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Research Article

Almost Bronze Structures on Differentiable Manifolds

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This study introduces a novel structure that is not included in the metallic structure family. This new structure, which is called an almost bronze structure, has been defined using a $(1, 1)$ type tensor field φ which fulfills the requirement $\varphi^2 = m\varphi - Id$ on a differentiable manifold. We investigated the parallelism and integrability conditions of these almost bronze structures by use of an almost product structure corresponding to them. Also, we have defined an almost bronze Riemannian manifold.

1. Introduction

Several polynomial structures on a differentiable manifold are defined using $(1, 1)$ type C^∞–tensor fields. The following structures can be listed as examples of these polynomial structures: almost tangent structures, almost complex structures, almost product structures, golden structures, silver structures, bronze structures, and metallic structures. These structures have been recently studied by many authors (see [1–12]).

The term “metallic ratio” has been defined by Spinadel [13] as a generalized form of the golden proposition in 1999 and coined the concept of the “metallic means family” or “metallic propositions.” The author has revealed the relationship between this metallic means family and the generalization of the Fibonacci numbers, i.e., the generalized secondary Fibonacci sequence.

There are several important generalizations of the Fibonacci and Lucas numbers such as Horadam, k-Fibonacci, m-Fibonacci, bivariate Fibonacci, m-Lucas, and p-Lucas numbers. As a result, members of the metallic ratio family have been obtained. The silver Fibonacci, silver Lucas, bronze Fibonacci, and bronze Lucas numbers [14] are examples of these generalizations. In [14], Kalia defined the silver Fibonacci numbers as a generalized form of the Fibonacci numbers and the silver Lucas numbers as a generalized form of the Lucas numbers. The author has revealed that silver Fibonacci numbers and silver Lucas numbers have been related to the golden ratio. Then, the author defined the bronze Fibonacci numbers and the bronze Lucas numbers as well.

As a member of this metallic ratio family, the bronze ratio was defined in [7]. In [12], the bronze structure was studied considering the bronze ratio, which is a member of this metallic ratio family. In [14], Kalia introduced a new bronze ratio which is related to the bronze Lucas and bronze Fibonacci numbers, which are not members of the metallic mean family. In [15], Şahin defined an almost poly-Norden structure by use of this new bronze ratio, examined several geometric properties of this structure by using a corresponding almost complex structure, and investigated poly-Norden manifolds in terms of their maps with other manifolds having different structures.

In this study, we investigate new almost bronze structures by using the new bronze mean. The new almost bronze structures are polynomial structures with a structure polynomial of $Q(\varphi) = \varphi^2 - m\varphi + Id$ for $m \in \mathbb{R}\setminus[-2, 2]$ on differentiable manifolds. The major novelty introduced by this paper is that an almost product structure has been used to examine a new almost bronze structure's geometry on a differentiable manifold. To the best of our knowledge, the new almost bronze structure on manifolds has not been studied yet before the current study in the literature. In the study, the method in [2] was used.

The present study has been designed as follows: Section 2 provides preliminary knowledge about the new bronze means, bronze Fibonacci numbers, and bronze Lucas numbers. In Section 3, the new almost bronze structure on a differentiable manifold is introduced. Several properties of these structures are obtained in relation to the bronze Fibonacci and bronze Lucas numbers. Moreover, the relationships among the bronze ratio, complex bronze ratio, and tangent real bronze ratio were determined. In Section 4, several examples of almost bronze structures are presented. The connections in principal fibre bundles and tangent bundles are explored in Section 5 in terms of almost bronze structure. Then in Section 6, the integrability feature of the almost bronze structure is studied, and the parallelism of the almost bronze structure is investigated considering the Schouten connection and the Vrănceanu connection. In the last section, an almost bronze Riemannian manifold is defined, and several features of the defined manifold are studied. This section also includes an illustration of the defined bronze structure on the manifold $\mathbb{R}^2$ manifold.

2. Preliminaries

This section provides brief information about the new mean, i.e., the bronze mean, using the related bronze Fibonacci and bronze Lucas numbers defined in [14].

The bronze Fibonacci numbers $(f_{m,n})$ are a family of sequencing numbers defined by the recurrence presented below:

$$f_{m,n+2} = mf_{m,n+1} - f_{m,n}; f_{m,0} = 0, f_{m,1} = 1. \tag{1}$$

On the other hand, the bronze Lucas numbers $(l_{m,n})$ refer to a family of sequencing numbers defined by the following recurrence:

$$l_{m,n+2} = ml_{m,n+1} - l_{m,n}; l_{m,0} = 2, l_{m,1} = m. \tag{2}$$

Different from the bronze means presented in [7, 10–12], a new bronze mean is defined as follows:

$$\rho_m = \frac{m + \sqrt{m^2 - 4}}{2}. \tag{3}$$

This is obtained as the positive root of the following equation:

$$x^2 - mx + 1 = 0. \tag{4}$$

For brevity's sake, we will refer to this mean as the bronze mean.

The bronze Fibonacci numbers and bronze Lucas numbers have the following relationship:

$$\begin{aligned} l_{m,n} &= f_{m,n+1} - f_{m,n-1} = 2f_{m,n+1} - mf_{m,n}, \\ \left(m^2 - 4\right)f_{m,n} &= l_{m,n+1} - l_{m,n-1}. \end{aligned} \tag{5}$$

The bronze means' continued fractions are defined as $\{m-1; \overline{1, m-2}\}$, while the recurrence relationship is defined as follows:

$$\rho_m^{n+2} = m\rho_m^{n+1} - \rho_m^n. \tag{6}$$

The following relationship is another one:

$$\rho_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2 - 4}}{2}. \tag{7}$$

3. Almost Bronze Structures on Manifolds

In this paper, M refers to a C^∞–class differentiable manifold, and all tensor fields and connections on this manifold are considered to be of class C^∞. We denote by $\mathcal{X}(M)$ the Lie algebra of the vector fields on M.

Definition 1 (see [16]). Let M be a differentiable manifold and F be a $(1,1)$ type tensor field on M. If F satisfies the following equation it is defined as a polynomial structure:

$$Q(X) = X^k + a_k X^{k-1} + \cdots + a_2 X + a_1 Id = 0. \tag{8}$$

In this equation, the identity operator on $\mathcal{X}(M)$ is denoted by Id, while $F^{k-1}(q)$, $F^{k-2}(q)$, $\ldots$, $F(q)$, Id are linear independent for each point q in M. In this case, the polynomial $Q(X)$ is said to be a structure polynomial.

As stated in [17], F, which is an almost product (resp., almost complex, almost tangent) structure, satisfies the condition of $F^2 - Id = 0$ (resp., $J^2 + Id = 0$, $T^2 = 0$). Then (M, F) is named as an almost product (resp., almost complex, almost tangent) manifold.

Being inspired by the bronze mean given in (3), we can introduce the almost bronze structure which is a new structure on a differentiable manifold M.

Definition 2. Let M be a differentiable manifold and φ be a $(1,1)$ tensor field that satisfies the equation below:

$$\varphi^2 = m\varphi - Id \tag{9}$$

where $m \in \mathbb{R} \setminus [-2, 2]$. Here, φ is said to be a new almost bronze structure on manifold M. For brevity's sake, we will refer to this structure as the almost bronze structure.

