equations, namely, KdV equations, for the first time in 1895, to describe the nonlinear shallow-water waves [20]. The KdV equations emerge in diverse phenomena of physics like the one-dimensional waves in shallow-water waves; the Ferma-Pasto-Ulam problem in the continuum limit; the evolution of long, ion-acoustic waves in a plasma, and so on. Time-fractional KdV equations are obtained by replacing the first-order time derivatives with fractional ones of the arbitrary orders. Many works have been done on the KdV and generalized KdV equations. For example, Bagheri and Khani used rational functions, trigonometric functions, and hyperbolic functions, to reach the exact solutions of a fractional model of the KdV equation [21]. A balance method was given to obtain some closed forms of solutions of the KdV equation in [22]. Authors in [23] applied the q-homotopy analysis transform method to study the modified coupled KdV equations. An extended tanh-function method was used in [24] to achieve soliton solutions of modified coupled KdV and generalized Hirota–Satsuma coupled KdV equations. Sahoo and Saha applied the (G'/G)-expansion method to solve the time-fractional KdV equation [25]. Kaya et al. applied radial basis functions to KdV and mKdV equations [26]. Momani et al. [27] utilized the variational iteration method for time-fractional KdV. The analytical traveling wave solutions of the nonlinear fractional KdV equation are obtained by introducing an approximate-analytical method in [28]. Authors in [29] dealt with obtaining exact solutions to the fractional KdV equation. In [30–34], the new iteration method, Adomian decomposition method, variational iteration method, and homotopy perturbation method were utilized to derive approximate solutions to different forms of the KdV equations.

The target of the current work is to achieve approximate solutions for two models of the KdV by means of the second-kind Chebyshev wavelets. From a viewpoint of comparison, the proposed method has a less computational size compared to some existing methods. The orthogonal second-kind Chebyshev polynomials are utilized as basis functions in diverse methods to obtain approximate solutions of integrodifferential equations [35], integral equations [36, 37], ordinary differential equations [38, 39], and partial differential equations [40, 41]. In the present paper, an approach based on the second-kind Chebyshev polynomials is presented to work out time-fractional inhomogeneous KdV and nonlinear time-fractional KdV equations. Finding analytic solutions to linear and especially nonlinear equations is hard; hence, presenting or modifying computational methods to find an approximate solution to these problems is noteworthy.

The main goal of this paper is to assess the numerical solutions of the linear inhomogeneous fractional KdV equation and nonlinear time-fractional KdV equations. An orthogonal collocation scheme is proposed based upon the SKCW functions. Two-dimensional integral operational matrices of fractional and integer orders are derived utilizing one-dimensional ones. Resultant matrices accompanied by the collocation method convert the main problem into an algebraic equation by collocating this algebraic equation at tensor points $\{(\theta_i, \vartheta_j)\}, i = 0, 1, \cdots, M_1, j = 0, 1, \cdots, M_2$ leading to a linear or nonlinear algebraic system. θ_i and ϑ_j are roots of the second-kind Chebyshev polynomials of degrees M_1 and M_2, respectively. By solving the resulted system, an approximate solution is achieved.

The organization of this paper is as follows: the fractional operators, one- and two-variable second-kind Chebyshev wavelets are introduced, and then, operational matrices of the integral are derived in Section 2. In Section 3, two models of the equations under study are presented. Then, it can be seen how using appropriate approximations results in a residual function. In Section 4, some error bounds of the resulted approximations are computed. The established approach is utilized for two equations in Section 5, and a conclusion is provided in the last section.

2. Fractional Operators and SKCWs

This section presented some definitions of the fractional calculus, the SKCWs are introduced, and their integral operational matrices of integer and fractional orders are gained.

2.1. Fractional Operators

Definition 1. The Caputo fractional derivative operator of $\mathfrak{g}(\theta, \vartheta) \in C^{\mathfrak{n}}(\Omega)$ with the order $\mu \in \mathbb{R}$ is given as the following [42]:

$$
{}_{0}^{C}\mathscr{D}_{\vartheta}^{\mu}\mathfrak{g}(\theta,\vartheta) = \begin{cases} \dfrac{1}{\Gamma(\mathfrak{n}-\mu)}\displaystyle\int_{0}^{\vartheta}(\vartheta-\eta)^{\mathfrak{n}-\mu-1}\frac{\partial^{\mathfrak{n}}\mathfrak{g}(\theta,\eta)}{\partial\eta^{\mathfrak{n}}}d\eta, & \mathfrak{n}-1<\mu<\mathfrak{n}, \mathfrak{n}\in\mathbb{N}, \\ \dfrac{\partial^{\mathfrak{n}}\mathfrak{g}(\theta,\vartheta)}{\partial\vartheta^{\mathfrak{n}}}, & \mu=\mathfrak{n}\in\mathbb{N}. \end{cases} \tag{1}
$$

Definition 2. The Riemann-Liouville fractional integral operator of $\mathfrak{g}(\theta, \vartheta) \in C(\Omega)$ with the order $\mu \in \mathbb{R}$ is given as [42]

$$
{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu}\mathfrak{g}(\theta,\vartheta) = \begin{cases} \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{0}^{\vartheta}(\vartheta-\eta)^{\mu-1}\mathfrak{g}(\theta,\eta)d\eta, & \mu>0, \\ {}_{0}^{RL}\mathscr{I}_{\vartheta}^{0}\mathfrak{g}(\theta,\vartheta)=\mathfrak{g}(\theta,\vartheta). \end{cases} \tag{2}
$$

Some features of the above-mentioned operators are as follows:

$$
\begin{aligned}
{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu_1}{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu_2}\mathfrak{g}(\theta,\vartheta) &= {}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu_2}{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu_1}\mathfrak{g}(\theta,\vartheta) = {}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu_1+\mu_2}\mathfrak{g}(\theta,\vartheta), \\
{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu}t^{\sigma} &= \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\mu+1)}\vartheta^{\sigma+\mu}, \quad \sigma>-1, \\
{}_{0}^{C}\mathscr{D}_{\vartheta}^{\mu}\vartheta^{\sigma} &= \begin{cases} 0, \mu>\lfloor\sigma\rfloor, \\ \dfrac{\Gamma(\sigma+1)}{\Gamma(\sigma-\mu+1)}\vartheta^{\sigma-\mu}, & \lfloor\sigma\rfloor\geq\mu, \end{cases} \\
{}_{0}^{C}\mathscr{D}_{\vartheta}^{\mu}\left({}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu}\mathfrak{g}(\theta,\vartheta)\right) &= \mathfrak{g}(\theta,\vartheta), \\
{}_{0}^{RL}\mathscr{I}_{\vartheta}^{\mu}\left({}_{0}^{C}\mathscr{D}_{\vartheta}^{\mu}\mathfrak{g}(\theta,\vartheta)\right) &= \mathfrak{g}(\theta,\vartheta)-\mathfrak{g}(\theta,0), \quad 0<\mu\leq 1.
\end{aligned} \tag{3}
$$

