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Research Article

On the Constant Edge Resolvability of Some Unicyclic and Multicyclic Graphs

Dalal Alrowaili,[1] **Zohaib Zahid**,[2] **Imran Siddique**,[2] **Sohail Zafar**,[2] **Muhammad Ahsan**,[2] **and Muhammad Sarwar Sindhu**[3]

[1]*Department of Mathematics, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia*
[2]*Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan*
[3]*Department of Mathematics, Virtual University of Pakistan, Lahore 54770, Pakistan*

Correspondence should be addressed to Imran Siddique; imransmsrazi@gmail.com

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Assume that $G = (V(G), E(G))$ is a connected graph. For a set of vertices $W_E \subseteq V(G)$, two edges $g_1, g_2 \in E(G)$ are distinguished by a vertex $x_1 \in W_E$, if $d(x_1, g_1) \neq d(x_1, g_2)$. W_E is termed edge metric generator for G if any vertex of W_E distinguishes every two arbitrarily distinct edges of graph G. Furthermore, the edge metric dimension of G, indicated by $\mathrm{edim}(G)$, is the cardinality of the smallest W_E for G. The edge metric dimensions of the dragon, kayak paddle, cycle with chord, generalized prism, and necklace graphs are calculated in this article.

1. Introduction and Preliminaries

A chemical compound's structure is typically seen as a collection of functional groups arranged on a substructure. The structure is a labeled graph from a graph-theoretic outlook, with the vertex and edge labels indicating the atom and bond types, respectively. Functional groupings and substructure are essentially subgraphs of the labeled graph representation from this perspective. A collection of compounds described by the substructure common to them is fundamentally defined by modifying the set of functional groups and permuting their locations. Chartrand et al. defined chemistry as the application of graphs to illustrate the structure of diverse chemical compounds (see [1, 2]). We employ graph principles to explain chemical structures in chemical graph theory. Chemical compounds' atomic structure can be presented using graphs. Johnson has worked on the application of structure-activity relationships by labeled graphs (see [3]).

By establishing the concept of metric dimension, Slater was able to locate the invader in a computer network (see [4]). Harary and Melter extended the work of Slater in [5]. Hosamani et al. studied the connection to chemical problems in [6]. Berhe and Wang calculated topological coindices for nanotubes and graphene sheets in [7]. Goyal et al. focused on the new composition of graphs in [8]. Ranjini et al. investigated the applications of molecular topology by degree sequence of graph operator in [9]. Imran et al. investigated chemistry problems in order to create mathematical representations of various chemical substances, with each compound having its own representation (see [10]). Navigation can be understood within a graphical structure in which the point robot or navigating agent moves from vertex to vertex of a graph. The robot can find its location by finding the distance from a fixed set of vertices also called landmarks. There is no idea of direction and visibility in a graph, but the point robot can find the distances from a fixed set of landmarks (see [11]). Melter and Tomescu worked on the metric distances used in image processing, for example, chessboard distance and the city block distance, and they also studied the applications to pattern recognition problems (see [12]). Raza investigated the chemistry application of polyphenyl and spiro chains (see [13]). Caceres et al. described the metric dimensions of graphs in coin weighing and mastermind games (see [14]).

Liu et al. worked on the application of cellular neural networks in [15]. Some applications of graphs to chemistry, biology, and physics are discussed in [16, 17]. Ali et al. studied the irregularity of graphs in detail in [18].

Let $w \in V(G)$ be a vertex with degree d_w and the total number of edges linking to w be the degree of vertex w. G's maximum and minimum degrees are indicated by the symbols $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. R is known as the metric generator of G if any two arbitrary elements of $V(G)$ can be distinguished by some element in $R \subseteq V(G)$. The metric dimension of G is the number of elements in the smallest $R \subseteq V(G)$.

Kelenc et al. [19] developed the concept of edge resolvability of graphs. The distance between edge g_1 and vertex x_1 is represented by $d(x_1, g_1)$, and it is defined as $d(x_1, g_1) = \min\{d(x_1, v_1), d(x_1, v_2)\}$, where $g_1 = v_1 v_2$ (see [19]). Edges g_1 and g_2 are distinguished by a vertex x_1 if $d(g_1, x_1) \neq d(g_2, x_1)$. For every two distinct elements $g_1, g_2 \in E(G)$, there always exists $v_1 \in W_E \subseteq V(G)$ such that$d(g_1, v_1) \neq d(g_2, v_1)$. The edge basis of G is thus known as the minimal W_E, and W_E is known as the edge metric generator. Furthermore, the minimum number of vertices in an edge basis is the edge metric dimension of G, which is denoted by $\mathrm{edim}(G)$. A graph with only one cycle is referred to as a unicyclic graph, while a graph having more than one cycle is referred to as a multicyclic graph. We will look at both unicyclic and multicyclic graphs in this article.