Several properties of almost bronze structures regarding number sequences are as follows:

Proposition 1. *The power of an almost bronze structure on the manifold M is defined as follows for any integer n:*

$$\varphi^n = f_{m,n}\varphi - f_{m,n-1}Id, \varphi^n = \frac{l_{m,n+1} - l_{m,n-1}}{m^2 - 4}\varphi - \frac{l_{m,n} - l_{m,n-2}}{m^2 - 4}Id, \tag{10}$$

where $(f_{m,n})$ stands for the bronze Fibonacci numbers, and $(l_{m,n})$ stands for the bronze Lucas numbers.

Let ρ_m be a bronze ratio. Binet's formulas of the bronze Fibonacci sequence and the bronze Lucas sequence are defined as follows:

$$f_{m,n} = \frac{\rho_m^n - (m - \rho_m)^n}{\sqrt{m^2 - 4}}, l_{m,n} = \rho_m^n + (m - \rho_m)^n, \tag{11}$$

respectively [14].

From (10) and (11), we have a new form for equality (10)

$$\begin{aligned}
\varphi^n &= \frac{\rho_m^n - (m - \rho_m)^n}{\sqrt{m^2 - 4}}\varphi + \frac{\rho_m^{n-1} - (m - \rho_m)^{n-1}}{\sqrt{m^2 - 4}} Id,\\
\varphi^n &= \frac{\rho_m^{n+1} - \rho_m^{n-1} + (m - \rho_m)^{n+1} - (m - \rho_m)^{n-1}}{m^2 - 4}\varphi\\
&\quad - \frac{\rho_m^{n} - \rho_m^{n-2} + (m - \rho_m)^{n} - (m - \rho_m)^{n-2}}{m^2 - 4} Id.
\end{aligned} \tag{12}$$

Unless otherwise stated, we will take $m \in \mathbb{R} \setminus [-2, 2]$ throughout the study.

A simple calculation results in the following.

Proposition 2. *The properties of an almost bronze structure φ are as follows:*

(i) The bronze ratio ρ_m and $\overline{\rho}_m = m - \rho_m$ are the eigenvalues of φ.

(ii) On the tangent space of the manifold T_pM, φ is an isomorphism for each $p \in M$.

(iii) φ is an invertible structure, and its inverse $\widehat{\varphi}$ is an almost bronze structure on M, and it can be calculated as follows: $\widehat{\varphi} = mId - \varphi$.

A polynomial structure on a manifold M induces a generalized almost product structure P, as described in [16]. Thus, the almost product structure and almost bronze structure on M are connected structures.

Theorem 1

(i) If φ is an almost bronze structure on M, then P_φ is an almost product structure on M, and it is defined as follows:

$$P_\varphi = \frac{1}{\sqrt{m^2 - 4}}(2\varphi - mId). \tag{13}$$

We call that P_φ is an almost product structure induced by φ.

(ii) If P is an almost product structure on M, then φ_P is an almost bronze structure on M, and it is defined as follows:

$$\varphi_P = \frac{1}{2}\left(mId + \sqrt{m^2 - 4}P\right). \tag{14}$$

Thus, φ_P is named as an almost bronze structure induced by P.

Since $\varphi_{P_\varphi} = \varphi$ and $P_{\varphi_P} = P$, almost product structures and almost bronze structures on M have a one-to-one correspondence.

Proof

(i) Assume that φ is an almost bronze structure on the manifold M. In this case, the structure $P_\varphi = 1/\sqrt{m^2 - 4}(2\varphi - mId)$ obtained from the almost bronze structure φ is an almost product structure since the following condition is satisfied:

$$P_\varphi^2 = \frac{4\varphi^2 - 4m\varphi + m^2 Id}{m^2 - 4} = \frac{4\left(\varphi^2 - m\varphi\right) + m^2 Id}{m^2 - 4} = \frac{\left(m^2 - 4\right)Id}{m^2 - 4} = Id. \tag{15}$$

(ii) Assume that P is an almost product structure on the manifold M. The structure $\varphi_P = 1/2(mId + \sqrt{m^2 - 4}P)$, which is induced by the almost product structure P, is an almost bronze structure since the following condition is satisfied:

$$\begin{aligned}
\varphi_P^2 &= \frac{m^2 Id + 2m\sqrt{m^2 - 4}P + \left(m^2 - 4\right)Id}{4} = \frac{\left(m^2 - 2\right)Id + m\sqrt{m^2 - 4}P}{2}\\
&= \frac{1}{2}\left(\left(m^2 - 2\right)Id + m\left(2\varphi_P - mId\right)\right) = m\varphi_P - Id.
\end{aligned} \tag{16}$$

We get $\varphi_{P_\varphi} = \varphi$ and $P_{\varphi_P} = P$ by straightforward calculations from (13) and (14).

Using the above-mentioned literature and Theorem 1, we can give the following definitions:

(i) Assume that (M, T) is an almost tangent manifold. Then, the tensor field φ_T, which is induced by T, is defined as follows:

$$\varphi_T = \frac{1}{2}\left(mId + \sqrt{m^2 - 4}\,T\right), \quad m \in \mathbb{R} \setminus [-2, 2], \tag{17}$$

and it is called an almost tangent bronze structure on manifold M.

Then, the equation verified by an almost tangent bronze structure

$$\varphi_T^2 - m\varphi_T + \frac{m^2}{4} Id = 0. \tag{18}$$

The tangent real bronze ratio $\rho_m^t = m/2$ is then calculated using the associated equation in the real field $\mathbb{R}$, that is, $x^2 - mx + m^2/4 = 0$.

(ii) Let J be an almost complex structure on manifold M. The tensor field φ_J which is induced by J is defined using the equation

$$\varphi_J = \frac{1}{2}\left(mId + \sqrt{m^2 - 4}\,J\right), \quad m \in \mathbb{R} \setminus [-2, 2], \tag{19}$$

is called an almost complex bronze structure on manifold M.

φ_J satisfies the following polynomial equation:

$$\varphi_J^2 - m\varphi_J + \frac{m^2 - 2}{2} Id = 0. \tag{20}$$

For $M = \mathbb{R}^2$, we obtain the following equation:

$$x^2 - mx + \frac{m^2 - 2}{2} = 0 \tag{21}$$

with solutions

$$\begin{aligned} x_1 &= \frac{m}{2} + \frac{\sqrt{m^2 - 4}}{2} i, \\ x_2 &= \frac{m}{2} - \frac{\sqrt{m^2 - 4}}{2} i. \end{aligned} \tag{22}$$

□

Definition 3. The complex number

$$\rho_m^c = \frac{m + \sqrt{m^2 - 4i}}{2} \tag{23}$$

will be called complex bronze ratio.

If we take $-2 < m < 2$ in (19), then we have an almost poly-Norden structure defined in [15].

$$\varphi_J^2 - m\varphi_J + Id = 0. \tag{24}$$

4. Examples of the Almost Bronze Structures

Several examples of the almost bronze structure will be presented in this section.

