Research Article

Two Kinds of Laplacian Spectra and Degree Kirchhoff Index of the Weighted Corona Networks

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Recently, the study related to network has aroused wide attention of the scientific community. Many problems can be usefully represented by corona graphs or networks. Meanwhile, the weight is a vital factor in characterizing some properties of real networks. In this paper, we give complete information about the signless Laplacian spectra of the weighted corona of a graph G_1 and a regular graph G_2 and the complete information about the normalized Laplacian spectra of the weighted corona of two regular graphs. The corresponding linearly independent eigenvectors of all these eigenvalues are also obtained. The spanning trees' total number and the degree Kirchhoff index of the weighted corona graph are computed.

1. Introduction

In recent years, the network research has attracted more and more attention from scientific communities [1–5], for example, chemical networks, social networks, disease transmission networks, and product networks. Especially, product networks can result some larger networks than before if through some given rules. We can synthesize new compounds through product networks and so on. However, previous researchers pay more attention to the binary networks [6–8]. However, the fact is that the weighted networks which are the extension of the binary networks are more in conformity to the networks in real world. Through the weighted networks, we can model the real world much better.

The definition of the corona and the edge corona about two graphs were given in [9] and some results about the corona have been illustrated in [10]. Azari [11–13] deduced the eccentric connectivity coindex, the Mostor index, and the edge Wiener index of the corona of two graphs. Barik et al. [14] completely represented the information about the spectrum of two graphs' corona. Hou and Shiu [15]

determined the eigenvalues of the edge corona graphs of two graphs and gave the eigenvectors corresponding to them. More information about the spectra of $G_1 \circ G_2$, such as Laplacian spectra and signless Laplacian spectra, were depicted in [16–18]. Chen and Liao [19] gave a complete description of the normalized Laplacian spectra of the corona and edge corona of two graphs.

Liu et al. [20] determined the normalized Laplacian spectra of a linear octagonal-quadrilateral network and computed the number of spanning trees of the octagonal-quadrilateral networks. The Kirchhoff index, multiplicative degree Kirchhoff index, and number of spanning trees of the linear crossed hexagonal chains were obtained in [21]. Recently, Dai et al. [22] deduced the complete conclusion of generalized adjacency eigenvalues and generalized Laplacian eigenvalues of the weighted corona graph with different structures. Liu et al. [23] determined the generalized adjacency, Laplacian, and signless Laplacian spectra of the weighted edge corona networks.

Motivated by this, we study two kinds of Laplacian spectra (the signless Laplacian spectrum and the normalized Laplacian spectrum) of the weighted corona of two graphs.

We calculate the spanning trees' number and the degree Kirchhoff index of the weighted corona graph as the application of the normalized Laplacian spectrum.

In this paper, we only pay attention to simple graphs. Let G be a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let $D(G)$ denote the diagonal degree matrix of G . Let $A(G)$ be the adjacency matrix of G . It is defined as $A(G) = (a_{ij})_n$, where $a_{ij} = 1$ if vertices i and j are adjacent in G and 0 otherwise.

The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. The signless Laplacian spectrum of the graph G is actually the spectrum of $Q(G)$. It is denoted by

$$S(G) = (\xi_1(G), \xi_2(G), \dots, \xi_n(G)), \quad (1)$$

where $\xi_1(G) \leq \xi_2(G) \leq \dots \leq \xi_n(G)$ are the eigenvalues of $Q(G)$.

The normalized Laplacian matrix of G is defined as

$$\mathcal{L}(G) = D(G)^{-1/2} (D(G) - A(G)) D(G)^{-1/2}, \quad (2)$$

which is closely correlated with spectrum geometry and the simple random walk on G . Given a simple graph G , let $P(G) = D(G)^{-1} A(G)$ be the matrix about the simple random walk on G (which is named the probability transition matrix); then,

$$\begin{aligned} \mathcal{L}(G) &= D(G)^{1/2} (I - D(G)^{-1} A(G)) D(G)^{-1/2} \\ &= D(G)^{1/2} (I - P(G)) D(G)^{-1/2}. \end{aligned} \quad (3)$$

So, if the spectrum of $\mathcal{L}(G)$ is denoted by

$$N(G) = (\eta_1(G), \eta_2(G), \dots, \eta_n(G)), \quad (4)$$

where $0 = \eta_1(G) \leq \eta_2(G) \leq \dots \leq \eta_n(G) \leq 2$ and $\eta_i(G)$, $i = 1, 2, \dots, n$, are the eigenvalues of $\mathcal{L}(G)$. The spectrum of $P(G)$ is denoted by

$$N_P(G) = (\mu_1(G), \mu_2(G), \dots, \mu_n(G)), \quad (5)$$

where $1 = \mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) \geq -1$ and $\mu_i(G)$, $i = 1, 2, \dots, n$, are the eigenvalues of $P(G)$. Then,

$$\eta_i(G) = 1 - \mu_i(G), \quad i = 1, 2, \dots, n. \quad (6)$$

More results of the normalized Laplacian spectra are depicted in [24, 25].