2.2. SKCWs. The one-variable second-kind Chebyshev wavelet $\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta)$ is defined on the interval $J = [0, 1)$ as

$$
\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta) = \begin{cases} 2^{\mathfrak{k}/2}\tilde{\mathbb{U}}_{\mathfrak{m}}\left(2^{\mathfrak{k}}\vartheta-2\mathfrak{n}+1\right), & \dfrac{\mathfrak{n}-1}{2^{\mathfrak{k}-1}}<\vartheta<\dfrac{\mathfrak{n}}{2^{\mathfrak{k}-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}
$$

where $\tilde{\mathbb{U}}_{\mathfrak{m}}(\vartheta)=\sqrt{2/\pi}\mathbb{U}_{\mathfrak{m}}(\vartheta), m=0,1,\cdots,\mathscr{M}-1$. $\mathbb{U}_{\mathfrak{m}}(\vartheta)$, $\mathfrak{m}=0,1,\cdots,\mathscr{M}-1$, are the Chebyshev polynomials of the second kind which are orthogonal with respect to the weight function $\omega(\vartheta)=(1-\vartheta^2)^{1/2}$ on the interval $[-1,1]$; on the other hand,

$$\int_{-1}^{1}\mathbb{U}_{\mathfrak{m}}(\vartheta)\mathbb{U}_{\mathfrak{k}}(\vartheta)\omega(\vartheta)d\vartheta=\begin{cases}\frac{\pi}{2}, & \mathfrak{m}=\mathfrak{k},\\ 0, & \mathfrak{m}\neq\mathfrak{k}.\end{cases} \quad (5)$$

These polynomials are obtained from the following formula:

$$\begin{aligned}&\mathbb{U}_{\mathfrak{m}+1}(\vartheta)=2\vartheta\mathbb{U}_{\mathfrak{m}}(\vartheta)-\mathbb{U}_{\mathfrak{m}-1}(\vartheta), \quad \mathfrak{m}=1,2,\cdots,\\ &\mathbb{U}_0(\vartheta)=1, \mathbb{U}_1(\vartheta)=2\vartheta.\end{aligned} \quad (6)$$

From (4), $\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta)$ involves four arguments, $\mathfrak{n}=1,\cdots,2^{\mathfrak{k}-1}$, $\mathfrak{k}\in\mathbb{N}$, $\mathfrak{m}$ is the degree of the second-kind Chebyshev polynomials, and ϑ is the time variable. The SKCWs are orthogonal with respect to the weight functions $\omega_{\mathfrak{n}}(\vartheta)=\omega(2^{\mathfrak{k}}\vartheta-2\mathfrak{n}+1)$, $\mathfrak{n}=1,2,\cdots,2^{\mathfrak{k}-1}$, over the interval $J_{\mathfrak{n}}=[(\mathfrak{n}-1)/2^{\mathfrak{k}-1},\mathfrak{n}/2^{\mathfrak{k}-1})$.

Every function $g\in L^2_{\omega_{\mathfrak{n}}}(J_{\mathfrak{n}})$ can be expanded as

$$g(\vartheta)=\sum_{\mathfrak{n}=1}^{\infty}\sum_{\mathfrak{m}=0}^{\infty}G_{\mathfrak{n}\mathfrak{m}}\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta), \quad (7)$$

where

$$G_{\mathfrak{n}\mathfrak{m}}=\int_0^1 g(\vartheta)\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta)\omega_{\mathfrak{n}}(\vartheta)d\vartheta. \quad (8)$$

Using a truncated form of the series in (7), an approximation to $g(\vartheta)$ is gained as follows:

$$g(\vartheta)\approx g_{\mathfrak{m}}(\vartheta)=\sum_{\mathfrak{n}=1}^{2^{\mathfrak{k}-1}}\sum_{\mathfrak{m}=0}^{M-1}G_{\mathfrak{n}\mathfrak{m}}\psi_{\mathfrak{n}\mathfrak{m}}(\vartheta)=\bar{G}^T\bar{\Psi}(\vartheta), \quad (9)$$

where $\bar{G}$ and $\bar{\Psi}(\vartheta)$ are $(2^{\mathfrak{k}-1}\mathscr{M})$-order vectors as follows:

$$\begin{aligned}\bar{G}&=\left[G_{10},G_{11},\cdots,G_{1(\mathscr{M}-1)},G_{20},G_{21},\cdots,G_{2(\mathscr{M}-1)},\cdots,G_{2^{\mathfrak{k}-1}0},G_{2^{\mathfrak{k}-1}1},\cdots,G_{2^{\mathfrak{k}-1}(\mathscr{M}-1)}\right]^T,\\ \bar{\Psi}(\vartheta)&=\left[\psi_{10}(\vartheta),\psi_{11}(\vartheta),\cdots,\psi_{1(\mathscr{M}-1)}(\vartheta),\psi_{20}(\vartheta),\psi_{21}(\vartheta),\cdots,\psi_{2(\mathscr{M}-1)}(\vartheta),\cdots,\psi_{2^{\mathfrak{k}-1}0}(\vartheta),\psi_{2^{\mathfrak{k}-1}1}(\vartheta),\cdots,\psi_{2^{\mathfrak{k}-1}(\mathscr{M}-1)}(\vartheta)\right]^T.\end{aligned} \quad (10)$$

The two-variable SKCWs can be defined on the interval $\mathbf{J}=[0,1)\times[0,1)$ using (4) as follows:

$$\psi_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(\theta,\vartheta)=\begin{cases}2^{(\mathfrak{k}_1+\mathfrak{k}_2)/2}\tilde{\mathbb{U}}_{\mathfrak{m}_1}\left(2^{\mathfrak{k}_1}\theta-2\mathfrak{n}_1+1\right)\tilde{\mathbb{U}}_{\mathfrak{m}_2}\left(2^{\mathfrak{k}_2}\vartheta-2\mathfrak{n}_2+1\right), & \frac{\mathfrak{n}_1-1}{2^{\mathfrak{k}_1-1}}<\theta<\frac{\mathfrak{n}_1}{2^{\mathfrak{k}_1-1}}, \frac{\mathfrak{n}_2-1}{2^{\mathfrak{k}_2-1}}<t<\frac{\mathfrak{n}_2}{2^{\mathfrak{k}_2-1}},\\ 0, \text{otherwise},\end{cases} \quad (11)$$

where $\mathfrak{n}_i=1,\cdots,2^{\mathfrak{k}_i-1}$, $\mathfrak{m}_i=0,1,\cdots,M_i-1$, $\mathfrak{k}_i\in\mathbb{N}$, $i=1,2$. It is clear that $\psi_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(\theta,\vartheta)=\psi_{\mathfrak{n}_1\mathfrak{m}_1}(\theta)\psi_{\mathfrak{n}_2\mathfrak{m}_2}(\vartheta)$. Every two-variable $g\in L^2_{W_{\mathfrak{n}_1\mathfrak{n}_2}}(\mathbf{J})$ can be written as follows:

$$g(\theta,\vartheta)=\sum_{\mathfrak{n}_1=1}^{\infty}\sum_{\mathfrak{m}_1=0}^{\infty}\sum_{\mathfrak{n}_2=1}^{\infty}\sum_{\mathfrak{m}_2=0}^{\infty}G_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}\psi_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(\theta,\vartheta), \quad (12)$$

where the coefficients $G_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}$ are computed as

$$G_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}=\int_0^1\int_0^1 g(\theta,\vartheta)\psi_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(\theta,\vartheta)W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)d\vartheta\,d\theta, \quad (13)$$