Example 1 (Clifford algebras). Let $\mathscr{C}\ell_n$ be the real Clifford algebra of the Euclidean space $\mathbb{R}^n$ [18]. The standard base $\{e_i\}_{1 \le i \le n}$ of $\mathbb{R}^n$ satisfies the multiplication rules according to the Clifford product

$$\begin{aligned} e_i e_i &= 1, \\ e_i e_j &= -e_j e_i \text{ for } i \neq j. \end{aligned} \tag{25}$$

Therefore, by using

$$\varphi_{e_i} = \frac{1}{2}\left(m + \sqrt{m^2 - 4}\, e_i\right), \tag{26}$$

where $m \in \mathbb{R} \setminus [-2, 2]$ and (25), we can obtain a new representation of the Clifford algebra as follows:

$$\begin{aligned} &\varphi_{e_i} \text{ almost bronze structure}, \\ &\varphi_{e_i}\varphi_{e_j} + \varphi_{e_j}\varphi_{e_i} = m\left(\varphi_{e_i} + \varphi_{e_j}\right) - \frac{m^2}{2} \text{ for } i \neq j. \end{aligned} \tag{27}$$

In [18], $\mathscr{C}\ell_2$ is constructed as

$$\begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ e_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ e_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{28}$$

and then we get

$$\begin{aligned} &(i)\ \varphi_{e_1} = \frac{1}{2}\left(m1 + \sqrt{m^2 - 4}\, e_1\right) = \begin{pmatrix} \rho_m & 0 \\ 0 & m - \rho_m \end{pmatrix}, \\ &(ii)\ \varphi_{e_2} = \frac{1}{2}\left(m1 + \sqrt{m^2 - 4}\, e_2\right) = \frac{1}{2}\begin{pmatrix} m & 2\rho_m - m \\ 2\rho_m - m & m \end{pmatrix}. \end{aligned} \tag{29}$$

Example 2 (Quaternion algebras). There is a quaternion algebra $\mathbb{H}$ with a base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying

$$\begin{aligned} \mathbf{i}^2 &= -1, \\ \mathbf{j}^2 &= -1, \\ \mathbf{k}^2 &= -1, \\ \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}. \end{aligned} \tag{30}$$

Any quaternion can be written as follows:

$$q = S_q + \overrightarrow{V}_q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \tag{31}$$

where $S_q = a_0$ denote the scalar part of q and $\overrightarrow{V}_q = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ denote the vectorial part of q.

For $q \neq 0$, $q_0 = q/N_q$ is called unit quaternion where $N_q = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$ is the norm of quaternion q. We can express each unit quaternion in the following form $q_0 = \cos\alpha + \overrightarrow{S}_0 \sin\alpha$, where $\overrightarrow{\varepsilon}_0$stands for a unit vector that satisfies $\overrightarrow{S}_0^2 = -1$.

Thus, inspired by [19], we have the following.

(a) An almost bronze hyperbolic quaternion structure can be defined as follows:

$$\varphi_{hq} = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\overrightarrow{S}_0, \langle \overrightarrow{S}_0, \overrightarrow{S}_0 \rangle = 1, \overrightarrow{S}_0^2 = 1, \tag{32}$$

where $\langle .,. \rangle_L$ is the inner product and $\overrightarrow{S}_0$ is a unit hyperbolic vector.

(b) An almost bronze biquaternion structure can be defined as follows:

$$\varphi_{bq} = \frac{m}{2} + \frac{\sqrt{m^2-4i}}{2}\overrightarrow{S}_0, \langle \overrightarrow{S}_0, \overrightarrow{S}_0 \rangle = 1, \overrightarrow{S}_0^2 = -1, \tag{33}$$

where $\langle .,. \rangle_L$ is the inner product and $i^2 = -1$.

(c) An almost bronze split quaternion structure can be defined as follows:

$$\varphi_{sq} = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\overrightarrow{S}_0, \langle \overrightarrow{S}_0, \overrightarrow{S}_0 \rangle_L = 1, \overrightarrow{S}_0^2 = 1, \tag{34}$$

where $\langle .,. \rangle_L$ is Lorentzian inner product and $\overrightarrow{S}_0$ is a spacelike unit vector in the Minkowski 3-space $\mathbb{E}_1^3$.

(d) An almost bronze dual split quaternion structure can be defined as follows:

$$\varphi_{dsq} = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\overrightarrow{\mathcal{S}}_0, \langle \overrightarrow{\mathcal{S}}_0, \overrightarrow{\mathcal{S}}_0 \rangle_L = 1, \overrightarrow{\mathcal{S}}_0^2 = 1, \tag{35}$$

where$\langle .,. \rangle$is Lorentzian inner product and $\overrightarrow{\mathcal{S}}_0$is a spacelike unit dual vector in $\mathbb{E}_1^3$.

(e) An almost bronze hyperbolic split quaternion structure can be defined as follows:

$$\varphi_{hsq} = \frac{m}{2} + \frac{\sqrt{m^2-4}}{2}\overrightarrow{S}_0, \langle \overrightarrow{S}_0, \overrightarrow{S}_0 \rangle_L = 1, \overrightarrow{S}_0^2 = 1, \tag{36}$$

where $\langle .,. \rangle$is Lorentzian inner product and $\overrightarrow{\mathfrak{S}}_0$is a spacelike unit hyperbolic vector in $\mathbb{E}_1^3$.

Example 3 (Bronze matrices). Let $\mathrm{Mat}(n,\mathbb{R})$ be a matrix algebra of real $n \times n$ –matrices and $\overline{\varphi} \in Mat(n,\mathbb{R})$. If $\overline{\varphi}$ satisfies the following equation:

$$\overline{\varphi}^2 = m\overline{\varphi} - \mathbb{I}_n, \tag{37}$$

where $\mathbb{I}_n$ is the identity matrix on $\mathrm{Mat}(n,\mathbb{R})$, then this matrix is called an almost bronze matrix.

By solving (37) for $n = 2$, we can obtain the almost bronze structure in $\mathrm{Mat}(2,\mathbb{R})$.