Kelenc et al. in [19] introduced the new concept of edim and made its comparison with metric dimension. Zubrilina in [20] calculated the $n-1$ edim of graphs which has order n. Filipovic et al. in [21] calculated the constant value of edim of graph $\mathrm{GP}(n, k)$ for fixed values of k and computed the lower bound for the rest of the values of k. Ahsan et al. in [22] worked on the bounded and unbounded edim of some graphs. In addition, Ahsan et al. in [23] calculated the constant edim of two regular graphs. Mufti et al. in [24] computed the $\mathrm{edim}(\mathrm{BS}(\mathrm{Cay}(z_n \oplus z_2)))$ of Cayley graphs. Wei et al. calculated the edim of chordal ring network and H graph in [25]. The edim of various polyphenyl chains was determined by Ahsan et al. in [26]. Alrowaili et al. computed the edim of very important classes of Toeplitz graphs in [27]. Xing et al. compared the edge resolvability with the mixed metric of wheel graphs in [28]. Koam and Ahmad calculated the edim of barycentric subdivision of Cayley graphs in [29]. Peterin and Yero studied the edim of graph operations in [30]. Zhang and Gao calculated the edim of various graphs in [31]. Liu et al. developed the idea of fault edim of some graphs in [32]. Deng et al. computed the edim of different families of mobius networks in [33].

These lemmas are helpful in determining edim values of graphs.

Lemma 1 (see [19]). *For any connected and simple graph G,*

(1) $1 + \lceil \log_2 \delta(G) \rceil \leq edim(G)$.

(2) $\log_2(\Delta(G)) \leq edim(G)$.

Lemma 2 (see [19]). *For any* $n \geq 2, \dim(P_n) = edim(P_n) = 1, dim(C_n) = edim(C_n) = 2, \dim(K_n) = edim(K_n) = n-1$. *Furthermore,* $edim(G) = 1 \Longleftrightarrow G$ *is path.*

The article is structured as follows.

We will study the edim of the dragon graph G_n^l, kayak paddle graph $G_{n,m}^l$, cycle with chord graph C_n^m, generalized prism graph $C_n * P_m$, and necklace graph Ne_n in Sections 2, 3, 4, 5, and 6, respectively. We will discuss two classes of graphs, first one is where the usual metric dimension is equal to the edge metric dimension and the second class is where the edge dimension is greater than the usual metric dimension. The graphs have symmetry according to cycles, first graph G_n^l has one cycle, second graph $G_{n,m}^l$ has two cycles, third graph C_n^m has three cycles, and last two graphs $C_n * P_m$ and Ne_n have n cycles.

2. Edge Metric Dimension of Dragon Graph G_n^l

The $\mathrm{edim}(G_n^l)$ will be computed in this section. The dragon graph G_n^l has $V(G_n^l) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_l\}$ and $E(G_n^l) = \left\{v_\xi v_{\xi+1}\colon 1 \leq \xi \leq n-1\right\} \cup \{u_s u_{s+1},\colon 1 \leq s \leq l-1\} \cup \{v_n u_1, u_1 v_1\}$. Figure 1 shows the graph G_n^l for $n = 8$ and $l = 6$. We will look at G_n^l's metric dimension in the next theorem.

Theorem 1 (see [34]). *For all* $n \geq 2, l \geq 3, \dim(G_n^l) = 2$.

The following theorem computes the $\mathrm{edim}(G_n^l)$.

Theorem 2. *For all* $n \geq 2$, $l \geq 3, edim(G_n^l) = 2$.

Proof. We must show that W_E is an edge basis for graph G_n^l if $W_E = \{v_1, u_l\} \subset V(G_n^l)$. We do this by computing each edge representation of G_n^l.

$$r\left(v_\xi v_{\xi+1} | W_E\right) = \begin{cases} (-1+\xi, l+\xi-1), & \text{if } 1 \leq \xi \leq \lfloor \frac{n}{2} \rfloor; \\ (\xi-1, n+l-1-\xi), & \text{if } \xi = \lfloor \frac{n}{2} \rfloor + 1; \\ (-\xi+1+n, n+l-1-\xi), & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \leq \xi \leq n-1; \end{cases} \tag{1}$$

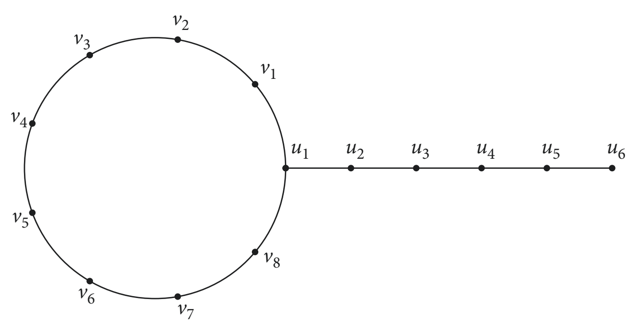

Figure 1: Dragon graph G_8^6.

