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Research Article

LP-Kenmotsu Manifolds Admitting η-Ricci Solitons and Spacetime

Yanlin Li,[1] **Abdul Haseeb**,[2] **and Musavvir Ali**[3]

[1]*School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China*
[2]*Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia*
[3]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*

Correspondence should be addressed to Musavvir Ali; musavvir.alig@gmail.com

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In the present paper, LP-Kenmotsu manifolds admitting η-Ricci solitons have been studied. Moreover, some results for η-Ricci solitons in LP-Kenmotsu manifolds in the spacetime of general relativity have also been proved. Through a nontrivial example, we have given a proof for the existence of η-Ricci solitons in a 5-dimensional LP-Kenmotsu manifold.

1. Introduction

About four decades ago, the Ricci flow on a Riemannian manifold was introduced by Hamilton[1] and is governed by the following partial differential equation:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}, \tag{1}$$

where R_{ij} shows the components of Ricci Tensor. This equation has been studied by many mathematicians and used as a tool to solve the famous Poincaré conjecture problem that every closed, simply connected three-manifold is a three-sphere [2, 3].

An Einstein metric is a special case of a Ricci soliton (abbreviated as RS). A triplet (g, U, Λ) on a manifold M with Riemannian metric g, potential vector field U, and a real number Λ is a Ricci soliton (RS) if

$$\pounds_U g + 2S + 2\Lambda g = 0, \tag{2}$$

where $\pounds_U$ is the Lie derivative operator along the vector field U, S is Ricci tensor, and Λ is constant. The RS is said to be shrinking if Λ is negative, steady if Λ is zero, and expanding if Λ is positive.

A study of η-Ricci solitons (abbreviated as η-RS) in the context of contact geometry was initiated by Cho and Kimura [4], which was dealt with as a generalization of RS. Also, for Hopf hypersurfaces in complex space forms, η-RS has been studied by Calin and Crasmareanu [5]. An η-RS is a tuple (g, U, Λ, ρ) satisfying

$$\pounds_U g + 2S + 2\Lambda g + 2\rho \eta \otimes \eta = 0, \tag{3}$$

where Λ and ρ are constants. Also, for $\rho = 0$, η-RS (g, U, Λ, ρ) reduces to RS (g, U, Λ). In this connection, we recommend the papers [6–12] for more detailed study of (g, U, Λ) and (g, U, Λ, ρ).

For the four-dimensional case with Lorentzian signature (preferred in general relativity), Akbar and Woolger [13] have given a local $k < 0$ soliton, named as Schwarzschild soliton, and the geometry of the Schwarzschild soliton has been studied by Ali and Ahsan [14]. They have used the characteristic equation of λ-tensor to determine the different kinds of gravitational fields. Furthermore, such a deformation of a metric of spacetime in general relativity is reduced to special cases of two and three-dimensional hypersurfaces, and then, the Gaussian curvature is expressed in terms of eigen values of the characteristic equation. The metric of the Schwarzschild soliton is obtained by deforming the original Schwarzschild metric for a proper substitution of functions and vector fields, for which the new metric tensor satisfies (2). The relationship between two concepts, the Ricci soliton and the symmetries of spacetimes, was established by Ali and Ahsan in their paper [15].

Motivated by the above researches, here, we give a study of (g, U, Λ, ρ) on LP-Kenmotsu manifolds under certain curvature conditions. We organize our paper as follows: in Section 2, we briefly gave the preliminary concepts, results, and definitions, which are required for the study of LP-Kenmotsu manifolds. Also, this section helps the readers for a better understanding of the subsequent sections in the paper. Section 3 is concerned with the study of LP-Kenmotsu manifolds admitting (g, U, Λ, ρ). In section 4, some results for (g, U, Λ, ρ) in LP-Kenmotsu manifolds in general and spacetime of general relativity have been proved. Also, through an example of a 5-dimensional LP-Kenmotsu manifold, we prove the existence of η-Ricci soliton on the manifold.

2. Preliminaries

Let the structure $(\varphi, \zeta, \eta, g)$ be admitted by a differentiable manifold M of dimension n, then it is called as a Lorentzian almost paracontact metric manifold, provided it admits a tensor field φ of type $(1,1)$, a vector field ζ of type $(1,0)$, a 1-form η, and a Lorentzian metric g satisfying [16, 17]

$$\eta(\zeta) = -1, \tag{4}$$

$$\varphi^2 U_1 = U_1 + \eta(U_1)\zeta, \tag{5}$$

$$\begin{aligned} \varphi\zeta &= 0, \\ \eta(\varphi U_1) &= 0, \end{aligned} \tag{6}$$

$$g(\varphi U_1, \varphi U_2) = g(U_1, U_2) + \eta(U_1)\eta(U_2), \tag{7}$$

$$g(U_1, \zeta) = \eta(U_1), \tag{8}$$

$$\Phi(U_1, U_2) = \Phi(U_2, U_1) = g(U_1, \varphi U_2), \tag{9}$$

for any vector fields$U_1, U_2 \in \chi(M)$; where $\chi(M)$ is the Lie algebra of vector fields on the manifold M.

The (para) contact structure will be called a K-(para) contact if ζ is a killing vector field. Then, we have

$$\nabla_{U_1}\zeta = \varphi U_1. \tag{10}$$

Definition 1. We call a Lorentzian almost paracontact manifold M as Lorentzian para-Kenmotsu (briefly, LP-Kenmotsu) manifold, provided [18, 19]

$$\left(\nabla_{U_1}\varphi\right)U_2 = -g(\varphi U_1, U_2)\zeta - \eta(U_2)\varphi U_1, \tag{11}$$

for any U_1, U_2 on M.