(i) For $a \in \mathbb{R}$, $b \in \mathbb{R}\setminus\{0\}$, and $d \in \mathbb{R}$

$$\begin{aligned} \overline{\varphi}_{a,b} &= \begin{pmatrix} a & -\frac{1}{b}\left(a^2 - ma + 1\right) \\ b & m-a \end{pmatrix}, \\ \overline{\varphi}_{d,b} &= \begin{pmatrix} m-d & -\frac{1}{b}\left(d^2 - m\,d + 1\right) \\ b & d \end{pmatrix}. \end{aligned} \tag{38}$$

(ii) For $a = \rho_m$ and $b \in \mathbb{R}$,

$$\begin{aligned} \overline{\varphi}_{\rho_m,b} &= \begin{pmatrix} \rho_m & 0 \\ b & m-\rho_m \end{pmatrix}, \overline{\varphi}_{m-\rho_m,b} = \begin{pmatrix} m-\rho_m & 0 \\ b & \rho_m \end{pmatrix} \\ \overline{\varphi}_{\rho_m,b} &= \begin{pmatrix} \rho_m & b \\ 0 & m-\rho_m \end{pmatrix}, \overline{\varphi}_{m-\rho_m,b} = \begin{pmatrix} m-\rho_m & b \\ 0 & \rho_m \end{pmatrix}. \end{aligned} \tag{39}$$

(iii) For $a = \rho_m$ and $b = 0$,

$$\begin{aligned} \overline{\varphi}_{\rho_m,0} &= \begin{pmatrix} \rho_m & 0 \\ 0 & m-\rho_m \end{pmatrix} \\ \overline{\varphi}_{m-\rho_m,0} &= \begin{pmatrix} m-\rho_m & 0 \\ 0 & \rho_m \end{pmatrix}. \end{aligned} \tag{40}$$

Then, from (29) and (38), we obtain

$$\begin{aligned} \overline{\varphi}_1 &= \lim_{b\longrightarrow 0} \overline{\varphi}_{\rho_m,b}, \\ \overline{\varphi}_2 &= \overline{\varphi}_{m/2,\sqrt{m^2-4}/2}. \end{aligned} \tag{41}$$

Also, from (38), we get the sequence of trace $(\mathrm{Tr}\overline{\varphi}_{a,b}^k)_{k\geq 0}$ is the bronze Lucas sequence: $2, m, m^2-2, m(m^2-3), \ldots$.

Example 4 (Bronze reflections). As also stated in [20], the equation of the reflection in accordance with a hyperplane $\mathscr{H}$ with the normal $u \in \mathbb{E}\setminus\{0\}$in Euclidean space $(\mathbb{E}, <, >)$ is as follows:

$$r_u(\alpha) = \alpha - \frac{2\langle \alpha, u \rangle}{\langle u, u \rangle} u \text{ for } \alpha \in \mathbb{E}. \tag{42}$$

In this equation, it is obvious that $r_u^2 = I_{\mathbb{E}}$ where $I_{\mathbb{E}}$ is the identity on $\mathbb{E}$.

Thus, the bronze reflection with respect to u can be defined as follows:

$$\varphi_u = \frac{mI_{\mathbb{E}} + \sqrt{m^2 - 4r_u}}{2}, \tag{43}$$

and then u is an eigenvector of φ_u with the corresponding eigenvalue $m-\rho_m$. Then, the following equation is obtained from [20, p.314]

$$X\varphi_u X^{-1} = \varphi_{X(u)}, \tag{44}$$

where X is an orthogonal transformation on $\mathbb{E}$. Thus, the following equation can be written as an explicit expression of the linear transformation

$$\varphi_u(\alpha) = \rho_m \alpha + (m-2\rho_m)\frac{\langle \alpha, u\rangle}{\langle u, u\rangle} u. \tag{45}$$

Example 5 (Triple structures with respect to almost bronze structures). Given two $(1,1)$ tensor fields $\mathscr{F}$ and $\mathscr{P}$ on the manifold M and $\mathscr{K} = \mathscr{P}\circ\mathscr{F}$, we called that the triple $(\mathscr{F}, \mathscr{P}, \mathscr{K} = \mathscr{P}\circ\mathscr{F})$ is as follows [21]:

(1) an almost hyperproduct structure: if $\mathscr{F}, \mathscr{P}$ are almost product structures and $\mathscr{P}\circ\mathscr{F} = \mathscr{F}\circ\mathscr{P}$, then $\mathscr{K}$ is an almost product structure,

(2) an almost biproduct complex structure: if $\mathscr{F}$, $\mathscr{P}$ are almost product structures and $\mathscr{P}\circ\mathscr{F} = -\mathscr{F}\circ\mathscr{P}$, then $\mathscr{K}$ is an almost complex structure,

(3) an almost product bicomplex structure: if $\mathscr{F}$, $\mathscr{P}$ are almost complex structures and $\mathscr{P}\circ\mathscr{F} = \mathscr{F}\circ\mathscr{P}$, then $\mathscr{K}$ is an almost product structure,

(4) an almost hypercomplex structure: if $\mathscr{F}$, $\mathscr{P}$ are almost complex structures and $\mathscr{P}\circ\mathscr{F} = -\mathscr{F}\circ\mathscr{P}$, then $\mathscr{K}$ is an almost complex structure.

Taking into account (14), we get

$$\begin{aligned}
\varphi_{\mathscr{F}} &= \frac{m}{2} Id + \left(\frac{2\rho_m - m}{2}\right)\mathscr{F},\\
\varphi_{\mathscr{P}} &= \frac{m}{2} Id + \left(\frac{2\rho_m - m}{2}\right)\mathscr{P},\\
\varphi_{\mathscr{K}} &= \frac{m}{2} Id + \left(\frac{2\rho_m - m}{2}\right)\mathscr{K}.
\end{aligned} \tag{46}$$

Then, we find a relation between $\varphi_{\mathscr{F}}$, $\varphi_{\mathscr{P}}$, and $\varphi_{\mathscr{K}}$ as

$$\sqrt{m^2-4}\varphi_{\mathscr{K}} = 2\varphi_{\mathscr{P}}\varphi_{\mathscr{F}} - m\varphi_{\mathscr{P}} - m\varphi_{\mathscr{F}} + \rho_m^2 Id + Id. \tag{47}$$

Hence, the triple $(\varphi_{\mathscr{F}}, \varphi_{\mathscr{P}}, \varphi_{\mathscr{K}})$ is as follows:

(i) an almost hyperproduct structure: if and only if $\varphi_{\mathscr{F}}$, $\varphi_{\mathscr{P}}$ are almost bronze structures and $\varphi_{\mathscr{F}}\varphi_{\mathscr{P}} - \varphi_{\mathscr{P}}\varphi_{\mathscr{F}} = 0$, then $\varphi_{\mathscr{K}}$ is an almost bronze structure,

(ii) an almost biproduct complex structure: if and only if $\varphi_{\mathscr{F}}$, $\varphi_{\mathscr{P}}$ are almost bronze structures and $\varphi_{\mathscr{P}}\varphi_{\mathscr{F}} + \varphi_{\mathscr{F}}\varphi_{\mathscr{P}} = m(\varphi_{\mathscr{P}} + \varphi_{\mathscr{F}}) - 1/2m^2 Id$, then $\varphi_{\mathscr{K}}$ is an almost complex bronze structure,

(iii) an almost product bicomplex structure: if and only if $\varphi_{\mathscr{F}}$, $\varphi_{\mathscr{P}}$ are almost complex bronze structures and $\varphi_{\mathscr{F}}\varphi_{\mathscr{P}} - \varphi_{\mathscr{P}}\varphi_{\mathscr{F}} = 0$, then $\varphi_{\mathscr{K}}$ is an almost bronze structure,

(iv) an almost hypercomplex structure: if and only if $\varphi_{\mathscr{F}}$, $\varphi_{\mathscr{P}}$ are almost complex bronze structures and $\varphi_{\mathscr{P}}\varphi_{\mathscr{F}} + \varphi_{\mathscr{F}}\varphi_{\mathscr{P}} = m(\varphi_{\mathscr{P}} + \varphi_{\mathscr{F}}) - 1/2m^2 Id$, then $\varphi_{\mathscr{K}}$ is an almost complex bronze structure.

