Research Article

(Q, T)-affine-periodic solutions and Pseudo (Q, T)-affine-periodic solutions for Dynamic Equations on Time Scales

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The aim of this paper is to study the existence of (Q, T)-affine-periodic solutions for affine-periodic systems on time scales of the type \( x_{Δ}(t) = A(t)x(t) + f(t) \) and \( x_{Δ}(t) = A(t)x(t) + g(t, x(t)) \), \( t \in T \), assuming that corresponding homogeneous equation of this system admits exponential dichotomy. The result is also extended to the case of pseudo (Q, T)-affine-periodic solutions. The main approaches are based on the Banach contraction mapping principle, but certain technical aspects on time scales are more complicated.

1. Introduction

In the mathematical theory of dynamical systems, exponential dichotomy plays a significant role in the analysis of nonautonomous dynamical systems. Exponential dichotomy was first studied by Lyapunov and Poincaré in the late nineteenth century. Perron [1] developed the exponential dichotomy of linear differential equations, particularly, it has attracted wide attention in the field of stable and unstable invariant manifolds, and therefore has attracted much attention, see, for example, Coppel [2], Chow and Leiva [3], and Fink [4], and references therein. Li [5] studied analogous results for nonautonomous discrete time dynamical systems. Pötzche [6] introduced the notion of the exponential dichotomy in the calculus on measure chains. Chu et al. [7] introduced a definition for nonuniform dichotomy spectrum and proved the reducibility result.

As we know, discrete time systems and continuous systems play important role in theory and application. The theory of time scales is a recent subject of research, which was introduced by Stefan Hilger in 1988 in order to unified continuous time and discrete time dynamic systems (see [8]). A time scale is a nonempty closed subset of real numbers, denoted by T. Since the theory of time scales can also describe continuous and discrete hybrid processes, it has an important role to model realistic problems, for instance, in the study of finance, economic, population models, neural networks, quantum physics, and technology. See [9–17] for more details.

Since Kepler and Newton studied the motion of celestial bodies, the research of periodic solutions has a long history. As we know, the problem of existence and uniqueness of a periodic solution of differential equations has been the main research topic of dynamic systems. Recently, the concept of affine periodicity has been introduced by Li et al. [18–23], which are not only periodic in time, but also symmetric in space. This solution is a kind of periodic (\( Q = I_{l} \)), anti-periodic (\( Q = I_{l} \)), and rotation periodic (\( Q \in O(l) \)), which are discussed in many papers like [24, 25]. For general affine matrix \( Q \), Li et al. [19, 20, 26] obtained the existence of affine-periodic solutions for several kinds of affine-periodic systems.

Recently, the qualitative properties of solutions of dynamic equations on time scales have been attracting of several researchers [27], specially concerning their periodicity [28–30]. On the contrary, Li and Wang in [31, 32] obtained almost periodicity on time scales. C. Lizama and
L. G. Mesquita (see [33]) introduced the existence of almost automorphic on time scales. After that, Wang and Li proved the existence of \((Q, T)\)-affine-periodic solutions on time scales via topological degree theory in [34].

As we know, exponential dichotomy is one of the most important methods and tools in the study of periodic solutions of difference equations and differential equations. C. Cheng and Y. Li considered nonhomogeneous linear differential equations and semilinear differential equations and proved the existence of \((Q, T)\)-affine-periodic solutions in [35]. Inspired by the above discussion, the main purpose of this paper is to investigate the existence of \((Q, T)\)-affine-periodic solutions and pseudo \((Q, T)\)-affine-periodic solutions when the corresponding homogeneous linear equations have exponential dichotomy on time scales.

The rest of this paper is arranged as follows: in Section 2, we give some preliminaries concerning the theory of time scales. We introduce the definitions of related concepts and some properties for dynamic equations on time scales. In Section 3, we first present some concepts for \((Q, T)\)-affine-periodic solutions on time scales. We show the existence of \((Q, T)\)-affine-periodic solutions of first-order linear dynamic equations on time scales and prove the existence and uniqueness of \((Q, T)\)-affine-periodic solutions for semilinear dynamic equations on time scales. In Section 4, we obtain the pseudo \((Q, T)\)-affine-periodic solutions for semilinear dynamic equations on time scales.

