Hindawi
Journal of Mathematics
Volume 2022, Article ID 6946424, 7 pages
https://doi.org/10.1155/2022/6946424

Research Article

Coefficient Bounds for a Certain Family of Biunivalent Functions Defined by Gegenbauer Polynomials

Isra Al-Shbeil,[1] **Abbas Kareem Wanas**,[2] **Abdelkader Benali**,[3] **and Adriana Cătaş**[4]

[1]*Department of Mathematics, Faculty of Sciences, University of Jordan, Amman 11942, Jordan*
[2]*Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah, Al-Qadisiyah, Iraq*
[3]*Mathematics Department, Faculty of the Exact Sciences and Computer, Universiy of Hassiba Benbouali, Laboratory LMA, Chlef 02000, Algeria*
[4]*Department of Mathematics and Computer Science, University of Oradea, 1 University Street, Oradea 410087, Romania*

Correspondence should be addressed to Adriana Cătaş; acatas@gmail.com

Received 20 May 2022; Revised 22 August 2022; Accepted 8 November 2022; Published 30 November 2022

Academic Editor: Barbara Martinucci

In the present work, by making use of Gegenbauer polynomials, we introduce and study a certain family of λ-pseudo bistarlike and λ-pseudo biconvex functions with respect to symmetrical points defined in the open unit disk. We obtain estimates for initial coefficients and solve the Fekete–Szegö problem for functions that belong to this family. Furthermore, we give connections to some of the earlier known results.

1. Introduction

In [1], Legendre studied orthogonal polynomials comprehensively. The importance of orthogonal polynomials for contemporary mathematics as well as for a wide range of their applications in physics and engineering is beyond any doubt. It is well known that these polynomials play an essential role in problems of approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [2, 3]. In practice, Gegenbauer polynomials are a special case of orthogonal polynomials. They are representatively related to typically real functions T_R as discovered in [4]. Typically, real functions play an important role in geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role in estimating coefficient bounds, where S_R indicates the family of univalent functions in the unit disk with real coefficients and $\overline{co}S_R$ denotes the closed convex hull of S_R.

On this subject in geometric function theory, so-called Fekete–Szegö type inequalities (or problems) estimate some upper bounds for $|a_3 - \mu a_2^2|$ for holomorphic univalent functions. Its origin was in the disproof by Fekete and Szego of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [5])

We consider $\mathscr{A}$, the set of functions f, which are holomorphic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$, having the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S be denoted as the subfamily of $\mathscr{A}$ consisting of the functions which are univalent in U.

According to the Koebe one-quarter theorem [6], each function $f \in S$ has an inverse f^{-1}, which fulfills

$$f^{-1}(f(z)) = z, \quad (z \in U), \tag{2}$$

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right), \tag{3}$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots. \tag{4}$$

We say that a function $f \in \mathscr{A}$ is biunivalent in U if both f and f^{-1} are univalent in U. Let Σ indicate the class of biunivalent functions in U given by (1). Starting with Srivastava et al. pioneering work [7] on the subject, a large number of works related to the subject have been presented (see, for example [4, 8–24]). We notice that the family Σ is not empty. Some examples of functions in the class Σ are

$$\begin{gathered} \frac{z}{1-z}, \\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \text{and} - \log(1-z), \end{gathered} \tag{5}$$

with the corresponding inverse functions

$$\begin{gathered} \frac{w}{1+w}, \\ \frac{e^{2w}-1}{e^{2w}+1}, \\ \frac{e^{w}-1}{e^{w}}, \end{gathered} \tag{6}$$

respectively. We recall here other common examples of functions that are not members of Σ, namely,

$$\begin{gathered} z - \frac{z^2}{2}, \text{ and} \\ \frac{z}{1-z^2}. \end{gathered} \tag{7}$$

So far, the coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

$$\left|a_n\right|, \quad (n = 3, 4, \ldots), \tag{8}$$

for functions $f \in \Sigma$ is still an open problem.