Definition 1 (see [9]). The corona of two graphs G_1 and G_2 , which is denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining the i th vertex of G_1 to every vertex in the i th copy of G_2 .

Definition 2 (see [9]). The edge corona of two graphs G_1 and G_2 , which is denoted by $G_1 \diamond G_2$, is defined as the graph obtained by taking one copy of G_1 and $|E(G_1)|$ copies of G_2 and then joining the two vertices of each edge of graph G_1 with all vertices of corresponding copy of graph G_2 (see Figure 1).

Dai et al. have defined the weighted corona graphs of two graphs [22]. Suppose that the initial graph G_1 and copy graph G_2 are simple graphs. In the weighted corona of G_1

FIGURE 1: Edge corona $G_1 \diamond G_2$ of G_1 and G_2 .

and G_2 (denoted by $\mathcal{G} = G_1 \tilde{\circ} G_2$), each edge of G_1 has a unit weight and each edge in the copy of G_2 has a weight factor r , $0 < r \leq 1$, and the edges which are newly generated have the weight r (see Figure 2). Let ω_{ij} be the weight of edge connecting vertex i and j and $W(G)$ be the generalized adjacency matrix of G , where its element is ω_{ij} if i and j are adjacent, 0 otherwise. Let $A = (a_{ij})$ and B be matrices. The partition matrix $(a_{ij}B)$ is defined as the Kronecker product of A and B , which are denoted by $A \otimes B$ [26]. If every multiplication makes sense, we can obtain

$$(A_1 \otimes A_2)(A_3 \otimes A_4) = A_1 A_3 \otimes A_2 A_4. \quad (7)$$

Let J_n denote the row vector of order n with all entries equal to 1, I_n denote the identity matrix of order n , $\mathbf{0}_n$ denote the zero matrix of order n , and e_i be the i th unit column vector.

The paper is organized as below. In Section 2, the signless Laplacian spectrum of weighted corona graph is given and the corresponding linearly independent eigenvectors of all these eigenvalues are also obtained. In Section 3, the analytical formulas of the normalized Laplacian eigenvalues are determined and the corresponding linearly independent eigenvectors of all these eigenvalues are depicted. In Section 4, we give the number of spanning trees and the degree Kirchhoff index of the weighted corona graph. Finally, we draw the conclusion.

2. The Signless Spectrum of Weighted Corona Graph $G_1 \tilde{\circ} G_2$

The strength of vertex i is $\tilde{d}_i = \sum_{j \in G} \omega_{ij}$ ($i = 1, 2, \dots, n$). Let $\tilde{D}(G) = \text{diag}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n)$ denote the diagonal intensity matrix of G . Then, the signless Laplacian matrix of the weighted graph $\mathcal{G}$ is defined by $Q(\mathcal{G}) = \tilde{D}(\mathcal{G}) + W(\mathcal{G})$. The spectrum of $Q(\mathcal{G})$ is said to be the signless Laplacian spectrum of $\mathcal{G}$. In this section, we will give the signless Laplacian eigenvalues of the weighted corona graph $\mathcal{G} = G_1 \tilde{\circ} G_2$.

Let G_1 be a graph with n_1 vertices and G_2 be a k -regular graph with n_2 vertices. Then, the signless Laplacian matrix of the weighted corona graph $G_1 \tilde{\circ} G_2$ is as follows:

$$Q\left(G_1 \tilde{\circ} G_2\right) = \begin{pmatrix} Q(G_1) + rn_2 I_{n_1} & rJ_{n_2} \otimes I_{n_1} \\ rJ_{n_2}^T \otimes I_{n_1} & r(Q(G_2) + I_{n_2}) \otimes I_{n_1} \end{pmatrix}, \quad (8)$$

FIGURE 2: The weighted corona graph, $G_1 \tilde{\circ} G_2$.

where $Q(G_1)$ and $Q(G_2)$ are the signless Laplacian matrices of the graphs G_1 and G_2 , respectively.

Theorem 1. Let G_1 be a graph with n_1 vertices and G_2 be a k -regular graph with n_2 vertices, and $S(G_1) = (\mu_1, \mu_2, \dots, \mu_{n_1})$ and $S(G_2) = (\eta_1, \eta_2, \dots, \eta_{n_2} = 2k)$. Then, the signless Laplacian spectra of the weighted corona $G_1 \tilde{\circ} G_2$ are as follows.