In an LP-Kenmotsu manifold, the following holds:

$$\nabla_{U_1}\zeta = -U_1 - \eta(U_1)\zeta, \tag{12}$$

$$\left(\nabla_{U_1}\eta\right)U_2 = -g(U_1, U_2) - \eta(U_1)\eta(U_2), \tag{13}$$

where ∇ represents the Levi-Civita connection with respect to g.

In an LP-Kenmotsu manifold, we have [18, 19]

$$\begin{aligned} g(R(U_1, U_2)U_3, \zeta) = \eta(R(U_1, U_2)U_3) = g(U_2, U_3)\eta(U_1) \\ - g(U_1, U_3)\eta(U_2), \end{aligned} \tag{14}$$

$$R(\zeta, U_1)U_2 = g(U_1, U_2)\zeta - \eta(U_2)U_1, \tag{15}$$

$$R(U_1, U_2)\zeta = \eta(U_2)U_1 - \eta(U_1)U_2, \tag{16}$$

$$R(\zeta, U_1)\zeta = U_1 + \eta(U_1)\zeta, \tag{17}$$

$$S(U_1, \zeta) = (n-1)\eta(U_1), \tag{18}$$

$$Q\zeta = (n-1)\zeta, \tag{19}$$

$$S(\varphi U_1, \varphi U_2) = S(U_1, U_2) + (n-1)\eta(U_1)\eta(U_2), \tag{20}$$

$$\left(\pounds_\zeta g\right)(U_1, U_2) = -2g(U_1, U_2) - 2\eta(U_1)\eta(U_2), \tag{21}$$

for any $U_1, U_2, U_3 \in \chi(M)$, where R, S, and Q stand for the curvature tensor, the Ricci tensor, and the Ricci operator, respectively.

Definition 2. An LP-Kenmotsu manifold M is called an η-Einstein manifold if its Ricci tensor S is expressed as [20]

$$S = \theta_1 g + \theta_2 \eta \otimes \eta, \tag{22}$$

for smooth functions $\theta_1, \theta_2 \in C^\infty(M)$, where $C^\infty(M)$ is the set of all smooth vector fields on M. In particular, if $\theta_2 = 0$, then M is named as an Einstein manifold.

Definition 3. The conformal curvature tensor C in an n-dimensional LP-Kenmotsu manifold is defined by [20]

$$\begin{aligned} C(U_1, U_2)U_3 = R(U_1, U_2)U_3 - \frac{1}{n-2}\begin{bmatrix} S(U_2, U_3)U_1 - S(U_1, U_3)U_2 + g(U_2, U_3)QU_1 \\ -g(U_1, U_3)QU_2 \end{bmatrix} \\ + \frac{r}{(n-1)(n-2)}\left[g(U_2, U_3)U_1 - g(U_1, U_3)U_2\right]. \end{aligned} \tag{23}$$

for any $U_1, U_2, U_3 \in \chi(M)$.

Lemma 1. *An n-dimensional LP-Kenmotsu manifold satisfies the following relations:*

$$\left(\nabla_{U_1} Q\right)\zeta = QU_1 - (n-1)U_1, \tag{24}$$

$$\left(\nabla_{\zeta} Q\right)U_1 = 2QU_1 - 2(n-1)U_1. \tag{25}$$

Proof. On differentiation of $Q\zeta = (n-1)\zeta$ with respect to U_1, then using (12), we obtain (24). Furthermore, by differentiating (16) and making use of (12) and (13), we lead to

$$\begin{aligned}\left(\nabla_{U_3} R\right)(U_1, U_2)\zeta = R(U_1, U_2)U_3 - g(U_3, U_2)U_1 \\ + g(U_3, U_1)U_2.\end{aligned} \tag{26}$$

Taking a frame field and then contracting equation (26), we get

$$\sum_{i=1}^{n} \varepsilon_i g\left(\left(\nabla_{e_i} R\right)(e_i, U_2)\zeta, U_4\right) = -S(U_2, U_4) + (n-1)g(U_2, U_4), \tag{27}$$

where $\varepsilon_i = g(e_i, e_i)$. In view of Bianchi's second identity, (27) takes the form

$$\sum_{i=1}^{n} \varepsilon_i g\left(\left(\nabla_{e_i} R\right)(U_4, \zeta)U_2, e_i\right) = \left(\nabla_{U_4} S\right)(\zeta, U_2) - \left(\nabla_{\zeta} S\right)(U_4, U_2). \tag{28}$$

From the last two equations, equation (25) follows.

For a chosen distribution of matter, we impose the symmetry assumptions on the geometry when the dynamics of matter distribution are compatible. These symmetry assumptions or usually known as geometrical symmmetries of spacetime can be explained through the following definitions (for more details on symmetries see [21–23]). □

Definition 4. A spacetime is said to admit a symmetry inheritance property along a vector field ζ if for any tensor field B

$$£_{\zeta} B = 2\Omega B, \tag{29}$$

where $£_{\zeta}$ stands for Lie derivative along the vector field ζ^i, "B" denotes a geometrical/physical quantity, and Ω is a scalar function. The vector field ζ^i is either space-like ($\zeta^i \zeta_i > 0$), time-like ($\zeta^i \zeta_i < 0$), or null ($\zeta^i \zeta_i = 0$).

Definition 5. The energy-momentum tensor is an important tool, and when dealing with the concepts in Einstein's theory, it is the tensor that explains the energy density/flux and momentum density/flux in the space.

The energy-momentum tensor T_{ij} for a perfect fluid is given by

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij}, \tag{30}$$

where μ is the energy density, p is the isotropic pressure, and u_i is the velocity of fluid such that $u_i u^i = -1$ and $g_{ij}u^i = u_j$.

