Research Article

Coefficient Bounds for a Certain Family of Biunivalent Functions Defined by Gegenbauer Polynomials

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In the present work, by making use of Gegenbauer polynomials, we introduce and study a certain family of \( \lambda \)-pseudobistarlike and \( \lambda \)-pseudobiconvex functions with respect to symmetrical points defined in the open unit disk. We obtain estimates for initial coefficients and solve the Fekete–Szegő problem for functions that belong to this family. Furthermore, we give connections to some of the earlier known results.

1. Introduction

In [1], Legendre studied orthogonal polynomials comprehensively. The importance of orthogonal polynomials for contemporary mathematics as well as for a wide range of their applications in physics and engineering is beyond any doubt. It is well known that these polynomials play an essential role in problems of approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [2, 3].

In practice, Gegenbauer polynomials are a special case of orthogonal polynomials. They are representatively related to typically real functions \( T_R \) as discovered in [4]. Typically, real functions play an important role in geometric function theory because of the relation \( T_R = \overline{\partial} S_R \) and its role in estimating coefficient bounds, where \( S_R \) indicates the family of univalent functions in the unit disk with real coefficients and \( \overline{\partial} S_R \) denotes the closed convex hull of \( S_R \).

On this subject in geometric function theory, so-called Fekete–Szegő type inequalities (or problems) estimate some upper bounds for \( |a_3 - \mu a_2^2| \) for holomorphic univalent functions. Its origin was in the disproof by Fekete and Szego of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [5]).

We consider \( \mathscr{A} \), the set of functions \( f \), which are holomorphic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), having the following form:

\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}
\]

Let \( S \) be denoted as the subfamily of \( \mathscr{A} \) consisting of the functions which are univalent in \( U \).

According to the Koebe one-quarter theorem [6], each function \( f \in S \) has an inverse \( f^{-1} \), which fulfills

\[
    f^{-1}(f(z)) = z, \quad (z \in U), \tag{2}
\]

\[
    f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}). \tag{3}
\]

where
\[ g(w) = f^{-1}(w) = w - a_3w^2 + (2a_2^2 - a_1)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (4) \]

We say that a function \( f \in A \) is biunivalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) indicate the class of biunivalent functions in \( U \) given by (1). Starting with Srivastava et al. pioneering work [7] on the subject, a large number of works related to the subject have been presented (see, for example [4, 8–24]). We notice that the family \( \Sigma \) is not empty. Some examples of functions in the class \( \Sigma \) are

\[
\begin{align*}
\frac{z}{1-z}, \\
\frac{1}{2} \log \left( \frac{1+z}{1-z} \right) \text{ and } -\log(1-z),
\end{align*}
\]

with the corresponding inverse functions

\[
\begin{align*}
\frac{w}{1+w}, \\
\frac{e^{2w} - 1}{e^{2w} + 1}, \\
\frac{e^w - 1}{e^w},
\end{align*}
\]

respectively. We recall here other common examples of functions that are not members of \( \Sigma \), namely,

\[
\begin{align*}
z - \frac{z^2}{2}, \\
\frac{z}{1-z^2},
\end{align*}
\]

So far, the coefficient estimate problem for each of the following Taylor–Maclaurin coefficients:

\[ |a_n|, \quad (n = 3, 4, \ldots), \]

for functions \( f \in \Sigma \) is still an open problem.

We say that a function \( f \in S \) is starlike with respect to symmetrical points if (see [25])

\[
\text{Re}\left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in U. \tag{9}
\]

The subset of all such functions is denoted by \( S^*_\times \).

The family of starlike functions with respect to symmetrical points obviously includes the family of convex functions with respect to symmetrical points \( C_\times \), satisfying the following condition:

\[
\text{Re}\left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in U. \tag{10}
\]

We say that a function \( f \in A \) is \( \lambda \)-pseudostarlike function in \( U \) if (see [26])

\[
\text{Re}\left\{ \frac{zf'(z)^\lambda}{f(z)} \right\} > 0, \quad (z \in U, \lambda \geq 1). \tag{11}
\]

We consider two functions \( f \) and \( g \) that are holomorphic in \( U \). We say that the function \( f \) is subordinate to \( g \) if there exists a Schwarz function \( w \) holomorphic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)) such that \( f(z) = g(w(z)) \). This subordination is denoted by \( f \prec g \). It is well known that (see [27]) if the function \( g \) is univalent in \( U \), then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Recently, Amourah et al. [28] have studied Gegenbauer polynomials \( H_\delta(z, t) \), which are given by the following recurrence relation.

For a nonzero real constant \( \delta \), a generating function of Gegenbauer polynomials is defined by

\[
H_\delta(z, t) = \frac{1}{\left(1 - 2tz + z^2\right)^\delta} \tag{12}
\]

where \( t \in [-1, 1] \) and \( z \in U \). For fixed \( t \), the function \( H_\delta \) is holomorphic in \( U \), so it can be expanded in a Taylor series, and note that if \( t = \cos \beta \), where \( \beta \in (-\pi/3, (\pi/3)) \), then

\[
H_\delta(z, t) = \frac{1}{\left(1 - 2tz + z^2\right)^\delta} = \sum_{n=0}^{\infty} G_\delta^n(t)z^n, \quad \delta \neq 0. \tag{13}
\]

where \( G_\delta^n(t) \) is the Gegenbauer polynomial of the degree \( n \). Obviously, \( H_\delta \) generates nothing when \( \delta = 0 \). Thus, the generating function of the Gegenbauer polynomial is set to be

\[
H_0(z, t) = 1 - \log\left(1 - 2tz + z^2\right) \tag{14}
\]

\[
= \sum_{n=0}^{\infty} G_0^n(t)z^n.
\]