- (i) $\mu_i + rn_2 + (2k+1)r \pm \sqrt{[\mu_i + rn_2 + (2k+1)r]^2 - 4(2k+1)r\mu_i - 8r^2n_2k/2} \in S(G_1 \tilde{\circ} G_2)$ with multiplicity 1 for $i = 1, 2, \dots, n_1$
- (ii) $r(\eta_j + 1) \in S(G_1 \tilde{\circ} G_2)$ with multiplicity n_1 for $j = 1, 2, \dots, n_2 - 1$

Proof. Let λ be the eigenvalue of $Q(G_1 \tilde{\circ} G_2)$, $X = (X_1 X_2 \dots X_{n_2+1})^T$, and $X_i \in R^{n_i}$ ($i = 1, 2, \dots, n_2 + 1$) be the eigenvector which corresponds to λ . Then,

$$Q(G_1 \tilde{\circ} G_2)X = \lambda X. \quad (9)$$

From equations (8) and (9), we can conclude that

$$(Q(G_1) + rn_2 I_{n_1})X_1^T + r(X_2^T + X_3^T + \dots + X_{n_2+1}^T) = \lambda X_1^T, \quad (10)$$

and

$$\begin{cases} lrX_1^T + rE_1((Q(G_2) + I_{n_2}) \otimes I_{n_1})(X_2 X_3 \dots X_{n_2+1})^T = \lambda X_2^T, \\ rX_1^T + rE_2((Q(G_2) + I_{n_2}) \otimes I_{n_1})(X_2 X_3 \dots X_{n_2+1})^T = \lambda X_3^T, \\ \vdots \\ rX_1^T + rE_{n_2}((Q(G_2) + I_{n_2}) \otimes I_{n_1})(X_2 X_3 \dots X_{n_2+1})^T = \lambda X_{n_2+1}^T, \end{cases} \quad (11)$$

where $E_i = (\underbrace{0}_{n_1}, \dots, \underbrace{0}_{n_1}, I_{n_1}, \underbrace{0}_{n_1}, \dots, \underbrace{0}_{n_1})$.

By adding the above equations in equation (11), we have

$$\begin{aligned} rn_2 X_1^T + (2k+1)r(X_2^T + X_3^T + \dots + X_{n_2+1}^T) \\ = \lambda(X_2^T + X_3^T + \dots + X_{n_2+1}^T), \end{aligned} \quad (12)$$

which is

$$rn_2 X_1^T = [\lambda - (2k+1)r](X_2^T + X_3^T + \dots + X_{n_2+1}^T). \quad (13)$$

Case 1: X_1 is the nonzero vector.

In this case, we immediately deduce that $\lambda - (2k+1)r \neq 0$. From equations (10) and (13), it gives

$$(Q(G_1) + rn_2 I_{n_1})X_1^T + \frac{r^2 n_2}{\lambda - (2k+1)r} X_1^T = \lambda X_1^T, \quad (14)$$

so

$$Q(G_1)X_1^T = \left[\lambda - rn_2 - \frac{r^2 n_2}{\lambda - (2k+1)r} \right] X_1^T. \quad (15)$$

Noticing that $S(G_1) = (\mu_1, \mu_2, \dots, \mu_{n_1})$ and according to equation (15), one obtains

$$\lambda - rn_2 - \frac{r^2 n_2}{\lambda - (2k+1)r} = \mu_i, \quad i = 1, 2, \dots, n_1, \quad (16)$$

which is

$$\begin{aligned} \lambda^2 - [(2k+1)r + rn_2 + \mu_i]\lambda + 2r^2 n_2 k \\ + (2k+1)r\mu_i = 0, \quad i = 1, 2, \dots, n_1. \end{aligned} \quad (17)$$

By solving this equation about λ , we obtain

$$\lambda_i, \bar{\lambda}_i = \frac{\mu_i + rn_2 + (2k+1)r \pm \sqrt{[\mu_i + rn_2 + (2k+1)r]^2 - 4(2k+1)r\mu_i - 8r^2 n_2 k}}{2}. \quad (18)$$

Case 2: X_1 is the zero vector.

From equations (10) and (11), it gives

$$X_1^T + X_2^T + \dots + X_{n_2+1}^T = \mathbf{0}, \quad (19)$$

and

$$\begin{aligned} & r(Q(G_2) + I_{n_2}) \otimes I_{n_1} (X_2 X_3 \dots X_{n_2+1})^T \\ & = \lambda (X_2 X_3 \dots X_{n_2+1})^T. \end{aligned} \quad (20)$$

Note that $(2k+1)r$ is not the eigenvalue of $Q(G_1 \tilde{\circ} G_2)$. Otherwise, we can deduce that the eigenvector $X = [X_1 X_2 \dots X_{k+1}]^T$ of the eigenvalue $(2k+1)r$ fits $X^T = J_{n_1} \otimes J_{n_2+1}$. Thus,

$$X_1 + X_2 + \dots + X_{n_2+1} = (n_2 + 1)J_{n_1}, \quad (21)$$

which contradicts with equation (19).

Note that the spectrum of $Q(G_2)$ is $S(G_2) = (\eta_1, \eta_2, \dots, \eta_{n_2} = 2k)$, so according to the equation (20), we can directly obtain

$$\lambda = r(\eta_j + 1), \quad j = 1, 2, \dots, n_2 - 1, \quad (22)$$

and the multiplicity of $\lambda = r(\eta_j + 1)$ is n_1 .

According to equations (18) and (22), we have $n_1(1 + n_2)$ eigenvalues of $Q(G_1 \tilde{\circ} G_2)$. The eigenvectors of $Q(G_1 \tilde{\circ} G_2)$ corresponding to these eigenvalues are presented as follows.