Definition 6 (see [24]). Any tensor, say E is said to be of Codazzi type if

$$\left(\nabla_{U_1} E\right)(U_2, U_3) = \left(\nabla_{U_2} E\right)(U_1, U_3) = \left(\nabla_{U_3} E\right)(U_1, U_2), \tag{31}$$

which in local expression can be written as

$$E_{ij;k} = E_{jk;i} = E_{ki;j}. \tag{32}$$

3. LP-Kenmotsu Manifolds Admitting η-RS

In this section, using the definitions and the concepts given in Section 2, we will prove some results on LP-Kenmotsu manifolds admitting the structure (g, U, Λ, ρ) under certain conditions.

Here, we start with equation (3), and we have

$$\begin{aligned}(£_U g)(U_1, U_2) = -2S(U_1, U_2) - 2\Lambda g(U_1, U_2) \\ - 2\rho\eta(U_1)\eta(U_2).\end{aligned} \tag{33}$$

From the covariant differentiation of (33) with respect to U_3 and using (13), we find

$$\begin{aligned}\left(\nabla_{U_3} £_U g\right)(U_1, U_2) = -2\left(\nabla_{U_3} S\right)(U_1, U_2) \\ + 2\rho[g(U_3, U_1)\eta(U_2) + g(U_3, U_2)\eta(U_1) \\ + 2\eta(U_1)\eta(U_2)\eta(U_3)].\end{aligned} \tag{34}$$

Following Yano [25], the following formula

$$\left(£_U \nabla_{U_1} g - \nabla_{U_1} £_U g - \nabla_{[U, U_1]} g\right)(U_2, U_3) = -g\left(\left(£_U \nabla\right)(U_1, U_2), U_3\right) - g\left(\left(£_U \nabla\right)(U_1, U_3), U_2\right), \tag{35}$$

is well-known for any U_1, U_2, U_3 on M. Since $\nabla g = 0$, the above relation takes the form

$$\left(\nabla_{U_1} £_U g\right)(U_2, U_3) - g\left(\left(£_U \nabla\right)(U_1, U_2), U_3\right) - g\left(\left(£_U \nabla\right)(U_1, U_3), U_2\right) = 0, \tag{36}$$

for any U_1, U_2, U_3. In fact, $£_U\nabla$ is a $(1,2)$ type of symmetric tensor, then from (36), it follows that

$$g((£_U\nabla)(U_1,U_2),U_3) = \frac{1}{2}\left(\nabla_{U_2}£_U g\right)(U_1,U_3) + \frac{1}{2}\left(\nabla_{U_1}£_U g\right)(U_2,U_3) - \frac{1}{2}\left(\nabla_{U_3}£_U g\right)(U_1,U_2). \tag{37}$$

Using (34) in (37), we obtain

$$\begin{aligned} g((£_U\nabla)(U_1,U_2),U_3) = &\left(\nabla_{U_3}S\right)(U_1,U_2) \\ &-5pt\left(\nabla_{U_2}S\right)(U_1,U_2) - \left(\nabla_{U_1}S\right)(U_2,U_3) \\ &+2\rho[g(U_1,U_2)\eta(U_3) \\ &+\eta(U_1)\eta(U_2)\eta(U_3)], \end{aligned} \tag{38}$$

which by putting $U_2 = \zeta$ reduces to

$$g((£_U\nabla)(U_1,\zeta),U_3) = \left(\nabla_{U_3}S\right)(U_1,\zeta) - \left(\nabla_{\zeta}S\right)(U_1,U_3) - \left(\nabla_{U_1}S\right)(\zeta,U_3). \tag{39}$$

Making use of (24) and (25), (39) gives

$$\left(£_U\nabla\right)(U_1,\zeta) = -2QU_1 + 2(n-1)U_1. \tag{40}$$

Differentiating (40) covariantly with respect to U_2, we have

$$\left(\nabla_{U_2}£_U\nabla\right)(U_1,\zeta) = (£_U\nabla)(U_1,U_2) + 2\eta(U_2)((n-1)U_1 - QU_1) - 2\left(\nabla_{U_2}Q\right)U_1. \tag{41}$$

Again from [25], we have

$$\left(£_U R\right)(U_1,U_2)U_3 + \left(\nabla_{U_2}£_U\nabla\right)(U_1,U_3) - \left(\nabla_{U_1}£_U\nabla\right)(U_2,U_3) = 0. \tag{42}$$

Thus, the last two equations give

$$\begin{aligned} \left(£_U R\right)(U_1,U_2)\zeta = &\, 2\left(\nabla_{U_2}Q\right)U_1 - 2\left(\nabla_{U_1}Q\right)U_2 \\ &+ 2\eta(U_1)((n-1)U_2 - QU_2) \\ &- 2\eta(U_2)((n-1)U_1 - QU_1). \end{aligned} \tag{43}$$

Fixing $U_2 = \zeta$ in (43), and using (24) and (25), it follows that

$$\left(£_U R\right)(U_1,\zeta)\zeta = 0. \tag{44}$$

The Lie differentiation of $R(U_1,\zeta)\zeta = -U_1 - \eta(U_1)\zeta$ along U leads to

$$(£_U R)(U_1,\zeta)\zeta - g(U_1,£_U\zeta)\zeta + \eta(£_U\zeta)U_1 = -(£_U\eta)(U_1)\zeta. \tag{45}$$

In view of (44), (45) takes the form

$$\left(£_U\eta\right)(U_1)\zeta = -2\eta(£_U\zeta)U_1 + g(U_1,£_U\zeta)\zeta. \tag{46}$$

Taking the Lie derivative of $g(U_1,\zeta) = \eta(U_1)$ along U, we find

$$(£_U\eta)U_1 = g(U_1,£_U\zeta) + \left(£_U g\right)(U_1,\zeta). \tag{47}$$

By replacing U_2 by ζ in equation (33)and then using (4), (8), and (18), we have

$$\left(£_U g\right)(U_1,\zeta) = 2(\rho - \Lambda - n + 1)\eta(U_1). \tag{48}$$

Again replacing U_1 by ζ in (48) leads to

$$\eta(£_U\zeta) = \rho - \Lambda - n + 1. \tag{49}$$

By using (47)–(49) in (46), we get

$$(\rho - \Lambda - n + 1)\varphi^2 U_1 = 0, \tag{50}$$

from which it follows that

$$\rho - \Lambda = n - 1, \tag{51}$$

where $\varphi^2 U_1 \neq 0$. Hence, we can write the following theorem:

Theorem 1. *If the structure (g,U,Λ,ρ) is admitted by an n-dimensional LP-Kenmotsu manifold, then the scalars Λ and ρ are related through $\rho - \Lambda = (n-1)$.*

In particular for $U = \zeta$, the authors have proved the following theorem.