$r(u_\xi u_{\xi+1}|W_E) = (\xi, l-\xi-1)$ where $1 \le \xi \le l-1$, $r(v_n u_1|W_E) = (1, l-1)$, and $r(u_1 v_1|W_E) = (0, l-1)$.

We see that any two tuples are not equal. This implies that $\mathrm{edim}(G_n^l) \le 2$ and now we try to show that $\mathrm{edim}(G_n^l) \ge 2$. Since by Lemma 2, G_n^l is not a path, $\mathrm{edim}(G_n^l) \ge 2$. As a result, $\mathrm{edim}(G_n^l) = 2$. □

3. Edge Metric Dimension of Kayak Paddle Graph $G_{n,m}^l$

The $\mathrm{edim}(G_{n,m}^l)$ will be computed in this section. The kayak paddle graph $G_{n,m}^l$ has $V(G_{n,m}^l) = \{u_1, u_2, \ldots, v_m, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_l\}$ and $E(G_{n,m}^l) = \{v_\xi v_{\xi+1}: 1 \le \xi \le n-1\} \cup \{w_\psi w_{\psi+1}: 1 \le \psi \le l-1\} \cup \{u_s u_{s+1}: 1 \le s \le m-1\} \cup \{v_n w_1, w_1 v_1, w_l u_1, u_m w_l\}$. Figure 2 shows the graph $G_{n,m}^l$ for $n=8$, $m=5$, and $l=4$. The metric dimension of $G_{n,m}^l$ is presented in the next theorem.

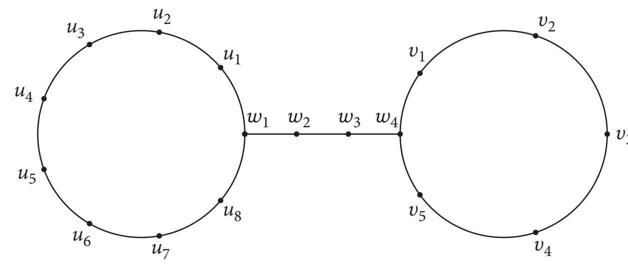

Figure 2: Kayak paddle graph $G_{8,5}^4$.

Theorem 3 (see [35]). *For $l \ge 4$, $m \ge 2$, and $n \ge 2$, $\dim(G_{n,m}^l) = 2$.*

The following theorem computes the $\mathrm{edim}(G_{n,m}^l)$.

Theorem 4. *For $l \ge 4$, $m \ge 2$, and $n \ge 2$, $\mathrm{edim}(G_{n,m}^l) = 2$.*

Proof. Let $W_E = \{v_1, u_1\} \subset V(G_{n,m}^l)$, and we have to prove that W_E is an edge basis for graph $G_{n,m}^l$. We can do it by computing each edge representation of $G_{n,m}^l$.

$$r\left(v_\xi v_{\xi+1}|W_E\right) = \begin{cases} (\xi-1, \xi+l), & \text{if } 1 \le \xi \le \lfloor \frac{n}{2} \rfloor, \\ (\xi-1, n+l-\xi), & \text{if } \xi = \lfloor \frac{n}{2} \rfloor + 1, \\ (n-\xi+1, n+l-\xi), & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \le \xi \le n-1, \end{cases} \tag{2}$$

$r(w_\xi w_{\xi+1}|W_E) = (\xi, l-\xi)$ where $1 \le \xi \le l-1$.

$$r\left(u_\xi u_{\xi+1}|W_E\right) = \begin{cases} (l+\xi, \xi-1), & \text{if } 1 \le \xi \le \lfloor \frac{m}{2} \rfloor, \\ (m+l-\xi, \xi-1), & \text{if } \xi = \lfloor \frac{m}{2} \rfloor + 1, \\ (m+l-\xi, m-\xi+1), & \text{if } \lfloor \frac{m}{2} \rfloor + 2 \le \xi \le m-1. \end{cases} \tag{3}$$

$r(v_n w_1|W_E) = (1, l)$, $r(w_1 v_1|W_E) = (0, l)$, $r(w_l u_1|W_E) = (l, 0)$, and $r(u_m w_l|W_E) = (l, 1)$.