and $W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)=\omega_{\mathfrak{n}_1}(\theta)\omega_{\mathfrak{n}_2}(\vartheta)$. By considering the truncated series of the infinite series in (12), one gets the following approximation to $g(\theta,\vartheta)$:

$$\begin{aligned}g(\theta,\vartheta)&\approx g_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)=\sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{m}_1=0}^{\mathscr{M}_1-1}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\sum_{\mathfrak{m}_2=0}^{\mathscr{M}_2-1}G_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}\psi_{\mathfrak{n}_1\mathfrak{m}_1\mathfrak{n}_2\mathfrak{m}_2}(\theta,\vartheta)\\ &=\mathbf{G}^T\Delta(\theta,\vartheta)=\mathbf{G}^T(\Psi(\theta)\otimes\Psi(\vartheta)),\end{aligned} \quad (14)$$

where $\mathbf{G}$ and Δ are $(2^{\mathfrak{k}_1-1}\mathscr{M}_1)(2^{\mathfrak{k}_2-1}\mathscr{M}_2)\times 1$ vectors and $\otimes$ denotes the Kronecker product.

2.3. Operational Matrices of the Integration. The integration of the one-variable basis in (10) can be approximated as

$$\int_0^{\vartheta}\bar{\Psi}(\eta)d\eta\approx\mathscr{P}^1\bar{\Psi}(\vartheta), \quad (15)$$

where $\mathscr{P}^1$ is the operational matrix of the integration, and its entries are calculated as

$$\mathscr{P}^1[i,j] := \left\langle \int_0^{\vartheta} \bar{\Psi}_i(\eta)d\eta, \bar{\Psi}_j(\vartheta) \right\rangle_{\omega_{\mathfrak{n}}}, \quad i,j = 1,2,\cdots,2^{\mathfrak{k}-1}\mathscr{M}. \tag{16}$$

If ${}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}$ is the fractional integral of order $\mu > 0$ [7], then the operational matrix of the integration of the fractional order μ, $\mathscr{P}^{(\mu)}$, is given as

$$ {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\bar{\Psi}(\vartheta) \approx \mathscr{P}^{(\mu)}\bar{\Psi}(\vartheta), \tag{17}$$

where

$$\begin{aligned} {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\bar{\Psi}(\vartheta) &= \left[{}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\psi_{10}(\vartheta), \cdots, {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\psi_{1(\mathscr{M}-1)}(\vartheta), \cdots, {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\psi_{2^{\mathfrak{k}-1}0}(\vartheta), \cdots, {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\psi_{2^{\mathfrak{k}-1}(\mathscr{M}-1)}(\vartheta)\right]^T, \\ {}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\psi_{\mathfrak{m}\mathfrak{n}}(\vartheta) &= \begin{cases} 2^{\mathfrak{k}/2}\,{}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\tilde{\mathbb{U}}_{\mathfrak{m}}\left(2^{\mathfrak{k}}\vartheta - 2\mathfrak{n} + 1\right), & \dfrac{\mathfrak{n}-1}{2^{\mathfrak{k}-1}} < \vartheta < \dfrac{\mathfrak{n}}{2^{\mathfrak{k}-1}}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{18}$$

Now, the two-dimensional operational matrices of the integration are constructed using $\mathscr{P}^1$ and $\mathscr{P}^{(\mu)}$:

$$\begin{aligned} &\int_0^{\theta} \Delta(\xi,\vartheta)d\xi \approx \mathbf{P}_{\theta}\Delta(\theta,\vartheta) = \left(\mathscr{P}^1 \otimes I\right)\Delta(\theta,\vartheta), \\ &\int_0^{\vartheta} \Delta(\theta,\eta)d\eta \approx \mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) = \left(I \otimes \mathscr{P}^1\right)\Delta(\theta,\vartheta), \\ &{}_0^{RL}\mathscr{I}_{\vartheta}^{\mu}\Delta(\theta,\eta) \approx \mathbf{P}_{\vartheta}^{(\mu)}\Delta(\theta,\vartheta) = \left(I \otimes \mathscr{P}^{(\mu)}\right)\Delta(\theta,\vartheta), \end{aligned} \tag{19}$$

where $\mathbf{P}_{\theta}$, $\mathbf{P}_{\vartheta}$, and $\mathbf{P}_{\vartheta}^{(\mu)}$ are two-dimensional operational matrices regarding the classic and fractional integral operators, respectively, and I is the $(\mathscr{M} \times \mathscr{M})$ identity matrix.

3. Methodology

To show the applicability of the proposed scheme, the time-fractional inhomogeneous KdV equation and nonlinear time-fractional KdV equation are considered.

3.1. Time-Fractional Inhomogeneous KdV Equation. A form of this model is given as follows [27]:

$$ {}_0^C\mathscr{D}_{\vartheta}^{\mu}\mathbf{u}(\theta,\vartheta) + p(\theta,\vartheta)\frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial\theta} + q(\theta,\vartheta)\frac{\partial^3 \mathbf{u}(\theta,\vartheta)}{\partial\theta^3} = \mathfrak{f}(\theta,\vartheta), (\theta,\vartheta) \in \mathbf{J}, \mu \in (0,1], \tag{20}$$

with

$$\mathbf{u}(\theta,0) = \rho_1(\theta), \mathbf{u}(0,\vartheta) = \phi_1(\vartheta), \frac{\partial \mathbf{u}(0,\vartheta)}{\partial\theta} = \phi_2(\vartheta), \frac{\partial^2 \mathbf{u}(0,\vartheta)}{\partial\theta^2} = \phi_3(\vartheta), \tag{21}$$

where functions $\rho_1, \phi_1, \phi_2, \phi_3$ are known continuous ones. By considering the highest orders of derivative operators regarding θ and ϑ, the following approximation is given:

$$\frac{\partial^4 \mathbf{u}(\theta,\vartheta)}{\partial t \partial\theta^3} \approx \mathbf{C}^T\Delta(\theta,\vartheta). \tag{22}$$

Triple integrating (22) regarding θ and using conditions (21) lead to the following approximations:

$$\begin{aligned} \frac{\partial^3 \mathbf{u}(\theta,\vartheta)}{\partial\vartheta\partial\theta^2} &\approx \mathbf{C}^T\mathbf{P}_{\theta}\Delta(\theta,\vartheta) + \frac{\partial^3 \mathbf{u}(0,\vartheta)}{\partial\vartheta\partial\theta^2} = \mathbf{C}^T\mathbf{P}_{\theta}\Delta(\theta,\vartheta) \\ &\quad + \phi_3'(\vartheta) \approx \mathbf{C}^T\mathbf{P}_{\theta}\Delta(\theta,\vartheta) + F_1^T\Delta(\theta,\vartheta), \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial^2 \mathbf{u}(\theta,\vartheta)}{\partial\vartheta\partial\theta} &\approx \mathbf{C}^T(\mathbf{P}_{\theta})^2\Delta(\theta,\vartheta) + F_1^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + \phi_2'(\vartheta) \\ &\approx \mathbf{C}^T(\mathbf{P}_{\theta})^2\Delta(\theta,\vartheta) + F_1^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_2^T\Delta(\theta,\vartheta), \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial t} &\approx \mathbf{C}^T(\mathbf{P}_{\theta})^3\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_{\theta})^2\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_{\theta}\Delta(\theta,\vartheta) \\ &\quad + \phi_1'(\vartheta) \approx \mathbf{C}^T(\mathbf{P}_{\theta})^3\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_{\theta})^2\Delta(\theta,\vartheta) \\ &\quad + F_2^T\mathbf{P}_{\theta}\Delta(\theta,\vartheta) + F_3^T\Delta(\theta,\vartheta). \end{aligned} \tag{25}$$