Example 6 (Almost bronze structures from symplectic distributions). Given any symplectic vector space (V,μ), we have $V = U + U^{\mu}$ where $U^{\mu} = \{v \in V\colon \mu(v,u) = 0 \text{ for every } u \in U\}$if U is a subspace of (V,μ). A subspace U of (V,μ) is symplectic if and only if $\mu|_{U\times U}$ is nondegenerate (or $U\cap U^{\mu} = \{0\}$) [22]. Consequently, if R is a symplectic distribution on a symplectic manifold (N,μ) (i.e., R_x is a symplectic subspace of the tangent space at $x \in N$), then another symplectic distribution $S = R^{\mu}$ is obtained complementary to R. In this case, $P = r - s$ is an almost product structure where r and s are the corresponding projection tensors. Then, an associated symplectic almost bronze structure is obtained as follows by using (14)

$$\varphi_R = \rho_m r + (m-\rho_m)s. \tag{48}$$

5. Connection as Almost Bronze Structure

5.1. Connections in the Principal Fibre Bundles. Assume that $\mathscr{P}(M,G)$ is a principal fibre bundle on a manifold M, where $\mathscr{P}$ is the total space, M is the base space, G is the structure group, and π is the projection. Let $\mathscr{V}$ denote a vertical distribution (i.e., $\mathscr{V} = \ker\pi_*$), $\mathscr{H}$ denote a horizontal distribution (i.e., $T\mathscr{P} = \mathscr{V}\oplus\mathscr{H}$), and $\mathscr{H}$ be a G-invariant. Thus, v and h become the corresponding projectors of $\mathscr{V}$ and $\mathscr{H}$, respectively. Therefore, $(1,1)$ type tensor field can be defined as follows:

$$P = v - h, \tag{49}$$

and it is an almost product structure on P. According to [2], P defines a connection if and only if the following conditions are satisfied:

(a) $P(X) = X \Leftrightarrow X$ is a vertical vector field,

(b) $dR_e \circ \mathrm{P_u} = \mathrm{P_{ue}} \circ \mathrm{dR_e}$ for each $u \in \mathscr{P}$ and $e \in G$.

We can get the following proposition by using the relation between the almost bronze structure and the almost product structure:

Proposition 3. *An almost bronze structure φ on $\mathscr{P}$ specifies a connection if and only if the following conditions are satisfied:*

(a) For $X \in \mathscr{X}(P)$, $\varphi(X) = \rho_m X$ if and only if $X \in \mathscr{V}$.

(b) $dR_e \circ \varphi_u = \varphi_{ue} \circ dR_e$ for each $u \in \mathscr{P}$ and $e \in G$.

Assumingthat $\varpi \in \wedge^1(\mathscr{P}, g)$ is a connection 1-form of horizontal distribution $\mathscr{H}$ and $\Omega \in \wedge^2(P, \mathrm{g})$ is the curvature form of ϖ where g stands for the Lie algebra of G, we can obtain the following relation [2]:

$$\Omega(X,Y) = -\frac{1}{4}\varpi(\mathscr{N}_P(X,Y)), \tag{50}$$

where N_P stands for the Nijenhuis tensor of P, i.e.,

$$\mathcal{N}_P(X,Y) = [X,Y] + [PX,PY] - P[PX,Y] - P[X,PY], \tag{51}$$

for all X, Y vector fields on M.

Thus, the following proposition can be stated by straightforward calculations from (13) and (51).

Proposition 4. *Let φ be an almost bronze structure on the manifold M and P_φ be an almost product structure induced by φ. Then*

$$\mathcal{N}_{P_\varphi}(X,Y) = \frac{4}{m^2-4}\mathcal{N}_\varphi(X,Y) \tag{52}$$

$$\Omega(X,Y) = -\frac{1}{m^2-4}\varpi\big(\mathcal{N}_\varphi(X,Y)\big), \tag{53}$$

where $\mathcal{N}_{P_\varphi}$ and $\mathcal{N}_\varphi$ stand for the Nijenhuis tensors of P_φ and φ, respectively.

Therefore, it can be stated that the integrability of the structures φ and P_φ is equivalent.

Proposition 5. *The connection is flat (i.e., $\Omega = 0$) if and only if the associated almost bronze structure is integrable, which means $\mathcal{N}_\varphi = 0$.*

Given two vector fields $\tilde{X}$, $\tilde{Y}$ on the manifold M and a connection, the lift $\mathcal{L}_\varpi\colon \mathcal{X}(M) \longrightarrow \mathcal{X}(P)$ is determined by this connection if the following condition is met [2]:

$$[\mathcal{L}_\varpi\tilde{X}, \mathcal{L}_\varpi\tilde{Y}] - \mathcal{L}_\varpi[\tilde{X},\tilde{Y}] = \mathcal{N}_P(\mathcal{L}_\varpi\tilde{X}, \mathcal{L}_\varpi\tilde{Y}). \tag{54}$$

Thus, considering (52) and (54), we have the following proposition.

Proposition 6. *The lift $\mathcal{L}_\varpi$, which is defined by ϖ, is a morphism if and only if the associated almost bronze structure is integrable.*

5.2. Connection in the Tangent Bundles. Let $TM = \cup_{p\in M} T_pM$ be the tangent bundle of the manifold M, π_M be the projection, π_{M*} be its differential, and $\mathcal{V}TM = \ker\pi_{M*}$ be the vertical distribution of M. For any coordinate neighborhood (U, x^i) in M, (TM, x^i, y^i) stands for the induced coordinate neighborhood in TM, i.e., $x^i(u) = x^i(\pi_M(u))$ and $y^i(u) = dx^i(u)$ for all $u \in \pi_M^{-1}(U)$. For an atlas on TM with these local coordinates, the almost tangent structure of TM is $T = \partial/\partial y^i \otimes dx^i$, i.e., $T^2 = 0$.

Definition 4 (see [2]). Given an almost tangent structure T of TM and a $(1,1)$ tensor field v on M, v is called a vertical projector when the following conditions are met:

$$\begin{aligned} v\circ T &= T,\\ T\circ v &= 0. \end{aligned} \tag{55}$$

Definition 5 (see [2]). N, which is complementary distribution to $\mathcal{V}TM$, i.e.,

$$\mathcal{X}(M) = \mathcal{V}TM \oplus N, \tag{56}$$

is called a normalization or a nonlinear connection or a horizontal distribution.

Knowing that a vertical projector v is $C^\infty(M)$ linear with $\mathrm{Im}v = \mathcal{V}TM$, we can state the following proposition.

Proposition 7 (see [2]). *We obtain a nonlinear connection $N(v)$ from the vertical projector v by using $\ker v = N(v)$. Otherwise, with the respect to the separation (56), v_N and h_N are vertical and horizontal projectors, respectively, if N is a nonlinear connection.*

Thus, the next proposition can be given as follow.