2. Preliminaries

In this section, we will introduce some basic notations, definitions, and results concerning the calculus on time scales which can be found in [36, 37].

A time scale \(\mathbb{T}\) is a closed and nonempty subset of \(\mathbb{R}\). It follows that forward jump operator \(\sigma: \mathbb{T} \rightarrow \mathbb{T}\), backward jump operator \(\rho: \mathbb{T} \rightarrow \mathbb{T}\), and the graininess function \(\mu: \mathbb{T} \rightarrow [0, \infty)\), respectively, as follows:

\[
\begin{align*}
\sigma(t) &= \inf\{s \in \mathbb{T}: s > t\}, \\
\rho(t) &= \sup\{s \in \mathbb{T}: s < t\}, \\
\mu(t) &= \sigma(t) - t.
\end{align*}
\]

In this definition, we supplement \(\inf\emptyset = \sup\emptyset = \sup\mathbb{T}\) and \(\sup\emptyset = \inf\mathbb{T}\). If \(\sigma(t) > t\), \(\sigma(t) = t\), \(\rho(t) < t\), \(\rho(t) = t\), we say that the point \(t \in \mathbb{T}\) is right-scattered, right-dense, left-scattered, left-dense, respectively. If \(\mathbb{T}\) has a left-scattered maximum \(m\), \(\mathbb{T}^\kappa = \mathbb{T}\setminus\{m\}\). Otherwise, \(\mathbb{T}^\kappa = \mathbb{T}\). Throughout this paper, we use \([a, b]_{\mathbb{T}}\) to denote a closed interval restricted on \(\mathbb{T}\). That is, \([a, b]_{\mathbb{T}} = \{t \in \mathbb{T}: a \leq t \leq b\}\).

**Definition 1.** Assume \(u: \mathbb{T} \rightarrow \mathbb{R}^n\) and \(t \in \mathbb{T}^\kappa\). The delta derivative of the vector \(u\) at \(t\) is \(u^\Delta(t)\) with the property if for given any \(\varepsilon > 0\), there is a neighborhood \(U \subset \mathbb{T}\) of \(t\) such that

\[
|u^\Delta(t) - u(s) - u^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,
\]

for all \(s \in U\).

Throughout this paper, we use \(C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{R}^n) = C_{rd}(\mathbb{T}, \mathbb{R}^n)\) to denote the set of \(rd\)-continuous functions. We use \(\mathcal{R} = \mathcal{R}(\mathcal{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})\) to denote the set of all regressive and \(rd\)-continuous functions from \(\mathbb{T} \rightarrow \mathbb{R}\).

Next we will introduce the definition of periodic time scale.

**Definition 2.** We say that a time scale \(\mathbb{T}\) is called \(T\)-periodic if there exists \(T > 0\) such that \(t \in \mathbb{T}\) implies \(t \pm T \in \mathbb{T}\). For \(\mathbb{T} \neq \mathbb{R}\), the smallest positive \(T\) is called the period of the time scale.

In what follows, we present the definition about the generalized exponential function \(e_p(t, s)\). For more details, see [36, 37].

**Definition 3.** Given \(p \in \mathcal{R}\), the exponential function is defined by

\[
e_p(t, s) = \begin{cases} 
\exp\left(\int_s^t \frac{1}{\mu(\tau)} \log(1 + p(\tau)) d\tau\right), & \mu(\tau) > 0, \\
\exp\left(\int_s^t p(\tau) d\tau\right), & \mu(\tau) = 0,
\end{cases}
\]

where \(\log\) is the principal logarithm function.