We say that a function $f \in S$ is starlike with respect to symmetrical points if (see [25])

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in U. \tag{9}$$

The subset of all such functions is denoted by S_s^*.

The family of starlike functions with respect to symmetrical points obviously includes the family of convex functions with respect to symmetrical points C_s, satisfying the following condition:

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0, \quad z \in U. \tag{10}$$

We say that a function $f \in \mathscr{A}$ is λ-pseudostarlike function in U if (see [26])

$$\mathrm{Re}\left\{\frac{z(f'(z))^{\lambda}}{f(z)}\right\} > 0, \quad (z \in U, \lambda \geq 1). \tag{11}$$

We consider two functions f and g that are holomorphic in U. We say that the function f is subordinate to g if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)\,(z \in U)$. It is well known that (see [27]) if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Recently, Amourah et al. [28] have studied Gegenbauer polynomials $H_\delta(z,t)$, which are given by the following recurrence relation.

For a nonzero real constant δ, a generating function of Gegenbauer polynomials is defined by

$$H_\delta(z,t) = \frac{1}{\left(1 - 2tz + z^2\right)^{\delta}}, \tag{12}$$

where $t \in [-1,1]$ and $z \in U$. For fixed t, the function H_δ is holomorphic in U, so it can be expanded in a Taylor series, and note that if $t = \cos\beta$, where $\beta \in (-(\pi/3), (\pi/3))$, then

$$H_\delta(z,t) = \frac{1}{\left(1 - 2tz + z^2\right)^{\delta}} = \sum_{n=0}^{\infty} \mathscr{G}_n^{\delta}(t) z^n, \tag{13}$$

where $\mathscr{G}_n^{\delta}(t)$ is the Gegenbauer polynomial of the degree n.

Obviously, H_δ generates nothing when $\delta = 0$. Thus, the generating function of the Gegenbauer polynomial is set to be

$$\begin{aligned} H_0(z,t) &= 1 - \log\left(1 - 2tz + z^2\right) \\ &= \sum_{n=0}^{\infty} \mathscr{G}_n^{0}(t) z^n. \end{aligned} \tag{14}$$

Furthermore, it is worth to mention that a normalization of δ to be greater than $-(1/2)$ is desirable [3, 29]. Gegenbauer polynomials can also be defined by the following recurrence relations:

$$\mathscr{G}_n^{\delta}(t) = \frac{1}{2}\left[2t(n + \delta - 1)\mathscr{G}_{n-1}^{\delta}(t) - (n + 2\delta - 2)\mathscr{G}_{n-1}^{\delta}(t)\right]. \tag{15}$$

The initial values are expressed as

$$\begin{aligned} \mathscr{G}_0^{\delta}(t) &= 1, \\ \mathscr{G}_1^{\delta}(t) &= 2\delta t, \\ \mathscr{G}_2^{\delta}(\mathrm{t}) &= 2\delta(\delta + 1)\mathrm{t}^2 - \delta. \end{aligned} \tag{16}$$

Remark 1. By choosing the particular values of δ, the Gegenbauer polynomial $\mathscr{G}_n^{\delta}(t)$ reduces to well-known polynomials. These special cases are as follows:

(1) Taking $\delta = 1$, we obtain Chebyshev polynomials

(2) Taking $\delta = (1/2)$, we obtain Legendre polynomials

2. Main Results

This section starts with defining the new family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$.