Theorem 2 (see [26]). *If an n-dimensional LP-Kenmotsu manifold M admits the structure (g,ζ,Λ,ρ), then*

(i) M is an η-Einstein manifold of the following form:

$$S(U_1,U_2) = (1-\Lambda)g(U_1,U_2) + (1-\rho)\eta(U_1)\eta(U_2). \tag{52}$$

(ii) M has the constant scalar curvature given by

$$r = -n\Lambda + \rho + n - 1, \tag{53}$$

and

(iii) The scalars Λ and ρ are related by

$$\rho - \Lambda = (n-1). \tag{54}$$

In particular, for $\rho = 0$, from (54), we get $\lambda = -(n-1)$. Thus, we have the following corollaries:

Corollary 1. *An n-dimensional LP-Kenmotsu manifold admitting (g,ζ,Λ) is shrinking.*

Now, let β be a $(0,2)$ type symmetric parallel tensor, that is, $\nabla\beta = 0$. Then, we have

$$\beta\left(R(U_1,U_2)U_3,U_4\right) + \beta\left(U_3,R(U_1,U_2)U_4\right) = 0, \tag{55}$$

for any $U_1,U_2,U_3,U_4 \in \chi(M)$.

Replacing $U_1 = U_3 = U_4 = \zeta$ in (55), we have

$$\beta\left(R(\zeta,U_2)\zeta,\zeta\right) + \beta\left(\zeta,R(\zeta,U_2)\zeta\right) = 0, \tag{56}$$

which by virtue of (17) yields

$$\beta(U_2,\zeta) = -\eta(U_2)\beta(\zeta,\zeta). \tag{57}$$

By differentiating (57) with respect to ∇ along U_1, we find

$$\beta\left(\nabla_{U_1}U_2,\zeta\right)+\beta\left(U_2,\nabla_{U_1}\zeta\right)=-\left[\left(\nabla_{U_1}\eta\right)U_2+\eta\left(\nabla_{U_1}U_2\right)\right]\alpha(\zeta,\zeta)-2\eta\left(U_2\right)\beta\left(\nabla_{U_1}\zeta,\zeta\right), \tag{58}$$

which by using (57) reduces to

$$\beta\left(U_2,\nabla_{U_1}\zeta\right)=-\left(\nabla_{U_1}\eta\right)U_2\beta(\zeta,\zeta)-2\eta\left(U_2\right)\beta\left(\nabla_{U_1}\zeta,\zeta\right). \tag{59}$$

By the use of (12) and (13) in (59), we obtain

$$\alpha\left(U_2,U_1\right)=-g\left(U_2,U_1\right)\alpha(\zeta,\zeta). \tag{60}$$

By differentiating $\alpha(\zeta,\zeta)$ with respect to any vector field on M, we can easily show that $\alpha(\zeta,\zeta)$ is constant. Thus, we have the following theorem:

Theorem 3. *On an LP-Kenmotsu manifold, any symmetric parallel covariant tensor of order two is a scalar multiple of g.*

Next, considering the equation

$$\beta\left(U_1,U_2\right)=\left(\pounds_{\zeta}g\right)\left(U_1,U_2\right)+2S\left(U_1,U_2\right)+2\rho\eta\left(U_1\right)\eta\left(U_2\right). \tag{61}$$

By using (21) and (52) in (61), we obtain

$$\beta\left(U_1,U_2\right)=-2\Lambda g\left(U_1,U_2\right). \tag{62}$$

Replacing $U_1=U_2=\zeta$ in (62), we get

$$\beta(\zeta,\zeta)=2\Lambda. \tag{63}$$

Thus, by virtue of (62), it follows that

$$\left(\pounds_{\zeta}g\right)\left(U_1,U_2\right)+2S\left(U_1,U_2\right)+2\rho\eta\left(U_1\right)\eta\left(U_2\right)=-2\Lambda g\left(U_1,U_2\right). \tag{64}$$

Therefore, we have the following theorem:

Theorem 4. *On an LP-Kenmotsu manifold M, if the second-order symmetric tensor $\beta=\pounds_{\zeta}g+2S+2\rho\eta\otimes\eta$ is parallel with respect to ∇, then M admits (g,ζ,Λ,ρ).*

4. η-RS on LP-Kenmotsu Manifolds Admitting Harmonic Conformal Curvature Tensor

In the last two decades, many efforts have been devoted to the study of RS, and the cases of expanding, shrinking, and steady solitons in different manifolds have been part of recent progress on the subject. Gradient RS has always been the matter of interest of many researchers in the field of differentiable manifolds. With shrinking property, the gradient RS is observed as a finite quotient of the round sphere $\mathbb{S}^3$ or the round cylinder $\mathbb{S}^2\times\mathbb{R}$. For higher-dimensional manifold, conformally flat gradient shrinking RS is studied by Cao et al. [27], and complete gradient shrinking RS with harmonic Weyl conformal tensor is studied by Fernández-López and García-Río [28]. A study of the vector fields associated with RS in the setting of spacetimes of general relativity and their geometrical symmetries has been given by Ali and Ahsan [15]. They established the relationship between the RS in Einstein manifold and conformal curvature symmetry (Lie derivative of conformal curvature tensor vanishes along a vector field associated with RS). Due to the importance of RS and their relation with spacetime symmetries, in this section, we have proved some results for η-RS in LP-Kenmotsu manifolds in general and spacetime of general relativity in particular. First, we have the following definitions:

Definition 7. The conformal curvature tensor C is said to be harmonic if $\operatorname{div} C=0$, where "div" denotes divergence.