We see that any two tuples are not equal. This implies that $\mathrm{edim}(G_{n,m}^l) \le 2$ and now we try to prove that $\mathrm{edim}(G_{n,m}^l) \ge 2$. By Lemma 2, $\mathrm{edim}(G_{n,m}^l) \ge 2$. This proves that $\mathrm{edim}(G_{n,m}^l) = 2$. □

4. Edge Metric Dimension of Cycle with Chord Graph C_n^m

The $\mathrm{edim}(C_n^m)$ will be computed in this section. The cycle with chord graph C_n^m has $V(C_n^m) = \{a_1, a_2, \ldots, a_n\}$ and $E(C_n^m) = \{a_\xi a_{\xi+1}: 1 \le \xi \le n-1\} \cup \{a_n a_1, a_1 a_m\}$. It suffices to

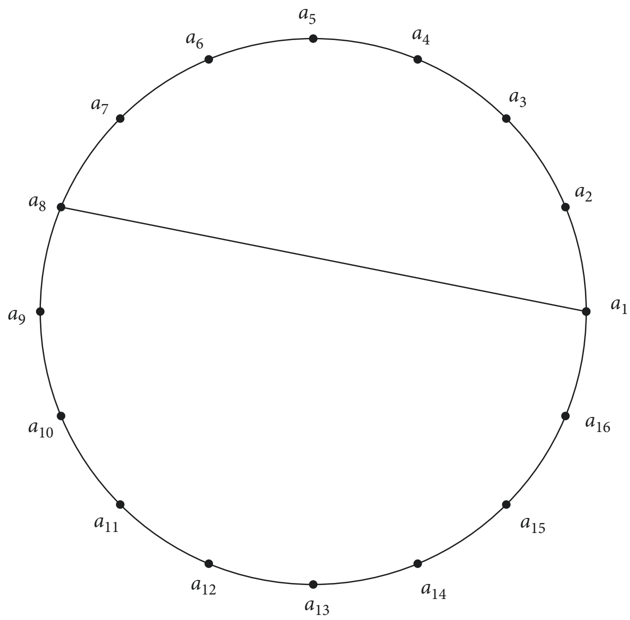

Figure 3: Cycle with chord graph C_{16}^{8}.

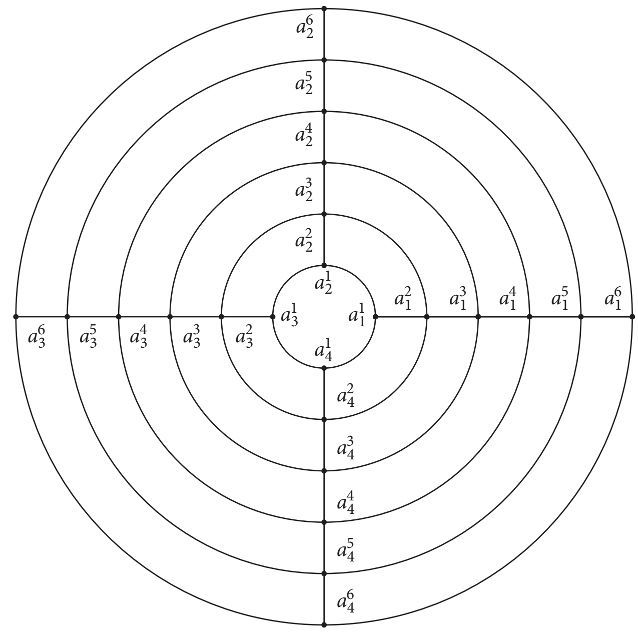

Figure 4: Graph of $C_4 * P_6$.

consider $2 < m \leq \lfloor n/2 \rfloor$. Figure 3 shows the graph C_n^m for $n = 16$ and $m = 8$. We will look at C_n^m's metric dimension in the next theorem.

Theorem 5 (see [35]). *For all* $n \geq 4$, *dim* $(C_n^m) = 2$.

The following theorem computes the edim (C_n^m).

Theorem 6. *For all* $n \geq 4$, *edim* $(C_n^m) = 2$.

Proof. To calculate the edge basis of C_n^m, the following are the four case scenarios.

Case (i). When both numbers n and m are even. With $n = 2\rho$ and $m = 2\phi$ and $W_E = \{a_{\phi+\rho}, a_{\phi+\rho+1}\} \subset V(C_n^m)$, we must show that W_E is an edge basis for graph C_n^m. We do this by computing each edge representation of C_n^m.

$$r\left(a_\xi a_{\xi+1} | W_E\right) = \begin{cases} (\xi + \rho - \phi, \xi + \rho - \phi - 1), & \text{if } 1 \leq \xi \leq \phi - 1, \\ (\xi + \rho - 1 - \phi, \xi + \rho - 1 - \phi), & \text{if } \xi = \phi, \\ (\xi + \phi - 1 + \rho, \phi - \xi + \rho), & \text{if } \phi + 1 \leq \xi \leq \rho + \phi - 1, \\ (\xi - \phi - \rho, \xi - \rho - \phi), & \text{if } \xi = \rho + \phi, \\ (\xi - \rho - \phi, \xi - 1 - \phi - \rho), & \text{if } \rho + \phi + 1 \leq \xi \leq n - 1. \end{cases} \tag{4}$$

$r(a_n a_1 | W_E) = (n - \phi - \rho, n - \phi - \rho - 1)$ and $r(a_1 a_m | W_E) = (-\phi + \rho, -\phi + \rho)$.