Definition 1. We consider that $\lambda\geq 1$, $0\leq\gamma\leq 1$, $t\in((1/2),1]$ and that δ is a nonzero real constant. A function $f\in\Sigma$ is said to be in the family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$ if both f and its inverse $g=f^{-1}$ given by (4) fulfill the following subordinations:

$$\left(\frac{2z(f'(z))^\lambda}{f(z)-f(-z)}\right)^\gamma\left(\frac{2((zf'(z))')^\lambda}{(f(z)-f(-z))'}\right)^{1-\gamma}\prec H_\delta(z,t)=\frac{1}{(1-2tz+z^2)^\delta}, \tag{17}$$

$$\left(\frac{2w(g'(w))^\lambda}{g(w)-g(-w)}\right)^\gamma\left(\frac{2((wg'(w))')^\lambda}{(g(w)-g(-w))'}\right)^{1-\gamma}\prec H_\delta(w,t)=\frac{1}{(1-2tw+w^2)^\delta}. \tag{18}$$

In particular, if we choose $\delta=1$ and $\lambda=1$ in Definition 1, the family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$ becomes the family $\mathscr{D}^s_\Sigma(\gamma,t)$, which Wanas defined in [30].

If we choose $\delta=1$ in Definition 1, the family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$ reduces to the family $T^s_\Sigma(\lambda,\gamma,t)$, which Wanas gave in [31]. As a consequence, we will generalize, in the following theorem. his main result given in [31, Theorem 2.1]

Theorem 1. *Let f, a holomorphic function, given by (1). If f is in the family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$, where $\lambda\geq 1$, $0\leq\gamma\leq 1$, $t\in((1/2),1]$, and $\delta\neq 0\in\mathbb{R}$, then the following are the upper bonds for the Taylor–Maclaurin coefficients a_2 and a_3:*

$$\begin{aligned}|a_2|&\leq\frac{|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\delta^2t^2(\lambda+2\gamma-3)-\delta\lambda^2(\gamma-2)^2(2t^2-1)\right|}},\\ |a_3|&\leq\frac{\delta^2t^2}{\lambda^2(\gamma-2)^2}+\frac{2|\delta|t}{(3\lambda-1)|3-2\gamma|}.\end{aligned} \tag{19}$$

Proof. We suppose that f is an element of the family $\mathbb{S}_\Sigma(\lambda,\gamma,t,\delta)$, and by Definition 1, f and its inverse g are biunivalent functions of Σ from U to U, satisfying subordinations (17) and (18), and hence, we have two holomorphic functions:

$$\begin{aligned}u&=\sum_{n=1}^\infty u_nz^n,\text{ and}\\ v&=\sum_{n=1}^\infty v_nw^n,\quad\text{for } z,\ w\in U\end{aligned} \tag{20}$$

$$\begin{aligned}&|u(z)|<1,\text{ and}\\ &|v(w)|<1,\\ &\text{for } z,w\in U.\end{aligned} \tag{21}$$

We have

$$\begin{aligned}&\left(\frac{2z(f'(z))^\lambda}{f(z)-f(-z)}\right)^\gamma\left(\frac{2((zf'(z))')^\lambda}{(f(z)-f(-z))'}\right)^{1-\gamma}\\ &=\sum_{n=0}^\infty\mathscr{G}_n^\delta(t)u^n(z)\\ &=1+\mathscr{G}_1^\delta(t)u_1z+\left[\mathscr{G}_1^\delta(t)u_2+\mathscr{G}_2^\delta(t)u_1^2\right]z^2+\cdots,\end{aligned} \tag{22}$$

$$\begin{aligned}&\left(\frac{2w(g'(w))^\lambda}{g(w)-g(-w)}\right)^\gamma\left(\frac{2((wg'(w))')^\lambda}{(g(w)-g(-w))'}\right)^{1-\gamma}\\ &=\sum_{n=0}^\infty\mathscr{G}_n^\delta(t)v^n(w)\\ &=1+\mathscr{G}_1^\delta(t)v_1w+\left[\mathscr{G}_1^\delta(t)v_2+\mathscr{G}_2^\delta(t)v_1^2\right]w^2+\cdots.\end{aligned} \tag{23}$$