Suppose that \(p, q \in \mathcal{R}\). Then \((p \oplus q)(t)\) and \((\oplus p)(t)\) can be defined as follows:

\[
(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)\text{ for all } t \in \mathbb{T}^k. \\
(\oplus p)(t) = \frac{-p(t)}{1 + \mu(t)p(t)}\text{ for all } t \in \mathbb{T}^k.
\]

**Definition 4.** Assume \(A(t)\) is an \(n \times n\) \(rd\)-continuous matrix-valued function on \(\mathbb{T}\). We say that the linear dynamic system

\[
X^\Delta(t) = A(t)X(t),
\]

admits an exponential dichotomy on \(\mathbb{T}\) if there exist a projection \(P\) on \(\mathbb{R}^n\) and positive constants \(K\) and \(\alpha\) satisfying

\[
|X(t)PX^{-1}(s)| \leq Ke_{\alpha}(t, s), s, t \in \mathbb{T}, t \geq s, \\
|X(t)(I - P)X^{-1}(s)| \leq Ke_{\alpha}(s, t), s, t \in \mathbb{T}, t \geq s,
\]

where \(X(t)\) is a fundamental solution matrix of (6) and \(I\) is the identity matrix.

The following theorem will be essential to our main result, for more details see [36, Theorem 2.39].

**Theorem 1.** If \(p \in \mathcal{R}\) and \(a, b, c \in \mathbb{T}\), then

\[
\begin{align*}
[e_p(c, \cdot)]^\Delta &= -p[e_p(c, \cdot)]^\sigma, \\
\int_a^b p(t)e_p(c, \sigma(t))d\tau &= e_p(c, a) - e_p(c, b).
\end{align*}
\]
The following result shows that $e_{op}(t,s)$ is a bounded function (see [33, Theorem 2.14]).

**Lemma 1.** If $p > 0, t > 0$, then $e_{op}(t,s) \leq 1$ for $t, s \in \mathbb{T}$ such that $t > s$.

### 3. Exponential Dichotomy and $(Q, T)$-Affine-Periodic Solutions of Dynamic Equations on Time Scales

We start by introducing definitions of $(Q, T)$-affine-periodic systems on time scales.

Consider the $(Q, T)$-affine-periodic dynamic system on time scales

$$x^\Delta = f(t, x),$$

(10)

where $f: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is rd-continuous, $\mathbb{T}$ is a $T$-periodic time scale and ensures the uniqueness of solutions with respect to initial value (for more details in [37]). We always assume $Q \in \text{GL}(\mathbb{R}^n)$ in this paper.

**Definition 5** (see [38]). The system (10) is said to be a $(Q, T)$-affine-periodic system, if there exists $Q$ such that

$$f(t + T, x) = Qf(t, Q^{-1}x),$$

(11)

holds for all $(t, x) \in \mathbb{T} \times \mathbb{R}^n$.

**Definition 6** (see [38]). A function $x: \mathbb{T} \rightarrow \mathbb{R}^n$ is called a solution of (10) if $x \in \{ y: y \in C(\mathbb{T}, \mathbb{R}^n), y^\Delta \in C_d(\mathbb{T}, \mathbb{R}^n) \}$ and $x(t)$ satisfies (10) for all $t \in \mathbb{T}$.

Definition 7 (see [38]). A function $x: \mathbb{T} \rightarrow \mathbb{R}^n$ is said to be an $(Q, T)$-affine-periodic solution of $(Q, T)$-affine-periodic system (10) if $x(\sigma(t))$ is a solution of (10) and for any $t \in \mathbb{T}$:

$$x(\sigma(t) + T) = Qx(\sigma(t)).$$

(12)

**Remark 1.** If $t$ is right-dense, then function $x: \mathbb{T} \rightarrow \mathbb{R}^n$ is an $(Q, T)$-affine-periodic solution of $(Q, T)$-affine-periodic system (10) provided $x(t)$ is a solution of (10) and for any $t \in \mathbb{T}$,

$$x(t + T) = Qx(t).$$

(13)

#### 3.1. $(Q, T)$-Affine-Periodic Solutions of First-Order Linear Dynamic Equations on Time Scales

Consider the linear nonhomogeneous dynamic equation on time scales

$$x^\Delta(t) = A(t)x(t) + f(t),$$

(14)

where $f: \mathbb{T} \rightarrow \mathbb{R}^n, A: \mathbb{T} \rightarrow \mathbb{R}^{n \times n}, \mathbb{T}$ is a $T$-periodic time scale, and its associated homogeneous equation is as follows:

$$x^\Delta(t) = A(t)x(t).$$

(15)

**Theorem 2.** Let $A(t), f(t)$ be $(Q, T)$-affine-periodic on $\mathbb{T}$, i.e., $A(t + T) = QA(t)Q^{-1}, f(t + T) = Qf(t)$. Also, suppose that linear dynamic equation (15) has an exponential dichotomy with projection $P$. Then nonhomogeneous linear differential equation (14) has a $(Q, T)$-affine-periodic solution.