Now, by integrating (23) regarding ϑ, an approximation to $\mathbf{u}(\theta,\vartheta)$ is obtained:

$$\begin{aligned} \mathbf{u}(\theta,\vartheta) &\approx \mathbf{C}^T(\mathbf{P}_{\theta})^3\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_{\theta})^2\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) \\ &\quad + F_2^T\mathbf{P}_{\theta}\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + \rho_1(\theta) \\ &\approx \mathbf{C}^T(\mathbf{P}_{\theta})^3\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_{\theta})^2\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) \\ &\quad + F_2^T\mathbf{P}_{\theta}\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_4^T\Delta(\theta,\vartheta). \end{aligned} \tag{26}$$

Again, by integrating (22) regarding ϑ and θ, approximations to $\mathbf{u}_{\theta\theta}$ and $\mathbf{u}_{\theta}$ are obtained:

$$\frac{\partial^3 \mathbf{u}(\theta,\vartheta)}{\partial\theta^3} \approx \mathbf{C}^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + \rho_1'''(\theta) \approx \mathbf{C}^T\mathbf{P}_{\vartheta}\Delta(\theta,\vartheta) + F_5^T\Delta(\theta,\vartheta), \tag{27}$$

$$\frac{\partial^2 \mathbf{u}(\theta,\vartheta)}{\partial\theta^2} \approx \mathbf{C}^T \mathbf{P}_\vartheta \mathbf{P}_\theta \Delta(\theta,\vartheta) + F_5^T \mathbf{P}_\vartheta \Delta(\theta,\vartheta) + \phi_3(\vartheta) \approx \mathbf{C}^T \mathbf{P}_\vartheta \mathbf{P}_\theta \Delta(\theta,\vartheta) + F_5^T \mathbf{P}_\theta \Delta(\theta,\vartheta) + F_6^T \Delta(\theta,\vartheta), \quad (28)$$

$$\frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial\theta} \approx \mathbf{C}^T \mathbf{P}_\vartheta (\mathbf{P}_\vartheta)^2 \Delta(\theta,\vartheta) + F_5^T (\mathbf{P}_\theta)^2 \Delta(\theta,\vartheta) + F_6^T \mathbf{P}_\theta \Delta(\theta,\vartheta) + \phi_2(\vartheta) \approx \mathbf{C}^T \mathbf{P}_\vartheta (\mathbf{P}_\theta)^2 \Delta(\theta,\vartheta) + F_5^T (\mathbf{P}_\theta)^2 \Delta(\theta,\vartheta) + F_6^T \mathbf{P}_\theta \Delta(\theta,\vartheta) + F_7^T \Delta(\theta,\vartheta). \quad (29)$$

Now, an approximation to ${}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}(\theta,\vartheta)$ is computed using (23):

$$\begin{aligned} {}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}(\theta,\vartheta) {}_0^{RL}\mathcal{J}_\vartheta^{1-\mu}\left(\frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial\vartheta}\right) &\approx {}_0^{RL}\mathcal{J}_\vartheta^{1-\mu}\big(\mathbf{C}^T(\mathbf{P}_\theta)^3\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_\theta)^2\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_\theta\Delta(\theta,\vartheta) + F_3^T\Delta(\theta,\vartheta)\big) \\ &\approx \mathbf{C}^T(\mathbf{P}_\theta)^3\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_\theta)^2\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_\theta\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta). \end{aligned} \quad (30)$$

Substituting approximations (27)–(30) into (20) yields $\mathcal{R}(\theta,\vartheta)$ as the residual function as follows:

$$\begin{aligned} \mathcal{R}(\theta,\vartheta) = {} & \mathbf{C}^T(\mathbf{P}_\theta)^3\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_\theta)^2\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_\theta\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) \\ & + p(\theta,\vartheta)\big(\mathbf{C}^T\mathbf{P}_\vartheta(\mathbf{P}_\theta)^2\Delta(\theta,\vartheta) + F_5^T(\mathbf{P}_\theta)^2\Delta(\theta,\vartheta) + F_6^T\mathbf{P}_\theta\Delta(\theta,\vartheta) + F_7^T\Delta(\theta,\vartheta)\big) + q(\theta,\vartheta)\big(\mathbf{C}^T\mathbf{P}_\vartheta\Delta(\theta,\vartheta) \\ & + F_5^T\Delta(\theta,\vartheta)\big) - \mathfrak{f}(\theta,\vartheta). \end{aligned} \quad (31)$$

3.2. Time-Fractional Nonlinear KdV Equation. In this paper, the following class of time-fractional nonlinear KdV equations is studied:

$${}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}(\theta,\vartheta) + 6\mathbf{u}(\theta,\vartheta)\frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial\theta} + \frac{\partial^3 \mathbf{u}(\theta,\vartheta)}{\partial\theta^3} = 0, (\theta,\vartheta) \in \mathbf{J}, \mu \in (0,1], \quad (32)$$

with the conditions in (21). Substituting approximations (26)–(30) into (32) yields the following residual function:

$$\begin{aligned} \mathcal{R}(\theta,\vartheta) = {} & \mathbf{C}^T(\mathbf{P}_\theta)^3\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_\theta)^2\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_\theta\mathbf{P}_\vartheta^{(1-\mu)}\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_\theta^{(1-\mu)}\Delta(\theta,\vartheta) \\ & + \big(\mathbf{C}^T(\mathbf{P}_\theta)^3\mathbf{P}_\vartheta\Delta(\theta,\vartheta) + F_1^T(\mathbf{P}_\theta)^2\mathbf{P}_\vartheta\Delta(\theta,\vartheta) + F_2^T\mathbf{P}_\theta\mathbf{P}_\vartheta\Delta(\theta,\vartheta) + F_3^T\mathbf{P}_\theta\Delta(\theta,\vartheta) + F_4^T\Delta(\theta,\vartheta)\big) \\ & \times \big(\mathbf{C}^T\mathbf{P}_\vartheta(\mathbf{P}_\theta)^2\Delta(\theta,\vartheta) + F_5^T(\mathbf{P}_\theta)^2\Delta(\theta,\vartheta) + F_6^T\mathbf{P}_\theta\Delta(\theta,\vartheta) + F_7^T\Delta(\theta,\vartheta)\big) + \mathbf{C}^T\mathbf{P}_\vartheta\Delta(\theta,\vartheta) + F_5^T\Delta(\theta,\vartheta). \end{aligned} \quad (33)$$

Collocating residual functions (31) and (33) at points $\{(\theta_i,\vartheta_j)\}, i = 1,2,\cdots,2^{\mathfrak{k}_1-1}\mathcal{M}_1, j = 1,2,\cdots,2^{\mathfrak{k}_2-1}\mathcal{M}_2$ results in a system of algebraic equations, where θ_i and ϑ_j are roots of $\tilde{\mathbb{U}}_{2^{\mathfrak{k}_1-1}\mathcal{M}_1}(\theta)$ and $\tilde{\mathbb{U}}_{2^{\mathfrak{k}_2-1}\mathcal{M}_2}(\vartheta)$, respectively. This algebraic system can be handled by the Newton scheme. Therefore, an approximate solution is acquired from (26).