From equation (15), we obtain that X_1^T is the eigenvector set corresponding to the eigenvalues of $Q(G_1)$. Let $X_1 = Y_i, X_2 = X_3 = \dots = X_{n_2+1} = rY_i/\lambda_i - (2k+1)r$, and $X_2' = X_3' = \dots = X_{n_2+1}' = rY_i/\lambda_i - (2k+1)r$ and $Y_i (i = 1, 2, \dots, n_1)$ be the nonzero eigenvectors corresponding to the eigenvalues $\mu_1, \mu_2, \dots, \mu_{n_1}$. Let $F_i = (X_1 X_2 \dots X_{n_2+1})^T$ and $F_i' = (X_1 X_2' \dots X_{n_2+1}')^T$, for $i = 1, 2, \dots, n_1$. We need to prove $Q(G_1 \tilde{\circ} G_2)F_i = \lambda_i F_i$ and $Q(G_1 \tilde{\circ} G_2)F_i' = \bar{\lambda}_i F_i'$ in order to confirm that F_i and F_i' are $2n_1$ eigenvectors corresponding to λ_i and $\bar{\lambda}_i$, respectively.

Given a vertex u in $G_1 \tilde{\circ} G_2$. If $u \in V(G_1)$, one has $F_i(u) = Y_i(u)$, and it obtains

$$\begin{aligned} & (Q(G_1) + rn_2 I_{n_1} r J_{n_2} \otimes I_{n_1}) F_i(u) \\ & = Q(G_1) Y_i(u) + rn_2 Y_i(u) + \frac{r^2 n_2 Y_i(u)}{\lambda_i - (2k+1)r} \\ & = \mu_i Y_i(u) + rn_2 Y_i(u) + \frac{r^2 n_2 Y_i(u)}{\lambda_i - (2k+1)r} \\ & = \lambda_i Y_i(u). \end{aligned} \quad (23)$$

If u is a vertex in the t th copy of G_2 , one has $F_i(u) = rY_i(t)/\lambda_i - (2k+1)r$, and it obtains

$$\begin{aligned} & (rI_{n_1} r E_t((Q(G_2) + I_{n_2}) \otimes I_{n_1})) F_i(u) \\ & = rY_i(t) + \frac{(2k+1)r^2}{\lambda_i - (2k+1)r} Y_i(t) \\ & = \lambda_i F_i(u). \end{aligned} \quad (24)$$

For each vertex in $G_1 \tilde{\circ} G_2$, it follows that $Q(G_1 \tilde{\circ} G_2)F_i = \lambda_i F_i$. Similarly, for each vertex in $G_1 \tilde{\circ} G_2$, we have $Q(G_1 \tilde{\circ} G_2)F_i' = \bar{\lambda}_i F_i'$.

Let $Z_j (j = 1, 2, \dots, n_2)$ be the orthogonal eigenvectors of $Q(G_2)$ corresponding to $\eta_1, \eta_2, \dots, \eta_{n_2} = 2k$. Noticing

that $Z_{n_2} = aJ_{n_2}^T$ and $Z_j \perp J_{n_2}^T$, for $j = 1, 2, \dots, n_2 - 1$. So, for $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2 - 1$, one knows

$$\begin{aligned} Q(G_1 \tilde{\circ} G_2) \begin{pmatrix} \mathbf{0}_{n_1} \\ Z_j \otimes e_i \end{pmatrix} & = \begin{pmatrix} \mathbf{0}_{n_1} \\ (r(Q(G_2) + I_{n_2}) \otimes I_{n_1})(Z_j \otimes e_i) \end{pmatrix} \\ & = \begin{pmatrix} \mathbf{0}_{n_1} \\ (r(Q(G_2) + I_{n_2})Z_j \otimes I_{n_1} e_i) \end{pmatrix} \\ & = r(\eta_j + 1) \begin{pmatrix} \mathbf{0}_{n_1} \\ Z_j \otimes e_i \end{pmatrix}. \end{aligned} \quad (25)$$

Now, we have obtained $n_1(1 + n_2)$ eigenvectors of $Q(G_1 \tilde{\circ} G_2)$, and they are linearly independent obviously.

3. The Normalized Laplacian Spectrum of Weighted Corona Graph $G_1 \tilde{\circ} G_2$

Let G_1 be an r_1 -regular graph with n_1 vertices and G_2 an r_2 -regular graph with n_2 vertices. Let $\mathcal{G} = G_1 \tilde{\circ} G_2$. Then,

$$W(\mathcal{G}) = \begin{pmatrix} W(G_1) & rJ_{n_2} \otimes I_{n_1} \\ rJ_{n_2}^T \otimes I_{n_1} & rW(G_2) \otimes I_{n_1} \end{pmatrix}, \quad (26)$$

$$\tilde{D}(\mathcal{G}) = \begin{pmatrix} (r_1 + rn_2)I_{n_1} & 0 \\ 0 & r(r_2 + 1)I_{n_1 n_2} \end{pmatrix}, \quad (27)$$