Case (ii). When the number n is odd and the number m is even. Let us say $n = 2\rho + 1$ and $m = 2\phi$, and let us take $W_E = \{a_{\phi+\rho}, a_{\phi+\rho+2}\} \subset V(C_n^m)$. We must show that W_E is an edge basis for graph C_n^m. We do this by computing each edge representation of C_n^m.

$$r\left(a_\xi a_{\xi+1} | W_E\right) = \begin{cases} (\xi + \rho - \phi, \xi + \rho - 1 - \phi), & \text{if } 1 \leq \xi \leq \phi - 1, \\ (\xi + \rho - \phi - 1, \xi + \rho - 1 - \phi), & \text{if } \xi = \phi, \\ (-\xi + \rho + \phi - 1, \rho + \phi - \xi), & \text{if } \phi + 1 \leq \xi \leq m - 1, \\ (-\xi + \rho + \phi - 1, \rho + \phi - \xi + 1), & \text{if } m \leq \xi \leq \rho + \phi - 1, \\ (\xi - \rho - \phi, \phi + \rho + 1 - \xi), & \text{if } \rho + \phi \leq \xi \leq \rho + \phi + 1, \\ (\xi - \phi - \rho, \xi - \phi - 2 - \rho), & \text{if } \rho + \phi + 2 \leq \xi \leq n - 1. \end{cases} \tag{5}$$

$r(a_n a_1|W_E) = (n-\phi-\rho, n-\phi-\rho-2)$ and $r(a_1 a_m|W_E) = (-\phi+\rho, -\phi+\rho)$.

Case (iii). When n is an even number and m is an odd number. Assuming $n = 2\rho$ and $m = 2\phi+1$, we must show that $W_E = \{a_{\phi+\rho}, a_{\phi+\rho+2}\} \subset V(C_n^m)$ is an edge basis for graph C_n^m. We do this by computing each edge representation of C_n^m.

$$r\left(a_\xi a_{\xi+1}|W_E\right) = \begin{cases} (\xi+\rho-\phi-1, \xi+\rho-2-\phi), & \text{if } 1 \le \xi \le \phi, \\ (-\xi+\rho-1+\phi, \xi+\rho-\phi-2), & \text{if } \xi = \phi+1, \\ (-\xi+\rho+\phi-1, -\xi+\rho+\phi), & \text{if } \phi+2 \le \xi \le m-1, \\ (-\xi+\rho-1+\phi, -\xi+\phi+\rho+1), & \text{if } m \le \xi \le \rho+\phi-1, \\ (\xi-\phi-\rho, -\xi+\rho+\phi+1), & \text{if } \rho+\phi \le \xi \le \rho+\phi+1, \\ (\xi-\rho-\phi, \xi-\rho-\phi-2), & \text{if } \rho+\phi+2 \le \xi \le n-1. \end{cases} \tag{6}$$

$r(a_n a_1|W_E) = (n-\phi-\rho, n-\phi-\rho-2)$ and $r(a_1 a_m|W_E) = (-\phi+\rho-1, -\phi+\rho-1)$.

Case (iv).When both numbers n and m are odd. Assuming $n = 2\rho+1$ and $m = 2\phi+1$, we must show that $W_E = \{a_{\rho+\phi+1}, a_{\rho+\phi+2}\} \subset V(C_n^m)$ is an edge basis for graph C_n^m. We do this by computing each edge representation of C_n^m.

$$r\left(a_\xi a_{\xi+1}|W_E\right) = \begin{cases} (\xi+\rho-\phi, \xi+\rho-\phi-1), & \text{if } 1 \le \xi \le \phi, \\ (-\xi+\rho+\phi, \xi+\rho-\phi-1), & \text{if } \xi = \phi+1, \\ (-\xi+\rho+\phi, -\xi+\rho+1+\phi), & \text{if } \phi+2 \le \xi \le \rho+\phi, \\ (\xi-\rho-\phi-1, -\xi+\rho+1+\phi), & \text{if } \xi = \rho+\phi+1, \\ (\xi-\rho-\phi-1, \xi-\rho-\phi-2), & \text{if } \rho+\phi+2 \le \xi \le n-1. \end{cases} \tag{7}$$

$r(a_n a_1|W_E) = (n-\phi-\rho-1, n-\phi-\rho-2)$ and $r(a_1 a_m|W_E) = (-\phi+\rho, -\phi+\rho)$.