Since the holomorphic functions u and v are from U to U with $u(0)=v(0)=0$, $|u(z)|<1$ and $|v(w)|<1$ for $z,w\in U$, and hence, we have the well-known fact

$$\begin{aligned}&|u_n|\leq 1,\text{ and}\\ &|v_n|\leq 1,\\ &\text{for all}\quad n\in\mathbb{N}.\end{aligned} \tag{24}$$

By comparing corresponding coefficients in (22) and (23), we obtain

$$-2\lambda(\gamma-2)a_2=\mathscr{G}_1^\delta(t)u_1, \tag{25}$$

$$2\left[\lambda^2(\gamma-2)^2+\lambda(3\gamma-4)\right]a_2^2+(3\lambda-1)(3-2\gamma)a_3=\mathscr{G}_1^\delta(t)u_2+\mathscr{G}_2^\delta(t)u_1^2, \tag{26}$$

$$2\lambda(\gamma-2)a_2=\mathscr{G}_1^\delta(t)v_1. \tag{27}$$

$$2\left[\lambda^2(\gamma-2)^2+\lambda(5-3\gamma)+(2\gamma-3)\right]a_2^2+(3\lambda-1)(2\gamma-3)a_3=\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2. \tag{28}$$

It follows from (25) and (27) that

$$u_1=-v_1, \tag{29}$$

$$8\lambda^2(\gamma-2)^2a_2^2=\left(\mathcal{G}_1^{\delta}(t)\right)^2\left(u_1^2+v_1^2\right). \tag{30}$$

By adding (26) to (28), we obtain

$$2\left(2\lambda^2(\gamma-2)^2+(\lambda+2\gamma-3)\right)a_2^2=\mathcal{G}_1^{\delta}(t)(u_2+v_2)+\mathcal{G}_2^{\delta}(t)\left(u_1^2+v_1^2\right). \tag{31}$$

By eliminating $u_1^2+v_1^2$ from (30) and (31), we obtain

$$2\left[2\lambda^2(\gamma-2)^2\left(1-\frac{2\mathcal{G}_2^{\delta}(t)}{\left(\mathcal{G}_1^{\delta}(t)\right)^2}\right)+(\lambda+2\gamma-3)\right]a_2^2=\mathcal{G}_1^{\delta}(t)(u_2+v_2). \tag{32}$$

Using (16), (24) and (32), we reach the first inequality

$$|a_2|\le\frac{|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\delta^2t^2(\lambda+2\gamma-3)-\delta\lambda^2(\gamma-2)^2\left(2t^2-1\right)\right|}}. \tag{33}$$

Next, by subtracting (28) from (26), we obtain

$$2(3\lambda-1)(3-2\gamma)\left(a_3-a_2^2\right)=\mathcal{G}_1^{\delta}(t)(u_2-v_2)+\mathcal{G}_2^{\delta}(t)\left(u_1^2-v_1^2\right). \tag{34}$$

Thus, in view of (29) and (30), (34) gives

$$a_3=\frac{\left(\mathcal{G}_1^{\delta}(t)\right)^2}{8\lambda^2(\gamma-2)^2}\left(u_1^2+v_1^2\right)+\frac{\mathcal{G}_1^{\delta}(t)}{2(3\lambda-1)(3-2\gamma)}(u_2-v_2). \tag{35}$$

Finally, by applying (16), we complete the proof by

$$|a_3|\le\frac{\delta^2t^2}{\lambda^2(\gamma-2)^2}+\frac{2|\delta|t}{(3\lambda-1)|3-2\gamma|}. \tag{36}$$

In particular, Theorem 1 gives the following corollaries for some specific values of γ,δ, and λ.