$$P(\mathcal{G}) = \tilde{D}(\mathcal{G})^{-1}W(\mathcal{G})$$

$$\begin{aligned} & = \begin{pmatrix} \frac{1}{r_1 + rn_2} W(G_1) & \frac{r}{r_1 + rn_2} J_{n_2} \otimes I_{n_1} \\ \frac{1}{r_2 + 1} J_{n_2}^T \otimes I_{n_1} & \frac{1}{r_2 + 1} W(G_2) \otimes I_{n_1} \end{pmatrix} \\ & = \begin{pmatrix} \frac{r_1}{r_1 + rn_2} P(G_1) & \frac{r}{r_1 + rn_2} J_{n_2} \otimes I_{n_1} \\ \frac{1}{r_2 + 1} J_{n_2}^T \otimes I_{n_1} & \frac{r_2}{r_2 + 1} P(G_2) \otimes I_{n_1} \end{pmatrix}. \end{aligned} \quad (28)$$

Theorem 2. Let G_1 be an r_1 -regular graph with n_1 vertices and G_2 an r_2 -regular graph with n_2 vertices. The spectra of $P(G_1)$ and $P(G_2)$ are

$$N_p(G_1) = (\beta_1, \beta_2, \dots, \beta_{n_1}) \text{ and } N_p(G_2) = (\rho_1, \rho_2, \dots, \rho_{n_2}), \quad (29)$$

respectively, where $-1 = \beta_1 \leq \beta_2 \leq \dots \leq \beta_{n_1} = 1$ and $-1 = \rho_1 \leq \rho_2 \leq \dots \leq \rho_{n_2} = 1$. Then, the spectra of $P(G_1 \tilde{\circ} G_2)$ are as follows:

$$\begin{aligned} & (i) \ r_2(r_1 + rn_2) + r_1\beta_i(r_2 + 1) \pm \sigma_i/2(r_1 + rn_2)(r_2 + 1) \in N_p(G_1 \tilde{\circ} G_2) \text{ with multiplicity 1, for } i = 1, 2, \dots, n_1, \\ & \text{where} \\ & \sigma_i = \sqrt{[r_2(r_1 + rn_2) - r_1\beta_i(r_2 + 1)]^2 + 4rn_2(r_1 + rn_2)(r_2 + 1)} \end{aligned}$$

(ii) $r_2\rho_j/r_2 + 1 \in N_p(G_1 \tilde{\circ} G_2)$ with multiplicity n_1 , for $j = 1, 2, \dots, n_2 - 1$

Proof. Let λ be the eigenvalue of $P(\mathcal{G})$, $X = (X_1 X_2 \dots X_{n_2+1})^T$, and $X_i \in R^{n_i}$ ($i = 1, 2, \dots, n_2 + 1$) be the eigenvector which corresponds to λ . Then,

$$P(\mathcal{G})X = \lambda X. \quad (30)$$

From equations (28) and (30), it follows that

$$\begin{aligned} r_1/r_1 + rn_2 P(G_1)X_1^T + r/r_1 \\ + rn_2(X_2^T + X_3^T + \dots + X_{n_2+1}^T) = \lambda X_1^T \end{aligned} \quad (31)$$

and

$$\left\{ \begin{aligned} l \frac{1}{r_2+1} X_1^T + \frac{r_2}{r_2+1} E_1 (P(G_2) \otimes I_{n_1}) (X_2 X_3 \dots X_{n_2+1})^T &= \lambda X_2^T, \\ \frac{1}{r_2+1} X_1^T + \frac{r_2}{r_2+1} E_2 (P(G_2) \otimes I_{n_1}) (X_2 X_3 \dots X_{n_2+1})^T &= \lambda X_3^T, \\ \vdots \\ \frac{1}{r_2+1} X_1^T + \frac{r_2}{r_2+1} E_{n_2} (P(G_2) \otimes I_{n_1}) (X_2 X_3 \dots X_{n_2+1})^T &= \lambda X_{n_2+1}^T, \end{aligned} \right. \quad (32)$$

where $E_i = (\underbrace{0_{n_1}, \dots, 0_{n_1}}_{i-1}, I_{n_1}, \underbrace{0_{n_1}, \dots, 0_{n_1}}_{n_2-i})$.

By equation (32), we obtain

$$\begin{aligned} \frac{n_2}{r_2+1} X_1^T + \frac{r_2}{r_2+1} (X_2^T + X_3^T + \dots + X_{n_2+1}^T) \\ = \lambda (X_2^T + X_3^T + \dots + X_{n_2+1}^T), \end{aligned} \quad (33)$$

that is,

$$\frac{n_2}{r_2+1} X_1^T + \left(\frac{r_2}{r_2+1} - \lambda \right) (X_2^T + X_3^T + \dots + X_{n_2+1}^T) = 0. \quad (34)$$

Case 1: X_1 is the nonzero vector.