**Proof.** By exponential dichotomy of (15), we have

$$x(t) = \int_{-\infty}^{t} X(t)P X^{-1}(\sigma(s))f(s)\Delta s = \int_{-\infty}^{t} X(t)(I - P) X^{-1}(\sigma(s))f(s)\Delta s.$$

(16)

$$= Q^{-1}X(\sigma(t) + T) - X(t + T)Q^{-1}(T)Q \mu(t)$$

(17)

$$= Q^{-1}X^\Delta(t + T)X^{-1}(T)Q \mu(t)$$

(18)

$$= A(t)\Psi(t).$$

(b) If $t$ is a right-dense point, then we have

$$\Psi^\Delta(t) = \lim_{\Delta t \rightarrow 0} \frac{\Psi(t + \Delta t) - \Psi(t)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{Q^{-1}X(t + \Delta t + T)X^{-1}(T)Q - Q^{-1}X(t + T)X^{-1}(T)Q}{\Delta t}$$

(19)
we get uniqueness theorem on time scales (see \([36], \text{Section 8.3}\)), with the initial value \(\Psi\).

Consider the following semilinear dynamic equation:

\[\Box\]

\[X(t + T) = QX(t)Q^{-1}X(T).\]  

By (19) and variable substitution, we get for all \(t \in T\),

By (a) and (b), we see that \(\Psi(t)\) is the solution of (15) with the initial value \(\Psi(0) = I\). By the existence and uniqueness theorem on time scales (see \([36], \text{Section 8.3}\)), we get \(\Psi(t) = X(t)\). Thus

\[x(\sigma(t) + T) = \int_{-\infty}^{\sigma(t)+T} X(\sigma(t) + T)P\chi_{1}(\sigma(s))f(s)ds - \int_{-\infty}^{\sigma(t)+T} X(\sigma(t) + T)(I - P)X^{-1}(\sigma(s))f(s)\Delta s.\]

Due to the exchangeability of matrix multiplication, i.e.,

\[Q^{-1}X(T)P = PQ^{-1}X(T),\]  

\[Q^{-1}X(T)(I - P) = (I - P)Q^{-1}X(T),\]

we have

\[x(\sigma(t) + T) = Qx(\sigma(t))\]

Thus, we confirm that \(x(\sigma(t) + T) = Qx(\sigma(t))\) and we get the desired result.

3.2. \((Q, T)\)-Affine-Periodic Solutions for Semilinear Dynamic Equations on Time Scales. Consider the following semilinear dynamic equation:

\[x^\Delta(t) = A(t)x(t) + f(t, x),\]  

where \(A: T \to \mathbb{R}^{n \times n}\) and \(f: T \to \mathbb{R}^n\), \(T\) is a \(T\)-periodic time scale, \(A(t), f(t, x)\) are \((Q, T)\)-affine-periodic, i.e.,

\[x(t) = \int_{-\infty}^{\sigma(t)} X(t)P\chi_{1}(\sigma(s))f(s, x(s))\Delta s - \int_{-\infty}^{\sigma(t)} X(t)(I - P)X^{-1}(\sigma(s))f(s, x(s))\Delta s.\]
We present an existence and uniqueness result of $(Q,T)$-affine-periodic solution of (24).

**Theorem 3.** Consider the equation (24). Let $A(t), f(t,x)$ be $(Q,T)$ affine-periodic, $f(t,x)$ be bounded. Assume that the following conditions hold:

(i) Equation (15) admits exponential dichotomy on $\mathbb{T}$.

(ii) There exist positive constants $N$ and $\gamma$ such that $0 < L < \gamma/2N(2 + \gamma^2)$.

(iii) For every $x, y \in \mathbb{R}^n$ and $t \in \mathbb{T}$,
\[
|f(t,x) - f(t,y)| \leq L|x - y|, \tag{27}
\]
where $\bar{v} = \sup_{t \in \mathbb{T}}|y(t)|$.