If $\gamma=1$, then the family $\mathbb{S}_{\Sigma}(\lambda,\gamma,t,\delta)=\mathbb{S}_{\Sigma}(\lambda,1,t,\delta)$, the class of λ-pseudo bistarlike functions with respect to symmetrical points. □

Corollary 1. *For $\lambda\ge1$, $t\in((1/2),1$, and δ is a nonzero real constant, let $f\in\mathscr{A}$ be in the family $\mathbb{S}_{\Sigma}(\lambda,1,t,\delta)$. Then,*

$$|a_2|\le\frac{|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\delta^2t^2(\lambda-1)-\delta\lambda^2\left(2t^2-1\right)\right|}}. \tag{37}$$

And we obtain

$$|a_3|\le\frac{\delta^2t^2}{\lambda^2}+\frac{2|\delta|t}{3\lambda-1}. \tag{38}$$

If $\gamma=0$, then the family $\mathbb{S}_{\Sigma}(\lambda,\gamma,t,\delta)=\mathbb{S}_{\Sigma}(\lambda,t,\delta)$, the class of λ-pseudo biconvex functions with respect to symmetrical points.

Corollary 2. *For $\lambda\ge1$, $t\in((1/2),1]$, and δ is a nonzero real constant, let $f\in\mathscr{A}$ be in the family $\mathbb{S}_{\Sigma}(\lambda,t,\delta)$. Then,*

$$|a_2|\le\frac{|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\delta^2t^2(\lambda-3)-4\delta\lambda^2\left(2t^2-1\right)\right|}}. \tag{39}$$

And we obtain

$$|a_3| \le \frac{\delta^2 t^2}{4\lambda^2} + \frac{2|\delta| t}{3(3\lambda - 1)}. \tag{40}$$

If $\gamma = 0$, Theorem 1 gives the following corollary by Wanas [31] for the well-known family $T_\Sigma^s(\lambda, \gamma, t)$.

Corollary 3 (see [31]). *For $\lambda \ge 1$, $0 \le \gamma \le 1$, and $t \in ((1/2), 1]$, let $f \in \mathscr{A}$ be in the family $T_\Sigma^s(\lambda, \gamma, t)$. Then,*

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{\left|(\lambda + 2\gamma - 3)t^2 - \lambda^2(\gamma - 2)^2\left(2t^2 - 1\right)\right|}}. \tag{41}$$

And we obtain

$$|a_3| \le \frac{t^2}{\lambda^2(\gamma - 2)^2} + \frac{2t}{(3\lambda - 1)|3 - 2\gamma|}. \tag{42}$$

If $\delta = \lambda = 1$, Theorem 1 the following corollary by Wanas [30] for the well-known family $\mathscr{D}_\Sigma^s(\gamma, t)$.

Corollary 4 (see [30]). *For $0 \le \gamma \le 1$ and $t \in ((1/2), 1]$, let $f \in \mathscr{A}$ be in the family $\mathscr{D}_\Sigma^s(\gamma, t)$. Then,*

$$|a_2| \le \frac{t\sqrt{2t}}{\sqrt{\left|2(\gamma - 1)t^2 - (\gamma - 2)^2\left(2t^2 - 1\right)\right|}}. \tag{43}$$

And we obtain

$$|a_3| \le \frac{t^2}{(\gamma - 2)^2} + \frac{t}{|3 - 2\gamma|}. \tag{44}$$

Furthermore, if $\delta = \lambda = 1$, Corollary 2 gives the following corollary by Wanas and Majeed [32] for the well-known family $F_\Sigma^{sc}(t)$.

Corollary 5 (see [32]). *For $t \in ((1/2), 1]$, let $f \in \mathscr{A}$ be in the family $F_\Sigma^{sc}(t)$. Then,*

$$|a_2| \le \frac{t\sqrt{t}}{\sqrt{\left|2 - 5t^2\right|}}. \tag{45}$$

And we obtain

$$|a_3| \le \frac{t(3t + 4)}{12}. \tag{46}$$

In the next theorem, we present the "Fekete–Szegö problem" for the family $\mathbb{S}_\Sigma(\lambda, \gamma, t, \delta)$.