In this case, we immediately deduce that $r_2/r_2 + 1 - \lambda \neq 0$. From equations (31) and (34), we have

$$\frac{r_1}{r_1 + rn_2} P(G_1)X_1^T + \frac{r}{r_1 + rn_2} \frac{n_2}{\lambda r_2 + \lambda - r_2} X_1^T = \lambda X_1^T, \quad (35)$$

so

$$P(G_1)X_1^T = \frac{1}{r_1} \left[\lambda(r_1 + rn_2) - \frac{rn_2}{\lambda r_2 + \lambda - r_2} \right] X_1^T. \quad (36)$$

Noting that $N_p(G_1) = (\beta_1, \beta_2, \dots, \beta_{n_1})$ and according to equation (36), we can obtain

$$\lambda((r_1 + rn_2) - \frac{rn_2}{\lambda r_2 + \lambda - r_2}) = r_1 \beta_i, \quad i = 1, 2, \dots, n_1. \quad (37)$$

By solving this equation about λ , we obtain

$$\lambda_i, \bar{\lambda}_i = \frac{r_2(r_1 + rn_2) + r_1 \beta_i(r_2 + 1) \pm \sigma_i}{2(r_1 + rn_2)(r_2 + 1)}, \quad (38)$$

$$\sigma_i = \sqrt{[r_2(r_1 + rn_2) - r_1 \beta_i(r_2 + 1)]^2 + 4rn_2(r_1 + rn_2)(r_2 + 1)}.$$

Case 2: X_1 is the zero vector.

From equations (31) and (32), we obtain

$$X_2^T + \dots + X_{n_2+1}^T = \mathbf{0} \quad (39)$$

and

$$\frac{r_2}{r_2+1} P(G_2) \otimes I_n (X_2 X_3 \dots X_{n_2+1})^T = \lambda (X_2 X_3 \dots X_{n_2+1})^T. \quad (40)$$

Notice that $r_2/r_2 + 1$ is not the eigenvalue of $P(G_1 \tilde{\circ} G_2)$. Otherwise, we can deduce that the eigenvector $X = (X_1 X_2 \dots X_{n_2+1})^T$ of the eigenvalue $r_2/r_2 + 1$ fits $X^T = J_{n_1} \otimes J_{n_2+1}$. Thus,

$$X_1 + X_2 + \dots + X_{n_2+1} = (n_2 + 1)J_n, \quad (41)$$

which contradicts with equation (39). Note that the spectrum of $P(G_2)$ is $N_p(G_2) = (-1 = \rho_1 \leq \rho_2 \leq \dots \leq \rho_{n_2} = 1)$, so according to equation (40), we can directly obtain

$$\lambda = \frac{r_2 \rho_j}{r_2 + 1}, \quad j = 1, 2, \dots, n_2 - 1, \quad (42)$$

and the multiplicity of $\lambda = r_2 \rho_j / r_2 + 1$ is n_1 .

According to equations (38) and (42), we have $n_1(1 + n_2)$ eigenvalues of $P(G_1 \tilde{\circ} G_2)$. The eigenvectors of $P(G_1 \tilde{\circ} G_2)$ which corresponding to these eigenvalues are presented as follows.

From equation (36), we obtain that X_1^T is the eigenvector set corresponding to the eigenvalues of $P(G_1)$. Let $X_1 = Y_i, X_2 = X_3 = \dots = X_{n_2+1} = rY_i/\lambda_i(r_2 + 1) - r_2$, and $X_2' = X_3' = \dots = X_{n_2+1}' = rY_i/\lambda_i(r_2 + 1) - r_2$, and $Y_i (i = 1, 2, \dots, n_1)$ be the nonzero eigenvectors which correspond to the eigenvalues $\mu_1, \mu_2, \dots, \mu_{n_1}$. Let $F_i = (X_1 X_2 \dots X_{n_2+1})^T$ and $F_i' = (X_1' X_2' \dots X_{n_2+1}')^T$, for $i = 1, 2, \dots, n_1$. We need to prove $P(G_1 \tilde{\circ} G_2)F_i = \lambda_i F_i$ and $P(G_1 \tilde{\circ} G_2)F_i' = \bar{\lambda}_i F_i'$ in order to confirm that F_i and F_i' are $2n_1$ eigenvectors corresponding to λ_i and $\bar{\lambda}_i$, respectively.

Given a vertex u in $G_1 \tilde{\circ} G_2$. If $u \in V(G_1)$, it obtain

$$\begin{aligned} & \left(\frac{r_1}{r_1 + rn_2} P(G_1) - \frac{r}{r_1 + rn_2} J_{n_2} \otimes I_{n_1} \right) F_i(u) \\ &= \frac{r_1}{r_1 + rn_2} P(G_1) Y_i(u) \\ &+ \frac{rn_2}{(r_1 + rn_2)(\lambda_i(r_2 + 1) - r_2)} Y_i(u) \\ &= \frac{r_1}{r_1 + rn_2} \beta_i Y_i(u) + \frac{rn_2}{(r_1 + rn_2)(\lambda_i(r_2 + 1) - r_2)} Y_i(u) \\ &= \lambda_i F_i(u). \end{aligned} \quad (43)$$