Definition 8. A smooth vector field V on a Riemannian manifold (M,g) of dimension n is conformal if

$$\pounds_V g=2\sigma g, \tag{65}$$

for smooth function $\sigma\in C^{\infty}(M)$, and V is closed if

$$\nabla_W V=\sigma W,\quad \text{for any } W\in\chi(M). \tag{66}$$

Conformal vector fields can be characterized by conharmonic and concircular vector fields as these have emerged in the study of conformal transformation that preserves harmonic functions and geodesic circles, respectively. Also, these are helpful tools in general relativity. In 2019–2020, Ali and others have studied spacetime symmetries defined for semiconformal curvature tensor, which is a linear combination of conformal and conharmonic curvature tensors, respectively. They have studied the divergence-free case in general spacetimes [29] and for perfect fluid spacetimes [30]. Recently, in 2021, Ali et al. [31] study the curvature inheritance symmetry of spacetimes for Ricci flat case, for Petrov type-N gravitational fields, and also in vacuum pp-waves.

Considering (g,ζ,Λ,ρ) in an n-dimensional LP-Kenmotsu manifold admitting harmonic conformal curvature tensor. Thus, (23) leads to

$$\left(\nabla_{U_1}S\right)\left(U_2,U_3\right)-\left(\nabla_{U_2}S\right)\left(U_1,U_3\right)=\frac{1}{2(n-1)}\left[\left(U_1r\right)g\left(U_2,U_3\right)-\left(U_2r\right)g\left(U_1,U_3\right)\right]. \tag{67}$$

From (52), we find

$$\left(\nabla_{U_1}S\right)\left(U_2,U_3\right)=(\rho-1)\left[g\left(U_1,U_2\right)\eta\left(U_3\right)+g\left(U_1,U_3\right)\eta\left(U_2\right)+2\eta\left(U_1\right)\eta\left(U_2\right)\eta\left(U_3\right)\right]. \tag{68}$$

By making use of (53) and (67), (68) reduces to

$$(\rho-1)g\left(\varphi U_2,\varphi U_3\right)=0, \tag{69}$$

which gives $\rho=1$, therefore, from the relation (54), we get $\Lambda=-(n-2)$. By using these values of Λ and ρ, (52) turns out to be

$$S\left(U_1,U_2\right)=(n-1)g\left(U_1,U_2\right). \tag{70}$$

Contraction of (70) over U_1 and U_2 yields

$$r=n(n-1). \tag{71}$$

Conversely, suppose that $S\left(U_1,U_2\right)=(n-1)g\left(U_1,U_2\right)$, then it is clear that $\left(\nabla_{U_1}S\right)\left(U_2,U_3\right)-\left(\nabla_{U_2}S\right)\left(U_1,U_3\right)=0$, and hence, the manifold satisfies $\operatorname{div} C=0$. Thus, we have

Theorem 5. *Let M be an n-dimensional LP-Kenmotsu manifold admitting $(g, \zeta, \Lambda, \rho)$. Then, M is of harmonic conformal curvature tensor if and only if it is an Einstein manifold. Moreover, the RS on M is always shrinking.*

Einstein's field equation without cosmological constant is given by

$$2S(U_1, U_2) = rg(U_1, U_2) + 2\kappa T(U_1, U_2), \tag{72}$$

for all $U_1, U_2 \in \chi(M)$, where T is the energy-momentum tensor and κ is the Einstein gravitational constant. By using (70) and (71), (72) takes the form

$$T(U_1, U_2) = -\frac{(n-1)(n-2)}{2\kappa} g(U_1, U_2). \tag{73}$$

Taking the Lie derivative of equation (73), we write

$$\pounds_\xi T(U_1, U_2) = \lambda \pounds_\xi g(U_1, U_2), \tag{74}$$

where $\lambda = -((n-1)(n-2)/2k)$ is constant.

Now if the contravariant vector ξ is killing, then by equation (74), we have

$$\pounds_\xi T(U_1, U_2) = 0. \tag{75}$$

If equation (75) holds good. Conversely, equation (74) confirms the existence of vector field ξ to be killing. Hence, we have a result.

Theorem 6. *If an n-dimensional LP-Kenmotsu manifold together with $(g, \zeta, \Lambda, \rho)$ satisfies Einstein's field equations without cosmological term, then the Lie derivative of the energy-momentum tensor vanishes along the vector field ξ if and only if ξ is Killing.*

As we know that the conformal motion is defined through the equation

$$\pounds_\xi g(U_1, U_2) = 2\Omega g(U_1, U_2), \tag{76}$$

the vector field ξ is then called as conformal Killing vector field.

Thus, if ξ is to be considered as conformal Killing vector, then we can write equation (74) as

$$\begin{aligned} \pounds_\xi T(U_1, U_2) &= \lambda 2\Omega g(U_1, U_2) \\ &= 2\Omega(\lambda g(U_1, U_2)) \\ &= 2\Omega\left[\frac{-(n-1)(n-2)}{2k} g(U_1, U_2)\right]. \end{aligned} \tag{77}$$

Using equation (73), we get

$$\pounds_\xi T(U_1, U_2) = 2\Omega T(U_1, U_2). \tag{78}$$

By virtue of (29), equation (78) explains the symmetry inheritance property of the energy-momentum tensor. Conversely, if equation (78) holds good then by the use of equations (73) and (74), we will get equation (75).