Theorem 2. *For $\lambda \ge 1$, $0 \le \gamma \le 1$, and $t \in ((1/2), 1]$, $\mu \in \mathbb{R}$, and δ is a nonzero real constant, let $f \in \mathscr{A}$ be in the family $\mathbb{S}_\Sigma(\lambda, \gamma, t, \delta)$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|}{(3\lambda - 1)(3 - 2\gamma)}; \\[2ex] \text{for } |\mu - 1| \le \dfrac{\left|\delta^2 t^2(\lambda + 2\gamma - 3) - \delta\lambda^2(\gamma - 2)^2\left(2t^2 - 1\right)\right|}{\delta^2 t^2(3\lambda - 1)(3 - 2\gamma)} \\[3ex] \dfrac{4t^3\left|\delta^3\right||\mu - 1|}{\left|\delta^2 t^2(\lambda + 2\gamma - 3) - \delta\lambda^2(\gamma - 2)^2\left(2t^2 - 1\right)\right|}; \\[2ex] \text{for } |\mu - 1| \ge \dfrac{\left|\delta^2 t^2(\lambda + 2\gamma - 3) - \delta\lambda^2(\gamma - 2)^2\left(2t^2 - 1\right)\right|}{\delta^2 t^2(3\lambda - 1)(3 - 2\gamma)}. \end{cases} \tag{47}$$

Proof. In the light of (32) and (34), we deduce that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1 - \mu)\frac{\left(\mathscr{G}_1^\delta(t)\right)^3(u_2 + v_2)}{2\left[\left(2\lambda^2(\gamma - 2)^2 + \lambda + 2\gamma - 3\right)\left(\mathscr{G}_1^\delta(t)\right)^2 - 4\lambda^2(\gamma - 2)^2\mathscr{G}_2^\delta(t)\right]} + \frac{\mathscr{G}_1^\delta(t)(u_2 - v_2)}{2(3\lambda - 1)(3 - 2\gamma)} \\ &= \mathscr{G}_1^\delta(t)\left[\left(\psi(\mu) + \frac{1}{2(3\lambda - 1)(3 - 2\gamma)}\right)u_2 + \left(\psi(\mu) - \frac{1}{2(3\lambda - 1)(3 - 2\gamma)}\right)v_2\right], \end{aligned} \tag{48}$$

where

$$\psi(\mu) = \frac{\left(\mathscr{G}_1^{\delta}(t)\right)^2 (1-\mu)}{2\left[\left(2\lambda^2(\gamma-2)^2+\lambda+2\gamma-3\right)\left(\mathscr{G}_1^{\delta}(t)\right)^2 - 4\lambda^2(\gamma-2)^2\mathscr{G}_2^{\delta}(t)\right]}. \tag{49}$$

According to (16), we deduce that

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|}{(3\lambda-1)(3-2\gamma)}, & 0\le|\psi(\mu)|\le\dfrac{1}{2(3\lambda-1)(3-2\gamma)}, \\ 4t|\delta||\psi(\mu)|, & |\psi(\mu)|\ge\dfrac{1}{2(3\lambda-1)(3-2\gamma)}. \end{cases} \tag{50}$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|}{(3\lambda-1)(3-2\gamma)}; \\ \text{for } |\mu-1| \le \dfrac{\left|\delta^2t^2(\lambda+2\gamma-3)-\delta\lambda^2(\gamma-2)^2\left(2t^2-1\right)\right|}{\delta^2t^2(3\lambda-1)(3-2\gamma)} \\ \dfrac{4t^3\left|\delta^3\right||\mu-1|}{\left|\delta^2t^2(\lambda+2\gamma-3)-\delta\lambda^2(\gamma-2)^2\left(2t^2-1\right)\right|}; \\ \text{for } |\mu-1| \ge \dfrac{\left|\delta^2t^2(\lambda+2\gamma-3)-\delta\lambda^2(\gamma-2)^2\left(2t^2-1\right)\right|}{\delta^2t^2(3\lambda-1)(3-2\gamma)} \end{cases} \tag{51}$$