If u is a vertex in one copy of G_2 , one has $F_i(u) = Y_i(t)/\lambda_i(r_2 + 1) - r_2$, and it obtains

$$\begin{aligned} & \left(\frac{1}{r_2 + 1} I_{n_1} - \frac{r_2}{r_2 + 1} E_t (P(G_2) \otimes I_{n_1}) \right) F_i(u) \\ &= \frac{1}{r_2 + 1} Y_i(t) + \frac{r_2}{(r_2 + 1)(\lambda_i(r_2 + 1) - r_2)} Y_i(t) \\ &= \lambda_i F_i(u). \end{aligned} \quad (44)$$

For each vertex in $G_1 \tilde{\circ} G_2$, it follows that $P(G_1 \tilde{\circ} G_2)F_i(u) = \lambda_i F_i(u)$. Similarly, for each vertex in $G_1 \tilde{\circ} G_2$, we have $P(G_1 \tilde{\circ} G_2)F_i(u) = \bar{\lambda}_i F_i(u)$.

Let $Z_j (j = 1, 2, \dots, n_2)$ be the orthogonal eigenvectors of $P(G_2)$ corresponding to $\rho_1, \rho_2, \dots, \rho_{n_2} = 2k$. Noticing that $Z_{n_2} = aJ_{n_2}^T$ and $Z_j \perp J_{n_2}^T$, for $j = 1, 2, \dots, n_2 - 1$. So, for $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2 - 1$, one knows

$$\begin{aligned} P(G_1 \tilde{\circ} G_2) \begin{pmatrix} \mathbf{0}_{n_1} \\ Z_j \otimes e_i \end{pmatrix} &= \begin{pmatrix} \mathbf{0}_{n_1} \\ r_2/r_2 + 1P(G_2) \otimes I_{n_1} (Z_j \otimes e_i) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{n_1} \\ r_2/r_2 + 1P(G_2) Z_j \otimes I_{n_1} e_i \end{pmatrix} \\ &= \frac{r_2 \rho_j}{r_2 + 1} \begin{pmatrix} \mathbf{0}_{n_1} \\ Z_j \otimes e_i \end{pmatrix}. \end{aligned} \quad (45)$$

Now, we have obtained $n_1(1 + n_2)$ eigenvectors of $P(G_1 \tilde{\circ} G_2)$, and they are linearly independent obviously. We easily deduce the following result according to relation (6).

Theorem 3. Let G_1 be an r_1 -regular graph with n_1 vertices and G_2 an r_2 -regular graph with n_2 vertices. The spectra of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ are

$$\begin{aligned} N(G_1) &= (0 = \gamma_1, \gamma_2, \dots, \gamma_{n_1}) \text{ and } N(G_2) \\ &= (0 = \delta_1, \delta_2, \dots, \delta_{n_2}), \end{aligned} \quad (46)$$

respectively. Then, the normalized Laplacian spectra of $G_1 \tilde{\circ} G_2$ are as follows:

- (i) $rn_2(r_2 + 2) + r_1 + \gamma_i r_1(r_2 + 1) \pm \tau_i/2(r_1 + rn_2)(r_2 + 1) \in N(G_1 \tilde{\circ} G_2)$ with multiplicity 1 for $i = 1, 2, \dots, n_1$, where
- $\tau_i = \sqrt{[rr_2 n_2 - r_1 + r_1 \gamma_i(r_2 + 1)]^2 + 4rn_2(r_1 + rn_2)(r_2 + 1)}$
- (ii) $1 + r_2 \delta_j / r_2 + 1 \in N(G_1 \tilde{\circ} G_2)$ with multiplicity n_1 for $j = 1, 2, \dots, n_2 - 1$

4. Application

In this section, the spanning trees' number and the degree Kirchhoff index of the weighted corona graph are obtained as applications of Theorem 2.

Given a graph G , the spanning tree is the tree which contains all vertices of G . Therefore, there are many spanning trees of graph G due to the complex structure of the tree. Let the graph G be a connected graph, the number of vertices be n , and the number of edges be m . The normalized Laplacian eigenvalues of G is $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$. Then, the number of spanning tree of G [21] can be expressed by

$$t(G) = \prod_{i \in V(G)} d_i \frac{\prod_{j=2}^n \lambda_j}{2m}, \quad (47)$$

where d_i is the degree of vertex i .

In [25], Chen and Zhang proposed a new graph invariant (called the degree Kirchhoff index) which is defined as $Kf(G) = \sum_{i < j} d_i d_j r_{ij}$. They have proved that this graph invariant is associated with the normalized Laplacian eigenvalues of G , that is,

$$Kf(G) = 2m \sum_{j=2}^n \frac{1}{\lambda_j}. \quad (48)$$

In the following theorem, we will give the number of spanning trees of the weighted corona graph $G_1 \circ G_2$ and its degree Kirchhoff index, by using the normalized Laplacian spectrum of graph $G_1 \circ G_2$.