This gives the following result:

Theorem 7. *In an n-dimensional LP-Kenmotsu manifold with $(g, \zeta, \Lambda, \rho)$ satisfying Einstein's field equation with cosmological constant, the energy-momentum tensor has the symmetry inheritance property if and only if the vector field ξ is conformally Killing.*

Furthermore, it is easy to write one more direct result by the use of equation (73).

Theorem 8. *Let $(g, \zeta, \Lambda, \rho)$ be an η-RS in an n-dimensional LP-Kenmotsu spacetime of harmonic conformal curvature tensor. If the manifold satisfies Einstein's field equations without cosmological constant, then the energy-momentum tensor is of the form of (73).*

The energy-momentum tensor in a perfect fluid spacetime is in the following form:

$$T(U_1, U_2) = pg(U_1, U_2) + (\sigma + p)A(U_1)A(U_2), \tag{79}$$

where σ stands for the energy density, p for the isotropic pressure and $A(X)$ is a nonzero 1-form such that $g(X, F) = A(X)$, and F is a unit time-like vector field. By making use of (70), (71), and (79), (72) leads to

$$\begin{aligned} (n-1)(n-2)g(U_1, U_2) + 2\kappa[&pg(U_1, U_2) \\ &+ (p+\sigma)\eta(U_1)\eta(U_2)] = 0. \end{aligned} \tag{80}$$

Replacing $U_1 = U_2 = \zeta$ in (80), and using (4) and (8), we get

$$2\kappa\sigma = (n-1)(n-2). \tag{81}$$

Taking a frame field and then contracting (80), we get

$$2\kappa[p(n-1) - \sigma] + n(n-1)(n-2) = 0. \tag{82}$$

By the use of (81) in (82), we get

$$p + \sigma = 0. \tag{83}$$

Thus, we have the following theorem.

Theorem 9. *Let $(g, \zeta, \Lambda, \rho)$ be an η-RS in an n-dimensional LP-Kenmotsu spacetime of harmonic conformal curvature tensor. If the manifold satisfies Einstein's field equations without cosmological constant, then the matter contents of $M^n\,(n \geq 4)$ satisfy the vacuum-like equation of state.*

If we suppose a dust in a perfect fluid, then we have

$$\sigma = 3p. \tag{84}$$

Thus, from (83) and (84), it follows that $p = 0$. Thus, we have

Theorem 10. *Let $(g, \zeta, \Lambda, \rho)$ be an η-RS in an LP-Kenmotsu spacetime of harmonic conformal curvature tensor, then the manifold admitting dust for a perfect fluid is filled with radiation.*

In a local coordinate system, equation (23) can be written as

$$C^h_{bcd} = R^h_{bcd} - \frac{1}{n-2}\left(\delta^h_c R_{bd} + g_{bd}R^h_c - \delta^h_d R_{bc} - g_{bc}R^h_d\right) + \frac{r}{(n-1)(n-2)}\left(\delta^h_c g_{bd} - \delta^h_d g_{bc}\right). \tag{85}$$

Taking the divergence, we get

$$C^h_{bcd;h} = R^h_{bcd;h} - \frac{1}{n-2}\left(R_{bd;c} - R_{bc;d}\right) - \frac{r}{(n-1)(n-2)}\left(g_{bc}R_{,d} - g_{bd}R_{,c}\right). \tag{86}$$

Using the Definition 7, equation (86) implies

$$0 = R^h_{bcd;h} - \frac{1}{n-2}\left(R_{bd;c} - R_{bc;d}\right) - \frac{r}{(n-1)(n-2)}\left(g_{bc}R_{,d} - g_{bd}R_{,c}\right), \tag{87}$$

and the Bianchi identities are

$$\nabla_e R^h_{bcd} + \nabla_c R^h_{bde} + \nabla_d R^h_{bec} = 0. \tag{88}$$

Contracting the above equation, we have

$$R^h_{bcd;h} = R_{bc;d} - R^h_{bd;c}. \tag{89}$$

From equation (87), we get

$$\left(1 - \frac{1}{n-2}\right)\left(R_{bc;d} - R_{bd;c}\right) - \frac{r}{(n-1)(n-2)}\left(g_{bc}R_{,d} - g_{bd}R_{,c}\right) = 0. \tag{90}$$

For 4-dimensional spacetime, the above equation reduces to

$$\left(R_{bc;d} - R_{bd;c}\right) - \frac{r}{3}\left(g_{bc}R_{,d} - g_{bd}R_{,c}\right) = 0. \tag{91}$$

Thus, we have the following theorem:

Theorem 11. *For η-RS in 4-dimensional LP-Kenmotsu spacetime $(M, g, \zeta, \Lambda, \rho)$ of harmonic conformal curvature tensor, the Ricci tensor is covariantly constant iff the spacetime is of constant curvature.*

For purely electromagnetic distribution, Einstein's field equation is given by

$$R_{ij} = kT_{ij}. \tag{92}$$

From equations (91) and (92), we get

$$k\left(T_{bc;d} - T_{bd;c}\right) - \frac{r}{3}\left(g_{bc}T_{,d} - g_{bd}T_{,c}\right) = 0. \tag{93}$$

Furthermore, equation (79) in local coordinates can be written as

$$T_{ab} = (\sigma + p)u_a u_b + p g_{ab}. \tag{94}$$

By contracting (94), we have

$$T = -\sigma + 3p. \tag{95}$$

If T_{ij} is Codazzi type (cf., [24]), then by using (94) and (95), (93) leads to

$$g_{bc}(-\sigma + 3p)_{,d} - g_{bd}(-\sigma + 3p)_{,c} = 0. \tag{96}$$

Contracting (96) with g^{bd}, we get

$$(-\sigma + 3p)_{,d} = 0, \tag{97}$$

or

$$\nabla_c(-\sigma + 3p) = 0, \tag{98}$$

which leads to the following result.