In all four case scenarios, we can observe that no two tuples have the same representations. This implies that $\mathrm{edim}(C_n^m) \le 2$. Because C_n^m is not a path according to Lemma 2, $\mathrm{edim}(C_n^m) \ge 2$. As a result, $\mathrm{edim}(C_n^m) = 2$. □

5. Edge Metric Dimension of Generalized Prism Graph $C_n * P_m$

In this part, the $\mathrm{edim}(C_n * P_m)$ will be calculated. The Cartesian product of path on m vertices and cycle on n vertices is the generalized prism graph $C_n * P_m$. $V(C_n * P_m) = \{a_\xi^\psi\colon 1 \le \xi \le n, 1 \le \psi \le m\}$ and $E(C_n * P_m) = \{a_\xi^\psi a_{\xi+1}^\psi\colon 1 \le \xi \le n-1, 1 \le \psi \le m\} \cup \{a_\xi^\psi a_\xi^{\psi+1}\colon 1 \le \xi \le n, 1 \le \psi \le m-1\} \cup \{a_n^\psi a_1^\psi\colon 1 \le \psi \le m\}$ in the generalized prism graph $C_n * P_m$. Furthermore, $C_n * P_2$ is referred to as a prism graph. Figure 4 shows the $C_n * P_m$ graph for $n = 4$ and $m = 6$. We will look at $C_n * P_m$'s metric dimension in the next theorem.

Theorem 7 (see [14]). *For all $n \ge 3$ and $m \ge 2$, we have*

$$\dim(C_n * P_m) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 3. *For $n \ge 2$ and $m \ge 3$, $\mathrm{edim}(C_n * P_m) \ge 3$.*

Proof. Since $\delta(C_n * P_m) = 3$, by Lemma 1, we have $\mathrm{edim}(C_n * P_m) \ge 3$.

In the following theorem, the $\mathrm{edim}(C_n * P_m)$ is determined. □

Theorem 8. *For all $n \ge 2$ and $m \ge 3$, $\mathrm{edim}(C_n * P_m) = 3$.*

Proof. The following two case scenarios are used to compute the edge basis of $C_n * P_m$.

Case (i). When the number n is even. Let $n = 2k$, $k \ge 2$, where k is a positive integer, and $W_E = \{a_1^1, a_2^1, a_{k+1}^1\} \subset V(C_n * P_m)$; we must show that

W_E is an edge basis for graph $C_n * P_m$. Each edge representation of $C_n * P_m$ is computed for this purpose.

$$r\left(a_{\xi}^{\psi}a_{\xi+1}^{\psi}|W_E\right) = \begin{cases} \left(\psi-1, \psi-1, \dfrac{n}{2}+\psi-2\right), & \text{if } \xi = 1, 1 \le \psi \le m, \\ \left(\xi+\psi-2, \xi+\psi-3, \dfrac{n}{2}-\xi+\psi-1\right), & \text{if } 2 \le \xi \le \dfrac{n}{2}, 1 \le \psi \le m, \\ \left(n-\xi+\psi-1, \xi+\psi-3, \xi-\dfrac{n}{2}+\psi-2\right), & \text{if } \xi = \dfrac{n}{2}+1, 1 \le \psi \le m; \\ \left(n-\xi+\psi-1, n-\xi+\psi, \xi-\dfrac{n}{2}+\psi-2\right), & \text{if } \dfrac{n}{2}+2 \le \xi \le n-1, 1 \le \psi \le m. \end{cases} \tag{9}$$

$r(a_n^{\psi}a_1^{\psi}|W_E) = (\psi-1, \psi, \psi+(n/2)-2)$ where $1 \le \psi \le m$.

$$r\left(a_{\xi}^{\psi}a_{\xi}^{\psi+1}|W_E\right) = \begin{cases} \left(\psi+\xi-2, \psi, \dfrac{n}{2}+\psi-\xi\right), & \text{if } \xi = 1, 1 \le \psi \le m-1, \\ \left(\psi+\xi-2, \psi+\xi-3, \dfrac{n}{2}+\psi-\xi\right), & \text{if } 2 \le \xi \le \dfrac{n}{2}+1, 1 \le \psi \le m-1, \\ \left(\psi+n-\xi, \psi+\xi-3, \psi+\xi-\dfrac{n}{2}-2\right), & \text{if } \xi = \dfrac{n}{2}+2, 1 \le \psi \le m-1, \\ \left(\psi+n-\xi, \psi+n-\xi+1, \psi+\xi-\dfrac{n}{2}-2\right), & \text{if } \dfrac{n}{2}+3 \le \xi \le n, 1 \le \psi \le m-1. \end{cases} \tag{10}$$

Case (ii). When n is odd. Let $n = 2k-1, k \ge 2$, where k is a positive integer and $W_E = \{a_1^1, a_2^1, a_{k+1}^1\} \subset V(C_n * P_m)$, and we have to prove that W_E is an edge basis for graph $C_n * P_m$. For this purpose, we compute each edge representation of $C_n * P_m$.