In particular, putting $\mu = 1$ in Theorem 2, we conclude the following result. □

Corollary 6. *For $\lambda \ge 1$, $0 \le \gamma \le 1$, and $t \in ((1/2), 1]$, and δ is a nonzero real constant, let $f \in \mathscr{A}$ be in the family $\mathbb{S}_{\Sigma}(\lambda,\gamma,t,\delta)$. Then,*

$$\left|a_3-a_2^2\right| \le \frac{2t|\delta|}{(3\lambda-1)(3-2\gamma)}. \tag{52}$$

3. Conclusion

In the present work, we obtain a family $\mathbb{S}_{\Sigma}(\lambda,\gamma,t,\delta)$ of λ-pseudo bistarlike and λ-pseudo biconvex functions with respect to symmetrical points defined by Gegenbauer polynomials. We generated Taylor–Maclaurin coefficient inequalities of functions belonging to this family and viewed the famous Fekete–Szegö problem. Furthermore, by specifying parameters, consequences of this family are mentioned.

Data Availability

The data are available upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Legendre, "Recherches sur la attraction sphéroides homog énes," vol. 10Mémoires présentes par divers savants a laAcad émie des Sciences de la Institut de France, Paris, France, 1785.

[2] H. Bateman, *Higher Transcendental Functions*, McGraw-Hill, New York, NY, USA, 1953.

[3] B. Doman, *The Classical Orthogonal Polynomials*, World Scientific, Singapore, 2015.

[4] K. Kiepiela, I. Naraniecka, and J. Szynal, "The Gegenbauer polynomials and typically real functions," *Journal of Computational and Applied Mathematics*, vol. 153, no. 1-2, pp. 273–282, 2003.

[5] M. Fekete and G. Szegö, "Eine bemerkung uber ungerade schlichte funktionen," *Journal of the London Mathematical Society*, vol. 1-8, pp. 85–89, 1933.

[6] P. L. Duren, *Univalent functions, grundlehren der Mathematischen wissenschaften*, Springer-Verlag, Berlin, Germany, 1983.

[7] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.

[8] E. A. Adegani, S. Bulut, and A. A. Zireh, "Coefficient estimates for a subclass of analytic bi-univalent functions," *Bull. Korean Math. Soc.*vol. 55, no. 2, pp. 405–413, 2018.

[9] A. G. Alamoush, "Coefficient estimates for a new subclasses of lambda-pseudo bi-univalent functions with respect to symmetrical points associated with the Horadam Polynomials," *Turkish Journal of Mathematics*, vol. 43, no. 6, pp. 2865–2875, 2019.

[10] A. G. Al Amoush and S. Bulut, "On a subclass of bi-close-to-convex functions by means of the Gegenbauer polynomial," *Mathematics for Application*, vol. 10, pp. 93–101, 2022.

[11] S. Altinkaya and S. Yalçin, "On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions," *Gulf Journal of Mathematics*, vol. 5, no. 3, pp. 34–40, 2017.

[12] M. Çaglar, "Chebyshev polynomial coefficient bounds for a subclass of bi-univalent functions," *Comptes Rendus De L'Academie Bulgare Des Sciences*, vol. 72, no. 12, pp. 1608–1615, 2019.

[13] M. Çaglar, H. Orhan, and N. Yagmur, "Coefficient bounds for new subclasses of bi univalent functions," *Filomat*, vol. 27, no. 7, pp. 1165–1171, 2013.

[14] H. O. Guney, G. Murugusundaramoorthy, and J. Sokol, "Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers," *Acta Universitatis Sapientiae, Mathematica*, vol. 10, no. 1, pp. 70–84, 2018.

[15] M. Kamali, M. Çaglar, E. Deniz, and M. Turabaev, "Fekete-Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials," *Turkish Journal of Mathematics*, vol. 45, no. 3, pp. 1195–1208, 2021.