Theorem 12. *For an η-RS in 4-dimensional LP-Kenmotsu perfect fluid spacetime $(M, g, \zeta, \Lambda, \rho)$ of harmonic conformal curvature tensor, if the energy-momentum tensor is of Codazzi type, then $(\sigma - 3p)$ is constant.*

Now, we construct an example of a 5-dimensional LP-Kenmotsu manifold to verify the existence of η-Ricci solitons and Theorem 2.

Example 1. Let the manifold $M = \{(x_1, x_2, y_1, y_2, z) \in R^5 \colon z > 0\}$ be a manifold of dimension 5, where (x_1, x_2, y_1, y_2, z) are the usual coordinates in R^5. If $\varrho_1, \varrho_2, \varrho_3, \varrho_4$, and ϱ_5 are the vector fields on M defined by

$$\begin{aligned} \varrho_1 &= z\frac{\partial}{\partial x_1}, \\ \varrho_2 &= z\frac{\partial}{\partial x_2}, \\ \varrho_3 &= z\frac{\partial}{\partial y_1}, \\ \varrho_4 &= z\frac{\partial}{\partial y_2}, \\ \varrho_5 &= z\frac{\partial}{\partial z} = \zeta, \end{aligned} \tag{99}$$

and these are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$\begin{aligned} &g(\varrho_i, \varrho_i) = 1, \quad \text{for } 1 \le i \le 4 \text{ and } g(\varrho_5, \varrho_5) = -1, \\ &g\left(\varrho_i, \varrho_j\right) = 0, \quad \text{for } i \ne j,\ 1 \le i,\ j \le 5. \end{aligned} \tag{100}$$

We define η, a 1-form as $\eta(U_1) = g(U_1, \varrho_5)$ for all $U_1 \in \chi(M)$, and let φ be the $(1,1)$-tensor field defined by

$$\begin{aligned} \varphi\varrho_1 &= -\varrho_2, \\ \varphi\varrho_2 &= -\varrho_1, \\ \varphi\varrho_3 &= -\varrho_4, \\ \varphi\varrho_4 &= -\varrho_3, \\ \varphi\varrho_5 &= 0. \end{aligned} \tag{101}$$

Using the linearity of φ and g, we yield

$$\begin{aligned}\eta(\zeta) &= g(\zeta,\zeta) = -1,\\ \varphi^2 U_1 &= U_1 + \eta(U_1)\zeta,\\ g(\varphi U_1,\varphi U_2) &= g(U_1,U_2) + \eta(U_1)\eta(U_2),\end{aligned} \tag{102}$$

for all $U_1, U_2 \in \chi(M)$. Thus, for $\varrho_5 = \zeta$, the structure (φ,ζ,η,g) defines a Lorentzian almost paracontact metric structure on manifold M. Then, we have

$$\begin{aligned}\left[\varrho_i,\varrho_j\right] &= 0, \quad \text{if } i \neq j,\ 1 \leq i,\ j \leq 4,\\ \left[\varrho_i,\varrho_5\right] &= -\varrho_i, \quad \text{for } 1 \leq i \leq 4.\end{aligned} \tag{103}$$

By using the well-known Koszul's formula, we find

$$\begin{aligned}
\nabla_{\varrho_1}\varrho_1 &= -\varrho_5,\\
\nabla_{\varrho_1}\varrho_2 &= 0,\\
\nabla_{\varrho_1}\varrho_3 &= 0,\\
\nabla_{\varrho_1}\varrho_4 &= 0,\\
\nabla_{\varrho_1}\varrho_5 &= -\varrho_1,\\
\nabla_{\varrho_2}\varrho_1 &= 0,\\
\nabla_{\varrho_2}\varrho_2 &= -\varrho_5,\\
\nabla_{\varrho_2}\varrho_3 &= 0,\\
\nabla_{\varrho_2}\varrho_4 &= 0,\\
\nabla_{\varrho_2}\varrho_5 &= -\varrho_2,\\
\nabla_{\varrho_3}\varrho_1 &= 0,\\
\nabla_{\varrho_3}\varrho_2 &= 0,\\
\nabla_{\varrho_3}\varrho_3 &= -\varrho_5,\\
\nabla_{\varrho_3}\varrho_4 &= 0,\\
\nabla_{\varrho_3}\varrho_5 &= -\varrho_3,\\
\nabla_{\varrho_4}\varrho_1 &= 0,\\
\nabla_{\varrho_4}\varrho_2 &= 0,\\
\nabla_{\varrho_4}\varrho_3 &= 0,\\
\nabla_{\varrho_4}\varrho_4 &= -\varrho_5,\\
\nabla_{\varrho_4}\varrho_5 &= -\varrho_4,\\
\nabla_{\varrho_5}\varrho_1 &= 0,\\
\nabla_{\varrho_5}\varrho_2 &= 0,\\
\nabla_{\varrho_5}\varrho_3 &= 0,\\
\nabla_{\varrho_5}\varrho_4 &= 0,\\
\nabla_{\varrho_5}\varrho_5 &= 0.
\end{aligned} \tag{104}$$

Also, one can easily verify that

$$\begin{aligned}\nabla_{U_1}\zeta &= -U_1 - \eta(U_1)\zeta,\\ \left(\nabla_{U_1}\varphi\right)U_2 &= -g(\varphi U_1,U_2)\zeta - \eta(U_2)\varphi U_1.\end{aligned} \tag{105}$$

Therefore, the manifold is an LP-Kenmotsu manifold.