Theorem 4. Let G_1 be an r_1 -regular graph with n_1 vertices and G_2 an r_2 -regular graph with n_2 vertices. Then, for the weighted corona $\mathcal{G} = G_1 \circ G_2$, we have

- (i) $t(\mathcal{G}) = r^{n_1 n_2} (rr_2 n_2 + 2rn_2 + r_1) / r_1 + r_2 n_2 + 2n_2 t(G_1) \prod_{j=1}^{n_2-1} (r_2 \delta_j + 1)^{n_1}$
- (ii) $Kf(\mathcal{G}) = (rr_2 n_2 + 2rn_2 + r_1)(r_2 n_2 + 2n_2 + r_1) / r_1^2 Kf(G_1) + 2|E(\mathcal{G})|(r_2 + 1)(r_1 + rn_2 / rr_2 n_2 + 2rn_2 + r_1 + n_1 - 1 + \sum_{j=1}^{n_2-1} n_1 / r_2 \delta_j + 1)$, where δ_j is the normalized Laplacian eigenvalue of G_2

Proof. Let the normalized Laplacian spectra of G_1 and G_2 be as depicted in Theorem 2. Then, by Theorem 2, $\gamma_1 = 0$ corresponds two eigenvalues 0 and $rr_2 n_2 + 2rn_2 + r_1 / (r_1 + rn_2)(r_2 + 1)$ of $\mathcal{L}(\mathcal{G})$; for each $\gamma_i \neq 0$, the product of the two corresponding eigenvalues is

$$\frac{[rn_2(r_2 + 2) + r_1 + \gamma_i r_1(r_2 + 1)]^2 - r_i^2}{4(r_1 + rn_2)^2(r_2 + 1)^2} \quad (49)$$

$$= \frac{r_1 \gamma_i}{(r_1 + rn_2)(r_2 + 1)}.$$

So,

$$\begin{aligned} \prod_{\lambda \in L(\mathcal{G}), \lambda \neq 0} \lambda &= \frac{rr_2 n_2 + 2rn_2 + r_1}{(r_1 + rn_2)(r_2 + 1)} \prod_{i=2}^{n_1} \frac{r_1 \gamma_i}{(r_1 + rn_2)(r_2 + 1)} \prod_{j=1}^{n_2-1} \left(\frac{r_2 \delta_j + 1}{r_2 + 1} \right)^{n_1} \\ &= \frac{(rr_2 n_2 + 2rn_2 + r_1) r_1^{n_1-1} \prod_{i=2}^{n_1} \gamma_i \prod_{j=1}^{n_2-1} (r_2 \delta_j + 1)^{n_1}}{(r_1 + rn_2)^{n_1} (r_2 + 1)^{n_1 n_2}}. \end{aligned} \quad (50)$$

Hence,

$$\begin{aligned} t(\mathcal{G}) &= \frac{(r_1 + rn_2)^{n_1} (r(r_2 + 1))^{n_1 n_2} \prod_{\lambda \in L(\mathcal{G}), \lambda \neq 0} \lambda}{n_1 (r_1 + r_2 n_2 + 2n_2)} \\ &= \frac{r^{n_1 n_2} (rr_2 n_2 + 2rn_2 + r_1)}{r_1 + r_2 n_2 + 2n_2} \frac{r_1^{n_1} \prod_{i=2}^{n_1} \gamma_i}{r_1 n_1} \prod_{j=1}^{n_2-1} (r_2 \delta_j + 1)^{n_1} \\ &= \frac{r^{n_1 n_2} (rr_2 n_2 + 2rn_2 + r_1)}{r_1 + r_2 n_2 + 2n_2} t(G_1) \prod_{j=1}^{n_2-1} (r_2 \delta_j + 1)^{n_1}. \end{aligned} \quad (51)$$

For the Kirchhoff index, we have

$$\begin{aligned} Kf(\mathcal{G}) &= 2|E(\mathcal{G})| \sum_{\lambda \in L(\mathcal{G}), \lambda \neq 0} \frac{1}{\lambda} \\ &= 2|E(\mathcal{G})| \left(\frac{(r_1 + rn_2)(r_2 + 1)}{rr_2 n_2 + 2rn_2 + r_1} + \sum_{i=2}^{n_1} \frac{rn_2(r_2 + 2) + r_1 + \gamma_i r_1(r_2 + 1)}{\gamma_i r_1} + \sum_{j=1}^{n_2-1} \frac{n_1(r_2 + 1)}{r_2 \delta_j + 1} \right) \\ &= \frac{(rr_2 n_2 + 2rn_2 + r_1)(r_2 n_2 + 2n_2 + r_1)}{r_1^2} Kf(G_1) + 2|E(\mathcal{G})|(r_2 + 1) \left(\frac{r_1 + rn_2}{rr_2 n_2 + 2rn_2 + r_1} + n_1 - 1 + \sum_{j=1}^{n_2-1} \frac{n_1}{r_2 \delta_j + 1} \right). \end{aligned} \quad (52)$$

5. Conclusion

In recent decades, the study of the normalized Laplacian spectrum has more practical significance, for some dynamical processes and structural aspects of various networks are related to it. This study explores the signless Laplacian spectrum and the normalized Laplacian spectrum of the weighted corona graphs.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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