Proposition 8 (see [2]).*v_N is a vertical projector provided that $N(v_N) = N$.*

Definition 6 (see [2]). A $(1,1)$ type tensor field Γ is called a nonlinear connection of an almost product type if the following relations are provided:

$$\begin{aligned} T\circ\Gamma &= T,\\ \Gamma\circ T &= -T. \end{aligned} \tag{57}$$

Proposition 9 (see [2]). *ie following assertions hold true if Γ is a nonlinear connection of an almost product type:*

(i) $v_\Gamma = 1/2(Id - \Gamma)$ is a vertical projector,

(ii) $\mathcal{V}TM$ is the (-1)-eigenspace of Γ when $N(v_\Gamma)$ is the $(+1)$ -eigenspace of Γ.

Corollary 1 (see [2]). *Any vertical projector v induces an almost product structure on manifold M as follows: $\Gamma = Id - 2v$.*

Thus,thisresult has been associated with the almost bronzestructure.

Proposition 10. *Obtained by the vertical projector v, a nonlinear connection N on M can also be defined by an almost bronze structure φ $(= \varphi_\Gamma)$*

$$\varphi = \rho_m Id - \sqrt{m^2-4}v \tag{58}$$

with $\mathcal{V}TM$ the $(m-\rho_m)$-eigenspace and N the ρ_m-eigenspace.

6. Integrability and Parallelism of Almost Bronze Structures

This section examines the almost bronze structure's integrability and parallelism.

Proposition 11. *Let (M, φ) be an almost bronze manifold. There are $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ complementary distributions on M corresponding to the following projection operators:*

$$\mathfrak{k} = \frac{1}{2\rho_m - m}\varphi - \frac{m-\rho_m}{2\rho_m - m}Id, \mathfrak{t} = -\frac{1}{2\rho_m - m}\varphi + \frac{\rho_m}{2\rho_m - m}Id. \quad (59)$$

Remark 1. The operators $\mathfrak{k}$ and $\mathfrak{t}$ obtained in Proposition 11 verify the following equations:

$$\mathfrak{k} + \mathfrak{t} = Id, \mathfrak{k}\mathfrak{t} = \mathfrak{k}\mathfrak{t} = 0, \mathfrak{k}^2 = \mathfrak{k}, \mathfrak{t}^2 = \mathfrak{t}, \quad (60)$$

$$\varphi\mathfrak{k} = \mathfrak{k}\varphi = \rho_m\mathfrak{k}, \varphi\mathfrak{t} = \mathfrak{t}\varphi = (m - \rho_m)\mathfrak{t}. \quad (61)$$

As a result, $\mathfrak{k}$ and $\mathfrak{t}$ operators define $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ complementary distributions corresponding to these projections.

From (61), we get

$$\begin{aligned} \mathfrak{t}[\mathfrak{k}X, \mathfrak{k}Y] &= \frac{1}{(2\rho_m - m)^2}\mathfrak{t}\mathcal{N}_{\varphi}(\mathfrak{k}X, \mathfrak{k}Y), \\ \mathfrak{k}[\mathfrak{t}X, \mathfrak{t}Y] &= \frac{1}{(2\rho_m - m)^2}\mathfrak{k}\mathcal{N}_{\varphi}(\mathfrak{t}X, \mathfrak{t}Y). \end{aligned} \quad (62)$$

As also stated by [23, 24], we have the following:

(i) A polynomial structure φ is integrable if and only if $\mathcal{N}_{\varphi} = 0$ or its equivalent $\nabla\varphi = 0$ where ∇ is a torsion-free linear connection.

(ii) For any vector fields X, Y in $\mathcal{X}(M)$, the distribution $\mathfrak{D}_{\mathfrak{k}}$ (resp. $\mathfrak{D}_{\mathfrak{t}}$) is integrable if and only if $\mathfrak{t}[\mathfrak{k}X, \mathfrak{k}Y] = 0$ (resp., $\mathfrak{k}[\mathfrak{t}X, \mathfrak{t}Y] = 0$).

The following proposition can be stated with the help of Proposition 4.

Proposition 12. *The almost bronze structure φ is integrable if and only if the almost product structure P_{φ} induced by φ is integrable.*

Using the above-mentioned literature and (62), we can give the following proposition.

Proposition 13. *The following claims are true:*

(i) $\mathfrak{D}_{\mathfrak{k}}$ is an integrable distribution if and only if $\mathfrak{t}\mathcal{N}_{\varphi}(\mathfrak{k}X, \mathfrak{k}Y) = 0$.

(ii) $\mathfrak{D}_{\mathfrak{t}}$ is an integrable distribution if and only if $\mathfrak{k}N_{\varphi}(\mathfrak{t}X, \mathfrak{t}Y) = 0$.

(iii) The almost bronze structure φ is integrable if and only if both of the distributions $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ are integrable.

Let us consider a fixed linear connection ∇ on manifold M. We can define the following two linear connections associated with the pair (φ, ∇)

$$\nabla_X^{Sc}Y = \mathfrak{k}(\nabla_X\mathfrak{k}Y) + \mathfrak{t}(\nabla_X\mathfrak{t}Y), \quad (63)$$

$$\nabla_X^{Vr}Y = \mathfrak{k}(\nabla_{\mathfrak{k}X}\mathfrak{k}Y) + \mathfrak{t}(\nabla_{\mathfrak{t}X}\mathfrak{t}Y) + \mathfrak{k}[\mathfrak{t}X, \mathfrak{k}Y] + \mathfrak{t}[\mathfrak{k}X, \mathfrak{t}Y], \quad (64)$$

for any X, Y vector fields of the manifold M. ∇^{Sc} and ∇^{Vr} are known as the Schouten connection and the Vrănceanu connection, respectively [25, 26].

Recall that a $(1, 1)$ tensor field F is parallel in accordance with the linear connection ∇ if its covariant derivative ∇F vanishes.

Theorem 2. *The following claims are true:*

(i) Both of the projectors $\mathfrak{k}$ and $\mathfrak{t}$ are parallel in accordance with the connections ∇^{Sc} and ∇^{Vr}.

(ii) The almost bronze structure φ is parallel regarding the connections ∇^{Sc} and ∇^{Vr}.

Proof

(i) With the help of (60), we can express the following equations for each vector field $X, Y \in \mathcal{X}(M)$:

$$\begin{aligned} \left(\nabla_X^{Sc}\mathfrak{k}\right)Y &= \nabla_X^{Sc}\mathfrak{k}Y - \mathfrak{k}\left(\nabla_X^{Sc}Y\right) = \mathfrak{k}(\nabla_X\mathfrak{k}Y) - \mathfrak{k}(\nabla_X\mathfrak{k}Y) = 0, \\ \left(\nabla_X^{Vr}\mathfrak{k}\right)Y &= \nabla_X^{Vr}\mathfrak{k}Y - \mathfrak{k}\left(\nabla_X^{Vr}Y\right) \\ &= \mathfrak{k}(\nabla_{\mathfrak{k}X}\mathfrak{k}Y) + \mathfrak{k}[\mathfrak{t}X, \mathfrak{k}Y] - \mathfrak{k}(\nabla_{\mathfrak{k}X}\mathfrak{k}Y) - \mathfrak{k}[\mathfrak{t}X, \mathfrak{k}Y] \\ &= 0. \end{aligned} \quad (65)$$

Therefore, the projector $\mathfrak{k}$ is parallel in accordance with the connections ∇^{Sc} and ∇^{Vr}.