Two models were solved by the variational iteration method in [27], and some figures of approximate solutions were depicted. The nonlinear time-fractional KdV equation (32) was solved by El-Wakil et al. in [43] using He's variational iteration method and presented a second-order solution including some parameters. Authors in [44] obtained an approximate solution utilizing the iteration method after spending many algebraic computational costs. Inc et al. acquired new numerical solutions of fractional-time KdV equation by a technique of fictitious time integration and group preserving [45]. Authors in [46–48] used algebraic computational methods such as the modified extended tanh method, Sardar-subequation method, and He's semi-inverse variation method and the ansatz method to construct some soliton solutions of the nonlinear time-fractional KdV equation.

4. Error Bound

In this section, error bounds are derived for the residual functions/perturbation terms for two given models in Section 3. First, some error bounds are computed for approximation errors.

4.1. Time-Fractional Inhomogeneous KdV Equation. Consider Equation (20) and suppose that $\mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)$ is its approximate solution obtained from the presented algorithm in Section 3. Thus, $\mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)$ satisfies the following equations:

$${}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta) + p(\theta,\vartheta)\frac{\partial \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)}{\partial\theta} + q(\theta,\vartheta)\frac{\partial^3 \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)}{\partial\theta^3} = \mathfrak{f}(\theta,\vartheta) - \mathcal{R}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta), \quad (34)$$

where $\mathcal{R}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)$ is called the residual function/perturbation term. By subtracting Equation (34) from Equation (20), one gets

$$\begin{aligned} \mathcal{R}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta) = {} & \left({}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}(\theta,\vartheta) - {}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)\right) \\ & + p(\theta,\vartheta)\left(\frac{\partial \mathbf{u}(\theta,\vartheta)}{\partial\theta} - \frac{\partial \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)}{\partial\theta}\right) \\ & + q(\theta,\vartheta)\left(\frac{\partial^3 \mathbf{u}(\theta,\vartheta)}{\partial\theta^3} - \frac{\partial^3 \mathbf{u}_{\mathcal{M}_1\mathcal{M}_2}(\theta,\vartheta)}{\partial\theta^3}\right). \end{aligned} \quad (35)$$

Suppose that $p(\theta,\vartheta), q(\theta,\vartheta)$ are continuous functions over $\mathbf{J}$. By taking L^2-norm on Equation (35), one has

$$\left\|\mathscr{R}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2} \le \left\|{}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u} - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2} + \|p\|_{L^2}\left\|\frac{\partial \mathfrak{u}}{\partial\theta} - \frac{\partial \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}}{\partial\theta}\right\|_{L^2} + \|q\|_{L^2}\left\|\frac{\partial^3 \mathfrak{u}}{\partial\theta^3} - \frac{\partial^3 \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}}{\partial\theta^3}\right\|_{L^2}. \tag{36}$$

First, error bounds are calculated for terms on the right-hand side in (36). Assume that $\mathscr{T}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)$ is the Taylor series expansion of $\mathfrak{u}(\theta,\vartheta)$, $\Theta_1 = \max_{(\theta,\vartheta)\in\mathbf{J}}\left|\mathfrak{u}^{(\mathscr{M}_1+\mathscr{M}_2-\mu)}(\theta,\vartheta)\right|$, and $\mathbf{J}_{\mathfrak{n}_1\mathfrak{n}_2} = [(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}, \mathfrak{n}_1/2^{\mathfrak{k}_1-1}] \times [(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}, \mathfrak{n}_2/2^{\mathfrak{k}_2-1}]$. One has,

$$\begin{aligned}
\left\|{}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u} - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2}^2 &= \int_0^1\int_0^1\left(\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)\right)^2 W(\theta,\vartheta)\,d\vartheta\,d\theta \\
&= \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) - {}_0^C\mathscr{D}_\vartheta^\mu \mathscr{T}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\frac{\max_{(\xi_{\mathfrak{n}_1},\eta_{\mathfrak{n}_2})\in\mathbf{J}_{\mathfrak{n}_1\mathfrak{n}_2}}\left|\mathfrak{u}^{(\mathscr{M}_1+\mathscr{M}_2-\mu)}\left(\xi_{\mathfrak{n}_1},\eta_{\mathfrak{n}_2}\right)\right|}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \left(\frac{\Theta_1}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2\int_0^1\int_0^1 \underbrace{W(\theta,\vartheta)}_{\omega(\theta)\omega(\vartheta)}\,d\vartheta\,d\theta \\
&= \left(\frac{\Theta_1}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2\int_0^1 \theta^{1/2}(1-\theta)^{1/2}d\theta\int_0^1 \vartheta^{1/2}(1-\vartheta)^{1/2}d\vartheta \\
&= \left(\frac{\Theta_1}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2\left(\frac{\pi}{8}\right)^2.
\end{aligned} \tag{37}$$

So, one gets

$$\left\|{}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u} - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2} \le \frac{\pi\Theta_1}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3}. \tag{38}$$

In a similar way, if $\Theta_{2,l} = \max_{(\theta,\vartheta)\in\mathbf{J}}\left|\mathfrak{u}^{(\mathscr{M}_1+\mathscr{M}_2-l)}(\theta,\vartheta)\right|$, $l=0,1,2,3$, one has

$$\begin{aligned}
\left\|\frac{\partial^l \mathfrak{u}}{\partial\theta^l} - \frac{\partial^l \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}}{\partial\theta^l}\right\|_{L^2}^2 &= \int_0^1\int_0^1\left(\frac{\partial^l \mathfrak{u}(\theta,\vartheta)}{\partial\theta^l} - \frac{\partial^l \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)}{\partial\theta^l}\right)^2 W(\theta,\vartheta)\,d\vartheta\,d\theta \\
&= \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\frac{\partial^l \mathfrak{u}(\theta,\vartheta)}{\partial\theta^l} - \frac{\partial^l \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)}{\partial\theta^l}\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\frac{\partial^l \mathfrak{u}(\theta,\vartheta)}{\partial\theta^l} - \frac{\partial^l \mathscr{T}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)}{\partial\theta^l}\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \sum_{\mathfrak{n}_1=1}^{2^{\mathfrak{k}_1-1}}\sum_{\mathfrak{n}_2=1}^{2^{\mathfrak{k}_2-1}}\int_{(\mathfrak{n}_1-1)/2^{\mathfrak{k}_1-1}}^{\mathfrak{n}_1/2^{\mathfrak{k}_1-1}}\int_{(\mathfrak{n}_2-1)/2^{\mathfrak{k}_2-1}}^{\mathfrak{n}_2/2^{\mathfrak{k}_2-1}}\left(\frac{\max_{(\xi_{\mathfrak{n}_1},\eta_{\mathfrak{n}_2})\in\mathbf{J}_{\mathfrak{n}_1\mathfrak{n}_2}}\left|\mathfrak{u}^{(\mathscr{M}_1+\mathscr{M}_2-l)}\left(\xi_{\mathfrak{n}_1},\eta_{\mathfrak{n}_2}\right)\right|}{(\mathscr{M}_1-l)!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2 W_{\mathfrak{n}_1\mathfrak{n}_2}(\theta,\vartheta)\,d\vartheta\,d\theta \\
&\le \left(\frac{\Theta_{2,l}}{(\mathscr{M}_1-l)!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}}\right)^2\left(\frac{\pi}{8}\right)^2.
\end{aligned} \tag{39}$$

Table 1: Maximum absolute errors for $\mu = 1$ and different values of M_1, M_2 for Example 1.