Then, the system (24) has a unique solution which is $(Q,T)$-affine-periodic.

**Proof.** Firstly, we consider the nonhomogeneous linear equation:
\[
x^\Delta(t) = A(t)x(t) + f(t,y(t)), \tag{28}
\]
where $y(t)$ is an rd-continuous function.

By Lemma 1, equation (28) has the following bounded solution:
\[
x(t) = \int_{-\infty}^{t} X(t)P X^{-1}(\sigma(s))f(s,y(s))\Delta s - \int_{t}^{\infty} X(t)(I - P) X^{-1}(\sigma(s)) f(s,y(s))\Delta s. \tag{29}
\]

We prove that $x(t)$ is $(Q,T)$-affine-periodic if $y(t)$ is $(Q,T)$-affine-periodic:

Let $s = r + T$. By the $(Q,T)$-affine periodicity of $f(t,y)$, $y(t)$ and (19), we obtain

\[
x(\sigma(t) + T) = \int_{-\infty}^{\sigma(t) + T} X(\sigma(t) + T)P X^{-1}(\sigma(r) + T)f(r + T,y(r + T))\Delta s - \int_{\sigma(t) + T}^{\infty} X(\sigma(t) + T)(I - P) X^{-1}(\sigma(r) + T) f(r + T,y(r + T))\Delta s.
\]

Thus we get $x(\sigma(t) + T) = Qx(\sigma(t))$ which means that $x(t)$ is $(Q,T)$-affine-periodic.

Let $Q \in \text{GL}(n)$ and
\[
C_T = \{y: \mathbb{T} \rightarrow \mathbb{R}^n: y(\sigma(t) + T) = Qy(\sigma(t)), \text{ for all } t \in \mathbb{T}\}, \tag{32}
\]

and define the norm as $\|y\| = \sup_{t \in [0,T]}|y(t)|$.

Define an operator $H: C_T \rightarrow C_T$ as follows:

\[
H(y)(t) = \int_{-\infty}^{t} X(t)P X^{-1}(\sigma(s))f(s,y(s))\Delta s - \int_{t}^{\infty} X(t)(I - P) X^{-1}(\sigma(s)) f(s,y(s))\Delta s. \tag{33}
\]

It is easy to see that $C_T$ is a Banach space with norm $\|\cdot\|$. $H$ is obviously well defined.
there exists a fixed point of time scale, in this section, we consider dynamic equations on time scales.

\[ \|H(y)(\cdot) - H(z)(\cdot)\| = \left\| \int_{-\infty}^{t} X(t) P X^{-1} (\sigma(s))[f(s, y) - f(s, z)] ds \right\|
\]

\[ \leq \sup_{t \in [0,T]} \left\{ \int_{-\infty}^{t} \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, \sigma(s)) L \| y(s) - z(s) \| ds + \int_{t}^{\infty} \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, \sigma(s)) L \| y(s) - z(s) \| ds \right\} \]

\[ \leq \frac{1}{|\phi|} \left[ N - \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, +\infty) \right] \| y - z \| + \frac{1}{y} \left[ N - \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, +\infty) \right] \| y - z \| \]

\[ \leq \frac{1}{y/1 + \bar{\psi}} \left[ N + \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, +\infty) \right] \| y - z \| + \frac{1}{y} \left[ N + \mathcal{N}_{\mathcal{E}_{\mathcal{D}}} (t, +\infty) \right] \| y - z \| \]

\[ \leq L \| y - z \| \left( \frac{2N (1 + \bar{\psi})}{y} + \frac{2N}{y} \right) = L \left( \frac{2N (2 + \bar{\psi})}{y} \right) \| y - z \| \leq \| y - z \|. \]

Thus \( H(y)(\cdot) \) is a contraction mapping on \( C_T \). By the Banach fixed point theorem, \( H \) has a unique fixed point \( x^* (t) \) in \( C_T \), which is the unique \((Q, T)\)-affine-periodic solution of (24). \( \square \)