$$r\left(a_{\xi}^{\psi}a_{\xi+1}^{\psi}|W_E\right) = \begin{cases} \left(\psi-1, \psi-1, \dfrac{n+1}{2}+\psi-2\right), & \text{if } \xi = 1, 1 \le \psi \le m, \\ \left(\xi+\psi-2, \xi+\psi-3, \dfrac{n+1}{2}-\xi+\psi-1\right), & \text{if } 2 \le \xi \le \dfrac{n+1}{2}, 1 \le \psi \le m, \\ \left(n-\xi+\psi-1, n-\xi+\psi, \xi-\dfrac{n+1}{2}+\psi-2\right), & \text{if } \dfrac{n+1}{2}+1 \le \xi \le n-1, 1 \le \psi \le m. \end{cases} \tag{11}$$

$r(a_n^{\psi}a_1^{\psi}|W_E) = (\psi-1, \psi, \psi+(n+1/2)-3)$ where $1 \le \psi \le m$.

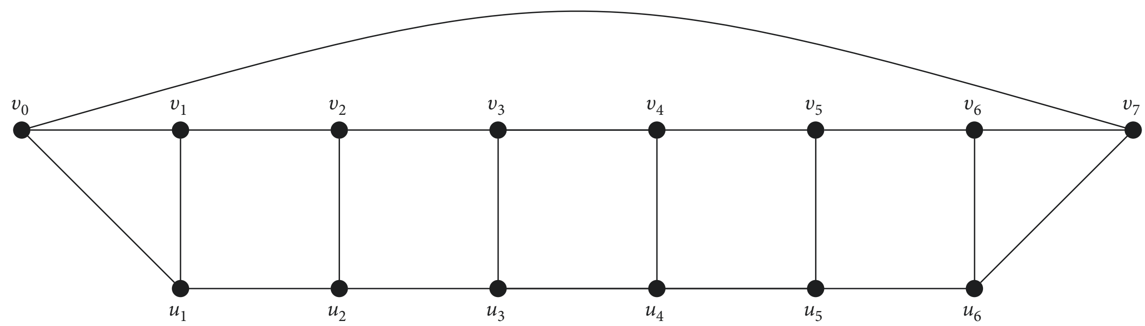

Figure 5: Necklace graph Ne_6.

$$r\left(a_{\xi}^{\psi} a_{\xi}^{\psi+1} | W_E\right) = \begin{cases} \left(\psi+\xi-2, \psi, \dfrac{n+1}{2}+\psi-\xi-1\right), & \text{if } \xi = 1, 1 \le \psi \le m-1, \\ \left(\psi+\xi-2, \psi+\xi-3, \dfrac{n+1}{2}+\psi-\xi\right), & \text{if } 2 \le \xi \le \dfrac{n+1}{2}, 1 \le \psi \le m-1, \\ \left(\psi+\xi-3, \psi+\xi-3, \psi-\xi+\dfrac{n+1}{2}\right), & \text{if } \xi = \dfrac{n+1}{2}+1, 1 \le \psi \le m-1, \\ \left(\psi+n-\xi, \psi+n-\xi+1, \psi+\xi-\dfrac{n+1}{2}-2\right), & \text{if } \dfrac{n+1}{2}+2 \le \xi \le n, 1 \le \psi \le m-1. \end{cases} \tag{12}$$

We see that any two tuples are not equal in both of the cases. This implies that $\mathrm{edim}(C_n * P_m) \le 3$. Now by Lemma 3, we have $\mathrm{edim}(C_n * P_m) \ge 3$. Hence, $\mathrm{edim}(C_n * P_m) = 3$. □

6. Edge Metric Dimension of Necklace Graph Ne_n

The $\mathrm{edim}(Ne_n)$ will be computed in this section. The necklace graph Ne_n has $V(Ne_n) = \{v_0, v_{n+1}, v_\varphi, u_\varphi\colon 1 \le \varphi \le n\}$ and $E(Ne_n) = \{v_\xi v_{\xi+1}, u_\xi u_{\xi+1}, u_\xi\ v_\xi\colon 1 \le \xi \le n-1\} \cup \{v_n v_{n+1}, v_{n+1} v_0, v_0 v_1, v_0 u_1, u_n v_{n+1}, u_n v_n\}$. Figure 5 shows the Ne_n family for $n = 9$. We will look at Ne_n's metric dimension in the next theorem.

Theorem 9 (see [36]). *For all $n \ge 1$, we have*

$$\dim(Ne_n) = \begin{cases} 3, & \text{if } n \text{ is odd;} \\ 2, & \text{otherwise.} \end{cases} \tag{13}$$

Lemma 4. *For $n \ge 2$, Ne_n is the necklace graph family; then, $edim(Ne_n) \ge 3$.*

Proof. Since $\delta(Ne_n) = 3$, by Lemma 1, we have $\mathrm{edim}(Ne_n) \ge 3$.