Likewise, it can be shown that the projector $\mathfrak{t}$ is parallel according to the connections ∇^{Sc} and ∇^{Vr}.

(ii) By direct computation, we get from (59) that φ is parallel in accordance with the connections ∇^{Sc} and ∇^{Vr}. □

As is known, a distribution $\mathfrak{D}$ on manifold M is said parallel in accordance with the linear connection ∇ provided that ∇_XY belongs to $\mathfrak{D}$ for each vector field $X \in \mathcal{X}(M)$ and $Y \in \mathfrak{D}$.

Definition 7 (see [27]). For any vector fields $X \in \mathfrak{D}_{\mathfrak{k}}$ (resp., $\mathfrak{D}_{\mathfrak{t}}$) and $Y \in \mathfrak{X}(M)$, if the vector field $(\Delta\varphi)(X, Y)$ belongs to $\mathfrak{D}_{\mathfrak{k}}$ (resp., $\mathfrak{D}_{\mathfrak{t}}$) where

$$(\Delta\varphi)(X, Y) = \varphi(\nabla_XY) - \varphi(\nabla_YX) - \nabla_{\varphi X}Y + \nabla_Y\varphi X \quad (66)$$

then the distribution $\mathfrak{D}_{\mathfrak{k}}$ (resp., $\mathfrak{D}_{\mathfrak{t}}$) is named half-parallel.

Definition 8 (see [27]). For any vector fields $X \in \mathfrak{D}_{\mathfrak{k}}$ (resp., $\mathfrak{D}_{\mathfrak{t}}$) and $Y \in \mathcal{X}(M)$, if the vector field $(\Delta\varphi)(X, Y)$ belongs to $\mathfrak{D}_{\mathfrak{t}}$ (resp., $\mathfrak{D}_{\mathfrak{k}}$) then $\mathfrak{D}_{\mathfrak{k}}$ (resp., $\mathfrak{D}_{\mathfrak{t}}$) is named anti-half-parallel.

Theorem 3. *According to the connections ∇^{Sc} and ∇^{Vr}, both distributions $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ are parallel.*

Proof. Given $X \in \mathcal{X}(M)$ and $Y \in \mathfrak{D}_{\mathfrak{k}}$ one has $\mathfrak{t}Y = 0$ and $\mathfrak{k}Y = Y$; then, taking into account (60), (63), and (64) we obtain

$$\mathfrak{t}\left(\nabla_X^{Sc} Y\right) = 0, \mathfrak{t}\left(\nabla_X^{Vr} Y\right) = 0. \tag{67}$$

Thus, the distribution $\mathfrak{D}_{\mathfrak{k}}$ is parallel in accordance with Schouten connection and Vrănceanu connection.

Likewise, it is seen that similar relations are satisfied by $\mathfrak{D}_{\mathfrak{t}}$. □

Proposition 14. *The connection ∇^{Sc} is equal to the connection ∇ if and only if the distributions of almost bronze structure φ (i.e., $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$) are parallel in terms of the connection ∇.*

Proof. If the connections ∇^{Sc} and ∇ are equal, then it follows from (63) that

$$\mathfrak{k}\left(\nabla_X \mathfrak{t}Y\right) + \mathfrak{t}\left(\nabla_X \mathfrak{k}Y\right) = 0 \tag{68}$$

and from (60)

$$\mathfrak{k}\left(\nabla_X \mathfrak{t}Y\right) = 0, \mathfrak{t}\left(\nabla_X \mathfrak{k}Y\right) = 0. \tag{69}$$

Therefore, $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ are parallel regarding the connection ∇.

The other direction of the proof can be shown easily. □

Proposition 15. *For any X vector field in $\mathfrak{D}_{\mathfrak{k}}$ and any Y vector field on M, if the vector field $[\mathfrak{k}X, \mathfrak{t}Y]$ belongs to the distribution $\mathfrak{D}_{\mathfrak{k}}$ then $\mathfrak{D}_{\mathfrak{k}}$ is half-parallel according to the connection ∇^{Vr}.*

Proof. Given $X \in \mathfrak{D}_{\mathfrak{k}}$ and $Y \in \mathcal{X}(M)$, we get the following equation by using (66) for ∇^{Vr}

$$\mathfrak{t}(\Delta\varphi)(X,Y) = \mathfrak{t}\varphi\left(\nabla_X^{Vr} Y\right) - \mathfrak{t}\varphi\left(\nabla_Y^{Vr} X\right) - \mathfrak{t}\left(\nabla_{\varphi X}^{Vr} Y\right) + \mathfrak{t}\left(\nabla_Y^{Vr} \varphi X\right). \tag{70}$$

Finally, we get the following equation by using (61) and (64),

$$\mathfrak{t}(\Delta\varphi)(X,Y) = \left(m - 2\rho_m\right)\mathfrak{t}[\mathfrak{k}X, \mathfrak{t}Y], \tag{71}$$

which proves the proposition. □

Likewise, the following proposition can be presented for the distribution $\mathfrak{D}_{\mathfrak{t}}$.

Proposition 16. *For any X vector field in $\mathfrak{D}_{\mathfrak{t}}$ and any Y vector field on M, if the vector field $[\mathfrak{t}X, \mathfrak{k}Y]$ belongs to the distribution $\mathfrak{D}_{\mathfrak{t}}$ then $\mathfrak{D}_{\mathfrak{t}}$ is half-parallel in accordance with the Vrănceanu connection ∇^{Vr}.*

Proposition 17. *According to the connection ∇^{Vr}, both distributions $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ are anti-half-parallel.*

Proof. Given $X \in \mathfrak{D}_{\mathfrak{k}}$ and $Y \in \mathcal{X}(M)$, taking into consideration equation (66) for ∇^{Vr}, we can obtain

$$\mathfrak{k}(\Delta\varphi)(X,Y) = \mathfrak{k}\varphi\left(\nabla_X^{Vr} Y\right) - \mathfrak{k}\varphi\left(\nabla_Y^{Vr} X\right) - \mathfrak{k}\left(\nabla_{\varphi X}^{Vr} Y\right) + \mathfrak{k}\left(\nabla_Y^{Vr} \varphi X\right). \tag{72}$$

By using the equations of (61) and (64), we have

$$\mathfrak{k}(\Delta\varphi)(X,Y) = \left(2\rho_m - m\right)\mathfrak{k}[\mathfrak{t}X, \mathfrak{k}Y]. \tag{73}$$

Because of $\mathfrak{t}X = 0$, one can obtain $\mathfrak{k}(\Delta\varphi)(X,Y) = 0$. Therefore, $(\Delta\varphi)(X,Y) \in \mathfrak{D}_{\mathfrak{t}}$.Similarly, it can be obtained that $\mathfrak{D}_{\mathfrak{t}}$ is anti-half-parallel regarding ∇^{Vr}. □

7. Almost Bronze Riemannian Metrics

Consider the fact that an almost product Riemannian structure is a $(P, \mathfrak{g})$ pair where P is an almost product structure on manifold M and $\mathfrak{g}$ is a Riemannian metric on M, which is related to

$$\mathfrak{g}(PX, Y) = \mathfrak{g}(X, PY) \tag{74}$$

or its equivalent

$$\mathfrak{g}(PX, PY) = \mathfrak{g}(X, Y), \tag{75}$$

for all X, Y vector fields on M. Thus, the Riemannian metric $\mathfrak{g}$ is called pure in accordance with the almost product structure P .