Thus, the above result simply leads to obtain the non-vanishing components of the curvature tensor as

$$\begin{aligned}
R(\varrho_1,\varrho_2)\varrho_1 &= -\varrho_2,\\
R(\varrho_1,\varrho_2)\varrho_2 &= \varrho_1,\\
R(\varrho_1,\varrho_3)\varrho_1 &= -\varrho_3,\\
R(\varrho_1,\varrho_3)\varrho_3 &= \varrho_1,\\
R(\varrho_1,\varrho_4)\varrho_1 &= -\varrho_4,\\
R(\varrho_1,\varrho_4)\varrho_4 &= \varrho_1,\\
R(\varrho_1,\varrho_5)\varrho_1 &= -\varrho_5,\\
R(\varrho_1,\varrho_5)\varrho_5 &= -\varrho_1,\\
R(\varrho_2,\varrho_3)\varrho_2 &= -\varrho_3,\\
R(\varrho_2,\varrho_3)\varrho_3 &= \varrho_2,\\
R(\varrho_2,\varrho_4)\varrho_2 &= -\varrho_4,\\
R(\varrho_2,\varrho_4)\varrho_4 &= \varrho_2,\\
R(\varrho_2,\varrho_5)\varrho_2 &= -\varrho_5,\\
R(\varrho_2,\varrho_5)\varrho_5 &= -\varrho_2,\\
R(\varrho_3,\varrho_4)\varrho_3 &= -\varrho_4,\\
R(\varrho_3,\varrho_4)\varrho_4 &= \varrho_3,\\
R(\varrho_3,\varrho_5)\varrho_3 &= -\varrho_5,\\
R(\varrho_3,\varrho_5)\varrho_5 &= -\varrho_3,\\
R(\varrho_4,\varrho_5)\varrho_4 &= -\varrho_5,\\
R(\varrho_4,\varrho_5)\varrho_5 &= -\varrho_4.
\end{aligned} \tag{106}$$

We calculate the Ricci tensors as follows:

$$\begin{aligned}&S(\varrho_1,\varrho_1) = S(\varrho_2,\varrho_2) = S(\varrho_3,\varrho_3) = S(\varrho_4,\varrho_4) = 4,\\ &S(\varrho_5,\varrho_5) = -4.\end{aligned} \tag{107}$$

Hence, we find

$$\begin{aligned}r = {} & S(\varrho_1,\varrho_1) + S(\varrho_2,\varrho_2) + S(\varrho_3,\varrho_3)\\ & + S(\varrho_4,\varrho_4) - S(\varrho_5,\varrho_5) = 20.\end{aligned} \tag{108}$$

Let U_1 and U_2 be the arbitrary vector fields of an LP-Kenmotsu manifold. Then, we can write U_1 and U_2 as

$$\begin{aligned}U_1 &= U_1^1\varrho_1 + U_1^2\varrho_2 + U_1^3\varrho_3 + U_1^4\varrho_4 + U_1^5\zeta,\\ U_2 &= U_2^1\varrho_1 + U_2^2\varrho_2 + U_2^3\varrho_3 + U_2^4\varrho_4 + U_2^5\zeta,\end{aligned} \tag{109}$$

where U_i^j, $i = 1,2$, $j = 1,2,3,4,5$ denotes the scalar on the LP-Kenmotsu manifold. From straightforward calculations, we have

$$\begin{cases} \left(\pounds_{\zeta} g\right)\left(U_1, U_2\right) = g\left(\nabla_{U_1}\zeta, U_2\right) + g\left(U_1, \nabla_{U_2}\zeta\right) = -2\left(U_1^1U_2^1 + U_1^2U_2^2 + U_1^3U_2^3 + U_1^4U_2^4\right), \\ S\left(U_1, U_2\right) = 4\left(U_1^1U_2^1 + U_1^2U_2^2 + U_1^3U_2^3 + U_1^4U_2^4 - U_1^5U_2^5\right), \\ g\left(U_1, U_2\right) = U_1^1U_2^1 + U_1^2U_2^2 + U_1^3U_2^3 + U_1^4U_2^4 - U_1^5U_2^5, \\ \eta\left(U_1\right) = -U_1^5. \end{cases} \tag{110}$$

From (52) and (107), we obtain $\Lambda = -3$ and $\rho = 1$. It is obvious that the metric g of an LP-Kenmotsu manifold satisfies the η-Ricci soliton equation (33), that is,

$$\begin{aligned} \left(\pounds_{\zeta} g\right)\left(U_1, U_2\right) + 2S\left(U_1, U_2\right) + 2\Lambda g\left(U_1, U_2\right) \\ + 2\rho\eta\left(U_1\right)\eta\left(U_2\right) = 0. \end{aligned} \tag{111}$$

Therefore, the data $(g, \zeta, -3, 1)$ define an η-RS on $(M, \varphi, \zeta, \eta, g)$. Next, by using $\Lambda = -3$, $\rho = 1$, and (110), it can be seen that Theorem 2 is verified in the case of a 5-dimensional LP-Kenmotsu manifold.

5. Conclusion

In the last few years, there have been extensive studies on the Ricci solitons and their generalizations on Riemannian (as well as semi-Riemannian) manifolds. The analysis of different types of solitons on these manifolds has tremendous importance in differential geometry, gravitational physics, and the general theory of relativity. Recently, in [8], η-Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu (LP-Kenmotsu) manifolds satisfying certain curvature conditions have been studied by Haseeb and Musawa. As a continuation of this study, we tried to look at η-Ricci solitons in LP-Kenmotsu manifolds with symmetries of the spacetime of general relativity. However, this is the beginning of dealing with η-Ricci solitons and spacetime physics. In the future, we will focus on studying various types of solitons in Lorentzian manifolds combined with singularity theory and submanifolds theory, presented in [11, 12, 20, 25, 32–35] to obtain new results and theorems. Many problems of spacetime symmetries and Ricci soliton are still unresolved, and we hope that the readers of the present paper can do a good amount of work on the subject.

Data Availability

The data that were utilised to support the conclusions of this research may be found inside the paper itself.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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