The $\mathrm{edim}(Ne_n)$ is computed in the following theorem. □

Theorem 10. *For all$n \ge 2$, $edim(Ne_n) = 3$.*

Proof. If $W_E = \{v_1, v_{\lceil n/2 \rceil}, v_n\} \subset V(Ne_n)$, we must show that W_E is an edge basis for graph Ne_n. We do this by computing each edge representation of Ne_n.

$$r\left(v_\xi v_{\xi+1}|W_E\right)=\begin{cases}\left(0,\lceil\frac{n}{2}\rceil-1,2\right), & \text{if } \xi=0,\\ \left(\xi-1,-\xi-1+\lceil\frac{n}{2}\rceil,\xi+2\right), & \text{if } 1\le\xi\le\lceil\frac{n}{2}\rceil-2,\\ \left(\xi-1,-\xi-1+\lceil\frac{n}{2}\rceil,-\xi+n-1\right), & \text{if } \xi=\lceil\frac{n}{2}\rceil-1,\\ \left(\xi-1,\xi-\lceil\frac{n}{2}\rceil,-\xi-1+n\right), & \text{if } \lceil\frac{n}{2}\rceil\le\xi\le\lceil\frac{n}{2}\rceil+1,\\ \left(-\xi+n+2,\xi-\lceil\frac{n}{2}\rceil,-\xi-1+n\right), & \text{if } \lceil\frac{n}{2}\rceil+2\le\xi\le n-1,\\ \left(-\xi+2+n,\xi-\lceil\frac{n}{2}\rceil,\xi-n\right), & \text{if } \xi=n,\end{cases}$$

$$r\left(u_n v_{n+1}|W_E\right)=\left(2,\lceil\frac{n}{2}\rceil+1,1\right),$$

$$r\left(u_\xi u_{\xi+1}|W_E\right)=\begin{cases}\left(\xi,\lceil\frac{n}{2}\rceil-\xi,\xi+2\right), & \text{if } 1\le\xi\le\lceil\frac{n}{2}\rceil-2,\\ \left(\xi,-\xi+\lceil\frac{n}{2}\rceil,n-\xi\right), & \text{if } \xi=\lceil\frac{n}{2}\rceil-1,\\ \left(\xi,\xi+1-\lceil\frac{n}{2}\rceil,-\xi+n\right), & \text{if } \xi=\lceil\frac{n}{2}\rceil,\\ \left(-\xi+n+2,\xi+1-\lceil\frac{n}{2}\rceil,-\xi+n\right), & \text{if } \lceil\frac{n}{2}\rceil+1\le\xi\le n-1,\end{cases} \tag{14}$$

$$r\left(v_0 u_1|W_E\right)=\left(1,\lceil\frac{n}{2}\rceil,2\right),$$

$$r\left(u_\xi v_\xi|W_E\right)=\begin{cases}\left(\xi-1,\lceil\frac{n}{2}\rceil-\xi,\xi+2\right), & \text{if } 1\le\xi\le\lceil\frac{n}{2}\rceil-2,\\ \left(\xi-1,-\xi+\lceil\frac{n}{2}\rceil,-\xi+n\right), & \text{if } \xi=\lceil\frac{n}{2}\rceil-1,\\ \left(\xi-1,\xi-\lceil\frac{n}{2}\rceil,-\xi+n\right), & \text{if } \lceil\frac{n}{2}\rceil\le\xi\le\lceil\frac{n}{2}\rceil+1,\\ \left(-\xi+3+n,\xi-\lceil\frac{n}{2}\rceil,n-\xi\right), & \text{if } \lceil\frac{n}{2}\rceil+2\le\xi\le n.\end{cases}$$

$$r\left(v_{n+1} v_0|W_E\right)=\left(1,\lceil\frac{n}{2}\rceil,1\right).$$

We see that any two tuples are not equal. This implies that $\operatorname{edim}(Ne_n)\le 3$. Now we only prove that $\operatorname{edim}(Ne_n)\ge 3$. So, by Lemma 4, we have $\operatorname{edim}(Ne_n)\ge 3$. Hence, $\operatorname{edim}(Ne_n)=3$. □

7. Conclusion

The edge metric dimension of the unicyclic and multicyclic graphs, dragon, kayak paddle, cycle with chord, generalized prism, and necklace, has been calculated in this paper. It should be underlined that the edge metric dimensions of all the described graphs are constant and do not rely on the number of vertices in the graph. The edge metric dimension of the dragon graph, kayak paddle graph, and cycle with chord graph is determined to be two, whereas the edge metric dimension of the generalized prism graph and necklace graph is determined to be three. We come to a halt here with an unsolved problem. The open problem is distinguishing between all forms of graphs in which the edge metric and metric dimensions are the same.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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