Definition 9. An almost bronze Riemannian structure is a pair $(\varphi, \mathfrak{g})$, which is satisfies

$$\mathfrak{g}(\varphi X, Y) = \mathfrak{g}(X, \varphi Y) \tag{76}$$

or its equivalent

$$\mathfrak{g}(\varphi X, \varphi Y) = m\mathfrak{g}(X, \varphi Y) - \mathfrak{g}(X, Y), \tag{77}$$

for all X, Y vector fields of the manifold M. Also, the triple $(M, \varphi, \mathfrak{g})$ is called an almost bronze Riemannian manifold.

From Theorem 1 and Definition 9, we get the followingproposition.

Proposition 18. *Let (P) be an almost product Riemannian structure and let (φ) be an almost bronze Riemannian structure on M. The Riemannian metric $\mathfrak{g}$ is pure in accordance with the operator φ if and only if $\mathfrak{g}$ is pure in accordance with the operator P_φ. Also, the Riemannian metric $\mathfrak{g}$ is pure in accordance with the operator P if and only if g is pure in accordance with the operator φ_P.*

Corollary 2. *On an almost bronze Riemannian manifold:*

(i) *With respect to the projectors $\mathfrak{k}$ and $\mathfrak{t}$, the Riemannian metric $\mathfrak{g}$ is pure, which means*

$$\mathfrak{g}(X, \mathfrak{k}Y) = \mathfrak{g}(\mathfrak{k}X, Y), \mathfrak{g}(X, \mathfrak{t}Y) = \mathfrak{g}(\mathfrak{t}X, Y). \tag{78}$$

(ii) *$\mathfrak{D}_{\mathfrak{k}}$, $\mathfrak{D}_{\mathfrak{t}}$ are $\mathfrak{g}$-orthogonal distributions, which means*

$$\mathfrak{g}(\mathfrak{k}X, \mathfrak{t}Y) = 0. \quad (79)$$

(iii) *The almost bronze structure is $\mathcal{N}_\varphi$-symmetric, which means*

$$\mathcal{N}_\varphi(\varphi X, Y) = \mathcal{N}_\varphi(X, \varphi Y). \quad (80)$$

If an almost product structure P is parallel in accordance with the Levi-Civita connection $\nabla^{\mathfrak{g}}$ of $\mathfrak{g}$, an almost product Riemannian structure is a locally product structure, which means $\nabla^{\mathfrak{g}}P = 0$. Also if the torsion tensor of linear connection ∇ vanishes then the Nijenhuis tensor of P satisfies the following equation:

$$\mathcal{N}_P(X, Y) = (\nabla_{PX}P)Y - P(\nabla_X P)Y - (\nabla_{PY}P)X + P(\nabla_Y P)X. \quad (81)$$

Thus, we have the following proposition.

Proposition 19. *The almost bronze structure φ is integrable if $(M, \varphi, \mathfrak{g})$ is a locally product bronze Riemannian manifold.*

Considering this finding, we can give the linear connections by making them parallel with the almost bronze structure given as follows.

Theorem 4. *For $\nabla\varphi = 0$, the set of linear connections ∇ is defined as follows:*

$$\nabla_X Y = \frac{1}{m^2 - 4}\left[\left(m^2 - 2\right)\breve{\nabla}_X Y + 2\varphi\left(\breve{\nabla}_X \varphi Y\right) - m\varphi\left(\breve{\nabla}_X Y\right) - m\left(\breve{\nabla}_X \varphi Y\right)\right] + O_{P_\varphi}Q(X, Y). \quad (82)$$

In this equation, $\breve{\nabla}$ stands for a linear connection while Q stands for $(1, 2)$ type tensor field where $O_{P_\varphi}Q$ is an associated Obata operator

$$O_{P_\varphi}Q(X, Y) = \frac{1}{2}\left[Q(X, Y) + P_\varphi Q\left(X, P_\varphi Y\right)\right], \quad (83)$$

for each field X, Y on the manifold M.

We complete the study of the almost bronze structure with the following example.

Example 7. For any C^∞ differentiable functions f and g depending on (x, y),

$$\begin{aligned} \mathfrak{k} &= \frac{f^2}{f^2 + g^2}\frac{\partial}{\partial x} \otimes dx + \frac{fg}{f^2 + g^2}\frac{\partial}{\partial x} \otimes dy + \frac{fg}{f^2 + g^2}\frac{\partial}{\partial y} \otimes dx + \frac{g^2}{f^2 + g^2}\frac{\partial}{\partial y} \otimes dy, \\ \mathfrak{t} &= \frac{g^2}{f^2 + g^2}\frac{\partial}{\partial x} \otimes dx - \frac{fg}{f^2 + g^2}\frac{\partial}{\partial x} \otimes dy - \frac{fg}{f^2 + g^2}\frac{\partial}{\partial y} \otimes dx + \frac{f^2}{f^2 + g^2}\frac{\partial}{\partial y} \otimes dy \end{aligned} \quad (84)$$

where $f^2 + g^2 \neq 0$, are projection operators in $\mathbb{R}^2$ and they satisfy the conditions in (60).

$$\mathfrak{D}_{\mathfrak{k}} = \mathrm{Sp}\left\{f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right\} \text{ and } \mathfrak{D}_{\mathfrak{t}} = \mathrm{Sp}\left\{g\frac{\partial}{\partial x} - f\frac{\partial}{\partial y}\right\} \quad (85)$$

are complementary distributions that correspond to the $\mathfrak{k}$ and $\mathfrak{t}$ projection operators, respectively. In terms of the Euclidean metric of $\mathbb{R}^2$, the distributions $\mathfrak{D}_{\mathfrak{k}}$ and $\mathfrak{D}_{\mathfrak{t}}$ are orthogonal. Furthermore, these distributions are connected to the almost bronze structure

$$\begin{aligned} \varphi\left(\frac{\partial}{\partial x}\right) &= \frac{\rho_m f^2 + \bar{\rho}_m g^2}{f^2 + g^2}\frac{\partial}{\partial x} + \frac{(m - 2\bar{\rho}_m)fg}{f^2 + g^2}\frac{\partial}{\partial y}, \\ \varphi\left(\frac{\partial}{\partial y}\right) &= \frac{(m - 2\bar{\rho}_m)fg}{f^2 + g^2}\frac{\partial}{\partial x} + \frac{\bar{\rho}_m f^2 + \rho_m g^2}{f^2 + g^2}\frac{\partial}{\partial y}, \end{aligned} \quad (86)$$

which is integrable since $\mathcal{N}_\varphi(\partial/\partial x, \partial/\partial y) = 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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