4. Pseudo \((Q, T)\)-Affine-Periodic Solutions of Dynamic Equations on Time Scales

In this section, we consider dynamic equations on time scales given by

\[ x^A (t) = A(t)x(t) + f(t, x), \quad (35) \]

where \( A: T \rightarrow \mathbb{R}^{m \times n} \) and \( f: T \rightarrow \mathbb{R}^n \), \( T \) is a \( T \)-periodic time scale, \( A(t), f(t, x) \) are \((Q, T)\)-affine-periodic, i.e.,

\[ A(t + T) = QA(t)Q^{-1}, \]

\[ f(t + T, x) = Qf(t, Q^{-1}x), \quad (36) \]

and its associated homogeneous equation is as follows:

\[ x^A (t) = A(t)x(t). \]

For \( T > 0 \), define

\[ C_T (T, \mathbb{R}^n) = \{ \varphi \in C_{r, d}(T, \mathbb{R}^n): \lim_{|t| \rightarrow \infty} |\varphi(t)| = 0 \}, \]

\[ C_0 (T, \mathbb{R}^n) = \left\{ \varphi \in C_{r, d}(T, \mathbb{R}^n): \lim_{|t| \rightarrow \infty} |\varphi(t)| = 0 \right\}. \]

Definition 8. A function \( \varphi \in C_{r, d}(T \times D, \mathbb{R}^n) \) is called asymptotically \((Q, T)\)-affine-periodic solution if there exist \( f \in C_T (T, \mathbb{R}^n), g \in C_0 (T, \mathbb{R}^n) \) such that \( \varphi = f + g \). Denote by \( \text{AAP}_T (T, \mathbb{R}^n) \) the set of asymptotically \((Q, T)\)-affine-periodic solution.

Definition 9. A function \( \varphi \in C_{r, d}(T \times D, \mathbb{R}^n) \) is called pseudo \((Q, T)\)-affine-periodic solution if there exist \( f \in C_T (T, \mathbb{R}^n), g \in \text{AAP}_T (T, \mathbb{R}^n) \) such that \( \varphi = f + g \). Denote by \( \text{PAP}_T (T, \mathbb{R}^n) \) the set of pseudo \((Q, T)\)-affine-periodic solution.

Remark 2. Note that \( f \) and \( g \) are uniquely determined.

Remark 3. It is obvious that \( C_T (T, \mathbb{R}^n) \subset \text{AAP}_T (T, \mathbb{R}^n) \subset \text{PAP}_T (T, \mathbb{R}^n) \).

Now, we present an existence and uniqueness result of pseudo \((Q, T)\)-affine-periodic solution of (35).

Theorem 4. Consider equation (35). Let \( T \) be a \( T \)-periodic time scale and \( f \in C_{r, d}(T \times \mathbb{R}^n, \mathbb{R}^n) \) be pseudo
(Q, T)-affine-periodic on time scales. Suppose also the following conditions are fulfilled:

(i) \( A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}) \) is a \((Q, T)\)-affine-periodic matrix function on time scales.

(ii) \( (37) \) has a exponential dichotomy on \( T \) with projection \( P \) and positive constants \( M \) and \( \gamma \).

(iii) There exists a constant \( 0 < L_f < \gamma/2M(2 + \bar{\gamma}) \) such that

\[
| f(t, x) - f(t, y) | \leq L_f |x - y|
\]

for every \( x, y \in \mathbb{R}^n \) and \( t \in \mathbb{T} \). Then, \( \sigma(t) = \frac{1}{t} \int_{t-1}^{t} f(s, x(s)) \Delta s \) is \( (Q, T) \)-affine-periodic.

We prove that \( (Q, T) \)-affine periodicity of \( f(t, x) \) is not difficult to show that \( I(t) \) is \((Q, T)\)-affine-periodic. We prove that \( I(t) \) is \((Q, T)\)-affine-periodic if \( x_1(t) \) is \((Q, T)\)-affine-periodic.