$M_1 = M_2$	2	3	4	5
MAE	5.2885×10^{-3}	7.0306×10^{-4}	2.3882×10^{-5}	2.0614×10^{-6}

Table 2: Absolute errors for $k_1 = k_2 = 1, M_1 = M_2 = 4$ at equally spaced points for Example 1.

$\theta_i = \vartheta_i$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1$
0	4.8935×10^{-7}	5.4663×10^{-7}	4.4250×10^{-7}	1.4522×10^{-11}
0.2	6.1889×10^{-7}	6.0321×10^{-7}	4.6254×10^{-7}	1.2627×10^{-7}
0.4	8.5473×10^{-7}	8.0229×10^{-7}	7.3242×10^{-7}	6.8164×10^{-7}
0.6	5.8306×10^{-6}	5.6182×10^{-6}	3.6174×10^{-6}	1.1872×10^{-6}
0.8	1.7757×10^{-5}	1.6597×10^{-5}	1.0738×10^{-5}	1.7657×10^{-6}
1	9.7134×10^{-5}	7.8075×10^{-5}	3.3765×10^{-5}	2.3881×10^{-5}

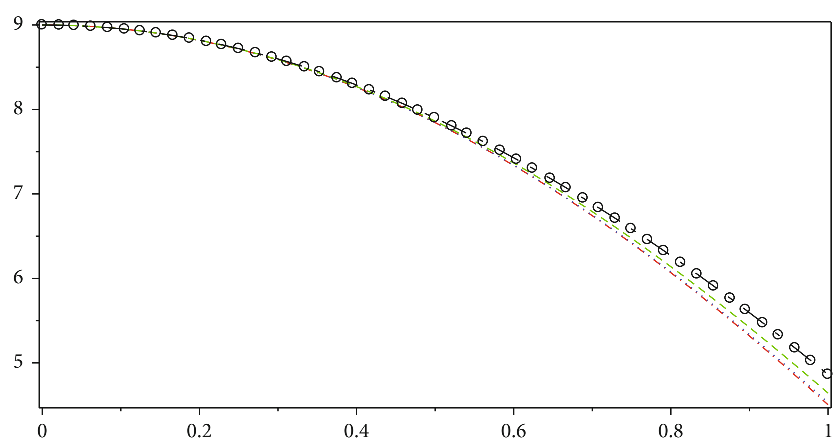

Figure 1: Exact and approximate solutions for $k_1 = k_2 = 1, M_1 = M_2 = 4$, and $\mu = 0.7, 0.8, 0.9, 1$ at time $\vartheta = 3$ for Example 1.

Thus, one gets

$$\left\| \frac{\partial^l \mathbf{u}}{\partial \theta^l} - \frac{\partial^l \mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}}{\partial \theta^l} \right\|_{L^2} \leq \frac{\pi \Theta_{2,l}}{(\mathcal{M}_1 - l)! \mathcal{M}_2! 2^{\mathcal{M}_1 (\mathfrak{k}_1 - 1)} 2^{\mathcal{M}_2 (\mathfrak{k}_2 - 1)} 2^3}, \quad l = 0, 1, 2, 3. \tag{40}$$

Therefore, a bound is obtained for inequality (36) using (37) and (39) as follows:

$$\begin{aligned} \|\mathcal{R}_{\mathcal{M}_1 \mathcal{M}_2}\|_{L^2} & \frac{\pi \Theta_1}{\mathcal{M}_1! (\mathcal{M}_2 - \mu)! 2^{\mathcal{M}_1 (\mathfrak{k}_1 - 1)} 2^{\mathcal{M}_2 (\mathfrak{k}_2 - 1)} 2^3} \\ & + \|p\|_{L^2} \frac{\pi \Theta_{2,1}}{(\mathcal{M}_1 - 1)! \mathcal{M}_2! 2^{\mathcal{M}_1 (\mathfrak{k}_1 - 1)} 2^{\mathcal{M}_2 (\mathfrak{k}_2 - 1)} 2^3} \\ & + \|q\|_{L^2} \frac{\pi \Theta_{2,3}}{(\mathcal{M}_1 - 3)! \mathcal{M}_2! 2^{\mathcal{M}_1 (\mathfrak{k}_1 - 1)} 2^{\mathcal{M}_2 (\mathfrak{k}_2 - 1)} 2^3}. \end{aligned} \tag{41}$$

It is evident from the right-hand side of (40) that $\|\mathcal{R}_{\mathcal{M}_1 \mathcal{M}_2}\|_{L^2} \longrightarrow 0$ when $\mathcal{M}_1, \mathcal{M}_2 \longrightarrow \infty$.

4.2. Time-Fractional Nonlinear KdV Equation. Consider Equation (32) and suppose that $\mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta)$ is its approximate solution obtained from the proposed method. Thus, $\mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta)$ satisfies the following equation:

$$\begin{aligned} {}_0^C\mathcal{D}_\vartheta^\mu \mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta) + 6 \mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta) \frac{\partial \mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta)}{\partial \theta} \\ + \frac{\partial^3 \mathbf{u}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta)}{\partial \theta^3} = -\mathcal{R}_{\mathcal{M}_1 \mathcal{M}_2}(\theta, \vartheta). \end{aligned} \tag{42}$$