[16] N. Magesh and J. Yamini, "Fekete- Szegö problem and second Hankel determinant for a class of bi-univalent functions," *Tbilisi Mathematical Journal*, vol. 11, no. 1, pp. 141–157, 2018.

[17] G. Murugusundaramoorthy and T. Bulboacă, "Subclasses of yamakawa-type Bi-starlike functions associated with gegenbauer polynomials," *Axioms*, vol. 11, no. 3, p. 92, 2022.

[18] H. Orhan, N. Magesh, and V. K. Balaji, "Initial coefficient bounds for a general class of bi-univalent functions," *Filomat*, vol. 29, no. 6, pp. 1259–1267, 2015.

[19] P. S. Á. Orsolya and A. K. Wanas, "Coefficient estimates for some new classes of bi- Bazilevic functions of Ma-Minda type involving the Salagean integro-differential operator," *Quaestiones Mathematicae*, vol. 44, no. 4, pp. 495–502, 2021.

[20] B. Seker, "On a new subclass of bi-univalent functions defined by using Salagean operator," *Turkish Journal of Mathematics*, vol. 42, no. 6, pp. 2891–2896, 2018.

[21] B. Seker and I. Taymur, "On subclasses of m-fold symmetric bi-univalent functions," *TWMS Journal of Applied and Engineering Mathematics*, vol. 11, no. 2, pp. 598–604, 2021.

[22] T. G. Shaba and A. B. Patil, "Coefficient estimates for certain subclasses of m-fold symmetric bi-univalent functions associated with pseudo-starlike functions," *Earthline Journal of Mathematical Sciences*, vol. 6, no. 2, pp. 2581–8147, 2021.

[23] H. M. Srivastava, S. Bulut, M. Çağlar, and N. Yagmur, "Coefficient estimates for a general subclass of analytic and bi univalent functions," *Filomat*, vol. 27, no. 5, pp. 831–842, 2013.

[24] H. M. Srivastava, S. Hussain, A. Raziq, and M. Raza, "The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination," *Carpathian Journal of Mathematics*, vol. 34, no. 1, pp. 103–113, 2018.

[25] K. Sakaguchi, "On a certain univalent mapping," *Journal of Mathematical Society of Japan*, vol. 11, no. 1, pp. 72–75, 1959.

[26] K. O. Babalola, "On λ-pseudo-starlike functions," *Journal of Classical Analysis*, vol. 3, no. 2, pp. 137–147, 2013.

[27] S. S. Miller and P. T. Mocanu, "Differential subordinations: theory and applications," *Series on monographs and textbooks in pure and applied mathematics*, Vol. 225, Marcel Dekker, New York, NY, USA, 2000.

[28] A. Amourah, A. Alamoush, and M. Al-Kaseasbeh, "Gegenbauer polynomials and bi-univalent functions," *Palestine Journal of Mathematics*, vol. 10, no. 2, pp. 625–632, 2021.

[29] M. Reimer, *Multivariate Polynomial Approximation*, Birkhäuser, Basel, Switzerland, 2012.

[30] A. K. Wanas, "Coefficient estimates of bi-starlike and bi-convex functions with respect to symmetrical points associated with the second kind Chebyshev polynomials," *Earthline Journal of Mathematical Sciences*, vol. 3, pp. 191–198, 2020.

[31] A. K. Wanas, "Applications of Chebyshev polynomials on λ -pseudo bi-starlike and λ -pseudo bi-convex functions with respect to symmetrical points," *TWMS Journal of Applied and Engineering Mathematics*, vol. 10, no. 3, pp. 568–573, 2020.

[32] A. K. Wanas and A. H. Majeed, "Chebyshev polynomial bounded for analytic and bi-univalent functions with respect to symmetric conjugate points," *Applied Mathematics E-Notes*, vol. 19, pp. 14–21, 2019.