Let \( I(t) = \int_{t}^{t+1} X(t)P^{-1} (\sigma(s)) f(s, x(s)) \Delta s \). Then, \( \sigma(t) = \frac{1}{t} \int_{t-1}^{t} f(s, x(s)) \Delta s \) is \( (Q, T) \)-affine-periodic.

\[
I(t) = \int_{t}^{t+1} X(t)P^{-1} (\sigma(s)) f(s, x(s)) \Delta s
\]

\[
I_{\Delta} (\sigma(t) + T) = \int_{t}^{t+1} X(\sigma(t) + T)P^{-1} (\sigma(s)) f(s, x(s)) \Delta s
\]

\[
= \int_{-\infty}^{\infty} X(\sigma(t) + T)P^{-1} (\sigma(s)) f(s) \Delta s
\]

\[
= \int_{-\infty}^{\infty} X(\sigma(t) + T)(I - P)X^{-1} (\sigma(s)) f(s) \Delta s
\]

\[
= \int_{-\infty}^{\infty} X(\sigma(t) + T)(I - P)X^{-1} (\sigma(s) + T) f(s + T) \Delta s
\]
Thus we get \( I(\sigma(t) + T) = QI(\sigma(t)) \) which means that \( I(t) \) is \((Q,T)\)-affine-periodic. Then we need to prove that

\[
\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} |U(t)| \Delta t = 0. \tag{47}
\]

By the exponential dichotomy and the Lipchitz condition, we have

\[
\frac{1}{2r} \int_{-r}^{r} |U(t)| \Delta t \leq \frac{1}{2r} \int_{-r}^{r} \int_{-\infty}^{t} M_{e_{qy}}(t, \sigma(s)) L_f (x_2(s)) |\Delta s\Delta t + \frac{1}{2r} \int_{-r}^{r} \int_{t}^{\infty} M_{e_{qy}}(t, \sigma(s)) L_f (x_2(s)) |\Delta s\Delta t \\
= ML_f \int_{-\infty}^{t} e_{qy}(t, \sigma(s)) \left( \frac{1}{2r} x_2(s) \right) 0 = 0 \tag{48}
\]

\[
\Phi_r(s) = \frac{1}{2r} \int_{-r}^{r} |x_2(s)|\Delta s. \tag{49}
\]

\[
\lim_{r \to +\infty} M_{\Phi_r}(\int_{-\infty}^{t} e_{qy}(t, \sigma(s)) \Phi_r(s) \Delta t + \int_{t}^{\infty} e_{qy}(t, \sigma(s)) \Phi_r(s) \Delta t) = 0. \tag{50}
\]

Then

\[
\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} |U(t)| \Delta t = 0. \tag{51}
\]

\[
\|H(z) - H(y)\| = \left\| \int_{-\infty}^{t} X(t) \frac{X}{P X^{-1}(\sigma(s))} [f(s, z) - f(s, y)] |\Delta s - \int_{t}^{\infty} X(t)(I - P) X^{-1}(\sigma(s)) [f(s, z) - f(s, y)] |\Delta s \right\|
\leq \sup_{t \in \{0, T\}} \left\{ \int_{-\infty}^{t} M_{e_{qy}}(t, \sigma(s)) L_f |y(s) - z| |\Delta s + \int_{t}^{\infty} M_{e_{qy}}(s, \sigma(s)) L_f |y(s) - z| |\Delta s \right\}
\leq \frac{1}{|\gamma|} \left[ M_{e_{qy}}(t, -\infty) - M_{e_{qy}}(t, t) \right] L_f \|y - y\| + \int_{t}^{\infty} M_{e_{qy}}(s, \sigma(s)) L_f \|y - y\| |\Delta s
\leq \frac{1}{|\gamma|} \left[ M_{e_{qy}}(t, -\infty) - M \right] L_f \|y - y\| + \frac{1}{\gamma} \left[ M - M_{e_{qy}}(t, +\infty) \right] L_f \|y - y\|
\leq \frac{1}{\gamma(1 + \gamma)} \left[ e_{qy}(t, -\infty) - M \right] L_f \|y - y\| + \frac{1}{\gamma} \left[ M - M_{e_{qy}}(t, +\infty) \right] L_f \|y - y\|
\]
Thus $H(y)(\cdot)$ is a contraction mapping on $C_T$. By the Banach fixed point theorem, $H$ has a unique fixed point $x^*(t) \in \text{PAP}_{\text{P}T}$, which is the unique pseudo $(Q,T)$ affine-periodic solution of (35).

Data Availability
Supporting the results of the study can be found in the paper.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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