Subtracting Equation (41) from (32) leads to the

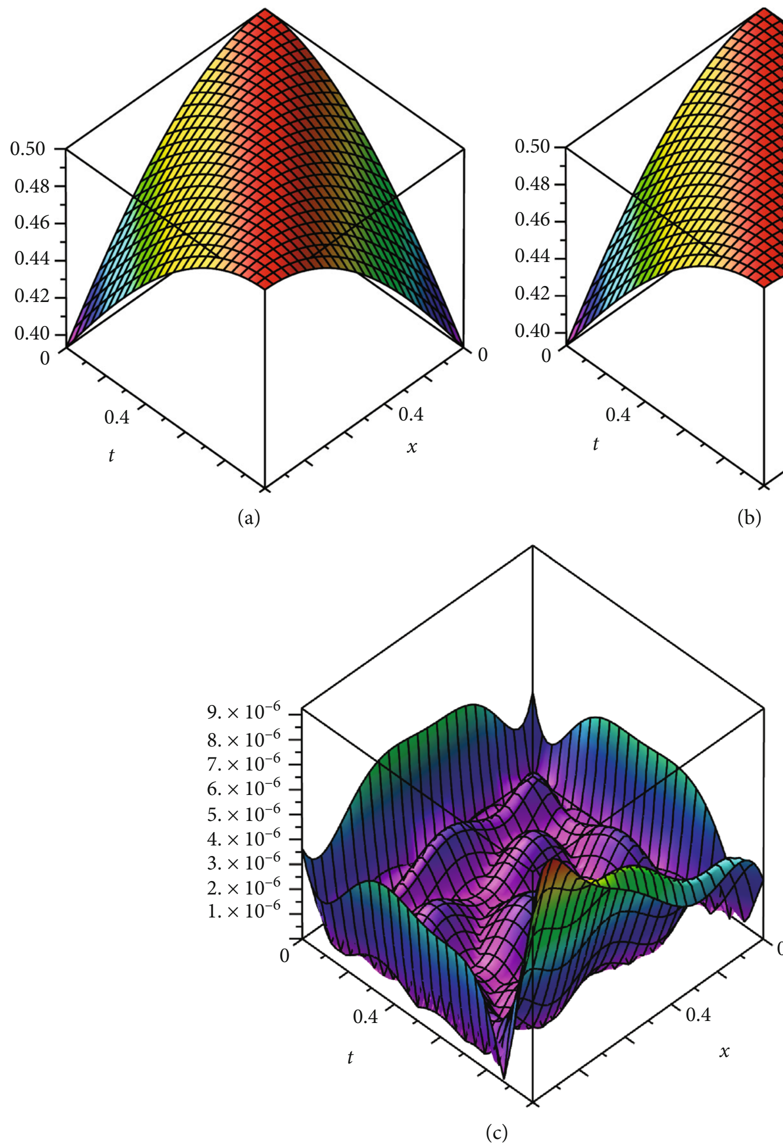

FIGURE 2: (a) Exact solution, (b) approximate solution, and (c) absolute error function for $k_1 = k_2 = 1$, $M_1 = M_2 = 5$, and $\mu = 1$ for Example 2.

following equation:

$$\begin{aligned}\mathscr{R}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta) &= \left({}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) - {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)\right)\\ &+ 6\left(\mathfrak{u}(\theta,\vartheta)\frac{\partial \mathfrak{u}(\theta,\vartheta)}{\partial\theta} - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)\frac{\partial \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)}{\partial\theta}\right)\\ &+ \left(\frac{\partial^3 \mathfrak{u}(\theta,\vartheta)}{\partial\theta^3} - \frac{\partial^3 \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}(\theta,\vartheta)}{\partial\theta^3}\right).\end{aligned} \tag{43}$$

The nonlinear term $\mathfrak{u}\mathfrak{u}_\theta - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}$ can be written as

$$\begin{aligned}\mathfrak{u}\mathfrak{u}_\theta - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2} &= \left(\mathfrak{u} - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right)\mathfrak{u}_\theta\\ &+ \left(\mathfrak{u}_\theta - \mathfrak{u}_{\theta\mathscr{M}_1 M_2}\right)\mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2} = \left(\mathfrak{u} - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right)\left(\mathfrak{u}_\theta - \mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2} + \mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}\right)\\ &+ \left(\mathfrak{u}_\theta - \mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}\right)\mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2} = \left(\mathfrak{u} - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right)\left(\mathfrak{u}_\theta - \mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}\right)\\ &+ \left(\mathfrak{u} - \mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right)\mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2} + \left(\mathfrak{u}_\theta - \mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}\right)\mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}.\end{aligned} \tag{44}$$

Now, using bounds obtained in the previous section, an error bound of (42) can be calculated as follows:

$$\begin{aligned}&\left\|\mathscr{R}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2} \frac{\pi\Theta_1}{\mathscr{M}_1!(\mathscr{M}_2-\mu)!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3}\\ &+ 6\left[\frac{\pi\Theta_{2,0}}{\mathscr{M}_1!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3}\right.\\ &\times\left(\frac{\pi\Theta_{2,1}}{(\mathscr{M}_1-1)!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3} + \left\|\mathfrak{u}_{\theta\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2}\right)\\ &\left.+ \frac{\pi\Theta_{2,1}\left\|\mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2}}{(\mathscr{M}_1-1)!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3}\right]\\ &+ \frac{\pi\Theta_{2,3}\left\|\mathfrak{u}_{\mathscr{M}_1\mathscr{M}_2}\right\|_{L^2}}{(\mathscr{M}_1-3)!\mathscr{M}_2!2^{\mathscr{M}_1(\mathfrak{k}_1-1)}2^{\mathscr{M}_2(\mathfrak{k}_2-1)}2^3}.\end{aligned} \tag{45}$$

Obviously, the right-hand side of (44) tends to zero, when $\mathscr{M}_1, \mathscr{M}_2$ are sufficiently large.

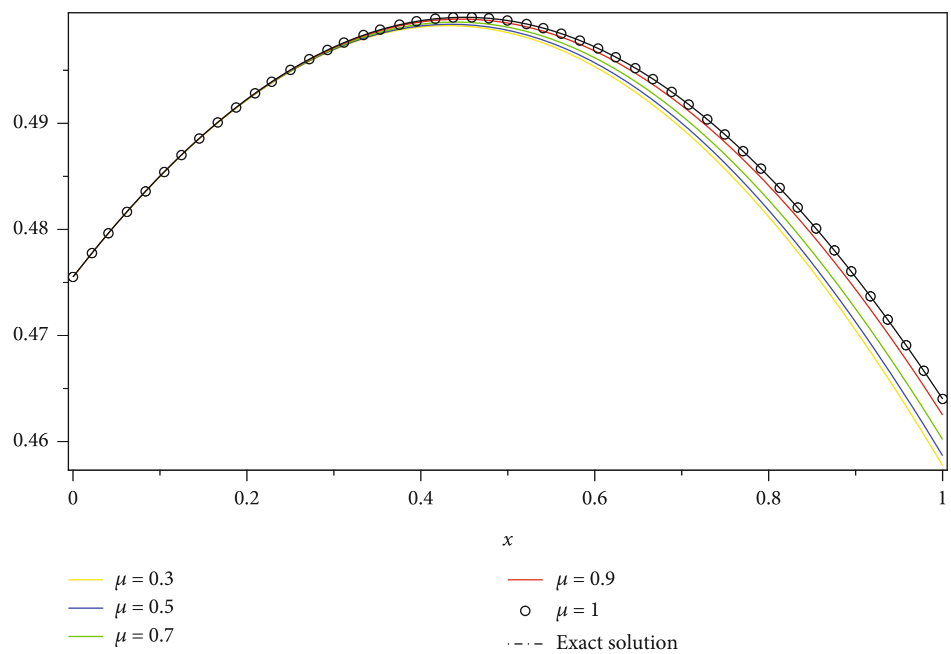

FIGURE 3: Exact and approximate solutions for $k_1 = k_2 = 1, M_1 = M_2 = 6$, and $\mu = 0.3, 0.5, 0.7, 0.9, 1$ at time $\vartheta = 0.45$ for Example 2.

TABLE 3: Absolute errors for $k_1 = k_2 = 1, M_1 = M_2 = 6$ at equally spaced points for Example 2.

$\theta_i = \vartheta_i$	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.7$	$\mu = 0.9$	$\mu = 1$
0	6.6378×10^{-8}	6.8466×10^{-8}	5.5505×10^{-8}	2.4525×10^{-8}	1.3170×10^{-9}
0.2	8.6724×10^{-5}	6.7852×10^{-5}	4.4262×10^{-5}	1.5860×10^{-5}	3.1335×10^{-9}
0.4	6.5214×10^{-4}	5.1748×10^{-4}	3.4307×10^{-4}	1.2522×10^{-4}	3.7353×10^{-10}
0.6	2.0195×10^{-3}	1.6274×10^{-3}	1.0975×10^{-3}	4.0827×10^{-4}	1.2169×10^{-8}
0.8	4.2767×10^{-3}	3.5051×10^{-3}	2.4073×10^{-3}	9.1284×10^{-4}	3.0611×10^{-8}
1	7.2448×10^{-3}	6.0506×10^{-3}	4.2372×10^{-3}	1.6379×10^{-3}	4.9306×10^{-8}

5. Numerical Examples

The two models given in Section 3 are considered to illustrate the accuracy and applicability of the proposed scheme. Maximum absolute errors are computed when the derivative order is an integer (classical case). The results are compared to the exact ones. Computations and simulations are handled by Maple 16.

Example 1. As a first example, the following linear inhomogeneous time-fractional KdV equation:

$$ {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) + \frac{\partial \mathfrak{u}(\theta,\vartheta)}{\partial\theta} + \frac{\partial^3 \mathfrak{u}(\theta,\vartheta)}{\partial\theta^3} = \frac{2\vartheta^{2-\mu}}{\Gamma(3-\mu)} \cos(\theta), (\theta,\vartheta) \in \mathbf{J}, \quad \mu \in (0,1], \tag{46} $$

subject to the initial and boundary conditions:

$$ \mathfrak{u}(\theta,0) = 0, \mathfrak{u}(0,\vartheta) = 0, \frac{\partial \mathfrak{u}(0,\vartheta)}{\partial\theta} = 0, \frac{\partial^2 \mathfrak{u}(0,\vartheta)}{\partial\theta^2} = 0. \tag{47} $$

The exact solution is $\mathfrak{u}(\theta,\vartheta) = \vartheta^2 \cos(\theta)$, if $\mu = 1$. Maximum absolute errors (MAE) are listed in Table 1 for $\mu = 1$, $\mathfrak{k}_1 = \mathfrak{k}_2 = 1, \mathscr{M}_1 = \mathscr{M}_2 = 2, 3, 4, 5$. As seen, the errors decrease when $\mathscr{M}_1, \mathscr{M}_2$ increase. Values of absolute errors of the exact and numerical solutions, at equally spaced points $\theta_i = \vartheta_j = 0.2i, i = 0, 1, \cdots, 5$, are seen in Table 2 for $\mathfrak{k}_1 = \mathfrak{k}_2 = 1$, $\mathscr{M}_1 = \mathscr{M}_2 = 4, \mu = 0.7, 0.8, 0.9, 1$. The results have more accuracy as $\mu \longrightarrow 1$. Plots of numerical solutions are depicted in Figure 1 for $\mathfrak{k}_1 = \mathfrak{k}_2 = 1, \mathscr{M}_1 = \mathscr{M}_2 = 4, \mu = 0.7, 0.8, 0.9, 1$, and $\vartheta = 3$. It can be found that the approximate solutions approach the exact one when $\mu \longrightarrow 1$.

Example 2. Consider the time-fractional nonlinear KdV equation as follows:

$$ {}_0^C\mathscr{D}_\vartheta^\mu \mathfrak{u}(\theta,\vartheta) + 6\mathfrak{u}(\theta,\vartheta)\frac{\partial \mathfrak{u}(\theta,\vartheta)}{\partial\theta} + \frac{\partial^3 \mathfrak{u}(\theta,\vartheta)}{\partial\theta^3} = 0, (\theta,\vartheta) \in \mathbf{J}, \quad \mu \in (0,1], \tag{48} $$

with

$$\mathbf{u}(\theta,0)=\frac{1}{2}\sec \mathrm{h}^2\left(\frac{\theta}{2}\right),\mathbf{u}(0,\vartheta)=\frac{1}{2}\sec \mathrm{h}^2\left(\frac{\vartheta}{2}\right),\frac{\partial \mathbf{u}(0,\vartheta)}{\partial\theta}=\frac{1}{2}\sec \mathrm{h}^2\left(\frac{\vartheta}{2}\right)\tanh\left(\frac{\vartheta}{2}\right),$$
$$\frac{\partial^2\mathbf{u}(0,\vartheta)}{\partial\theta^2}=\frac{1}{2}\sec \mathrm{h}^2\left(\frac{\vartheta}{2}\right)\tanh^2\left(\frac{\vartheta}{2}\right)+\frac{1}{2}\sec \mathrm{h}^2\left(\frac{\vartheta}{2}\right)\left(-\frac{1}{2}+\frac{1}{2}\tanh^2\left(\frac{\vartheta}{2}\right)\right). \tag{49}$$

The exact solution is $\mathbf{u}(\theta,\vartheta)=(1/2)\sec \mathrm{h}^2(\theta/2-\vartheta/2)$, if $\mu=1$. Plots of exact and approximate solutions and the absolute error function are depicted in Figure 2 for $\mathfrak{k}_1=\mathfrak{k}_2=1$, $\mathscr{M}_1=\mathscr{M}_2=5$, and $\mu=1$. A graphical comparison between exact and approximate solutions is observed in Figure 3 for $\mathfrak{k}_1=\mathfrak{k}_2=1$; $\mathscr{M}_1=\mathscr{M}_2=4$; $\mu=0.3,0.5,0.7,0.9,1$; and $\vartheta=0.45$. It can be found that the approximate solutions approach the exact one when $\mu\longrightarrow 1$. Values of absolute errors are listed in Table 3 for $\mathscr{M}_1=\mathscr{M}_2=6$; $\mu=0.3,0.5,0.7,0.9,1$; and $\theta_i=\vartheta_i=0.2i, i=0,1,\cdots,5$. It can be seen from Figure 3 and Table 3 that the approximate solutions approach the exact one when $\mu\longrightarrow 1$.

6. Conclusion

In this paper, the second-kind Chebyshev wavelets were employed to solve time-fractional inhomogeneous KdV and time-fractional nonlinear KdV equations. Using the presented scheme, the main problem was converted into a system of algebraic equations wherein obtaining its solution is easier than finding the solution of the problem under study. In comparison with the Adomian decomposition, homotopy analysis, and homotopy perturbation methods, the SKCW method possesses fewer computational costs. The few numbers of the basis functions lead to an approximate solution with appropriate accuracy. As seen from Table 1, by increasing values of $\mathscr{M}_i, i=1,2$, the maximum absolute errors decrease. In Tables 2 and 3, values of absolute errors at equally spaced points decrease as $\mu\longrightarrow 1$; then, approximate solutions are getting close to exact ones. This can be seen in Figures 1 and 3. It was seen from illustrative examples that the method is an efficient numerical scheme to find an approximate solution for linear or nonlinear PDEs. The authors intend to test the proposed approach on other nonlinear fractional partial differential equations such as Newell–Whitehead–Segel and Phi-four.

Data Availability

Data are available on request.

Conflicts of Interest

The authors declare that they have no competing interests.

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