Furthermore, it is worth to mention that a normalization of \( \delta \) to be greater than \(-1/2\) is desirable [3, 29]. Gegenbauer polynomials can also be defined by the following recurrence relations:

\[
G_\delta^n(t) = \frac{1}{2} \left[ 2t(n + \delta - 1)G_\delta^{n-1}(t) - (n + 2\delta - 2)G_\delta^{n-1}(t) \right]. \tag{15}
\]

The initial values are expressed as

\[
G_\delta^0(t) = 1, \\
G_\delta^1(t) = 2\delta t, \\
G_\delta^2(t) = 2\delta(\delta + 1)t^2 - \delta.
\]

**Remark 1.** By choosing the particular values of \( \delta \), the Gegenbauer polynomial \( G_\delta^n(t) \) reduces to well-known polynomials. These special cases are as follows:

1. Taking \( \delta = 1 \), we obtain Chebyshev polynomials
2. Taking \( \delta = (1/2) \), we obtain Legendre polynomials
2. Main Results

This section starts with defining the new family \( \mathcal{S}_2 (\lambda, \gamma, t, \delta) \).

\[
\left( \frac{2z (f' (z))^{\lambda}}{f (z) - f (-z)} \right)^{t} \left( \frac{2((zf'(z))^{\lambda})}{(f(z) - f(-z))^{y}} \right)^{-y} < H_\delta (z, t) = \frac{1}{(1-2tz+z^2)^{\delta}} \quad (17)
\]

\[
\left( \frac{2w(g' (w))^{\lambda}}{g(w) - g(-w)} \right)^{t} \left( \frac{2((wg'(w))^{\lambda})}{(g(w) - g(-w))^{y}} \right)^{-y} < H_\delta (w, t) = \frac{1}{(1-2tw+w^2)^{\delta}} \quad (18)
\]

In particular, if we choose \( \delta = 1 \) and \( \lambda = 1 \) in Definition 1, the family \( \mathcal{S}_2 (\lambda, \gamma, t, \delta) \) becomes the family \( \mathcal{D}_2 ^{(2)}(\lambda, t, \delta) \), which Wanas defined in [30].

If we choose \( \delta = 1 \) in Definition 1, the family \( \mathcal{S}_2 (\lambda, \gamma, t, \delta) \) reduces to the family \( \mathcal{T}_2 ^{(2)}(\lambda, \gamma, t) \), which Wanas gave in [31]. As a consequence, we will generalize, in the following theorem. His main result given in [31, Theorem 2.1]

**Theorem 1.** Let \( f \) be a holomorphic function, given by (1). If \( f \) is in the family \( \mathcal{S}_2 (\lambda, \gamma, t, \delta) \), where \( \lambda \geq 0, 0 \leq \gamma \leq 1 \), and \( \delta \neq 0 \in \mathbb{R} \), then the following are the upper bounds for the Taylor-Maclaurin coefficients \( a_2 \) and \( a_3 \):

\[
|a_2| \leq \frac{4|t|}{\sqrt{2}} \frac{3\lambda + 2\gamma - 3}{2(2t^2 - 1)}
\]

\[
|a_3| \leq \frac{\delta^2 t^2}{\lambda^2 (y - 2)^2} + \frac{2|t|}{(3\lambda - 1)[3 - 2\gamma]}
\]

**Proof.** We suppose that \( f \) is an element of the family \( \mathcal{S}_2 (\lambda, \gamma, t, \delta) \), and by Definition 1, \( f \) and its inverse \( g \) are biunivalent functions of \( \Sigma \) from \( U \) to \( U \), satisfying subordinations (17) and (18), and hence, we have two holomorphic functions:

\[
u = \sum_{n=1}^{\infty} v_n z^n, \quad \text{for } z, w \in U
\]

By comparing corresponding coefficients in (22) and (23), we obtain

\[
-2\lambda \gamma - 2\lambda a_2 = \mathcal{D}_1 \gamma (t) v_1, \quad (25)
\]

\[
2\gamma (\lambda - 2) a_2 = \mathcal{D}_1 \gamma (t) v_1, \quad (27)
\]
2\left[\lambda^2 (y - 2)^2 + \lambda (5 - 3y) + (2y - 3)\right] \alpha_2^2 + (3\lambda - 1)(2y - 3)\alpha_3 = \mathcal{G}^\delta_1(t)\nu_2 + \mathcal{G}^\delta_2(t)\nu_1^2. \tag{28}

It follows from (25) and (27) that
\[ u_1 = -\nu_1, \tag{29} \]
\[ 8\lambda^2 (y - 2)^2 \alpha_2^2 = \left(\mathcal{G}^\delta_1(t)\right)^2 (u_1^2 + \nu_1^2). \tag{30} \]

By eliminating $u_1^2 + \nu_1^2$ from (30) and (31), we obtain

\[ 2\left[2\lambda^2 (y - 2)^2 + (\lambda + 2y - 3)\right] \alpha_2^2 = \mathcal{G}^\delta_1(t)(u_2 + \nu_2) + \mathcal{G}^\delta_2(t)(u_1^2 + \nu_1^2). \tag{31} \]

Next, by subtracting (28) from (26), we obtain

\[ |a_3| \leq \frac{\delta t^2}{\lambda^2 (y - 2)^2 + \frac{2\|\delta t\|}{(3\lambda - 1)(3 - 2y)}}. \tag{33} \]

Thus, in view of (29) and (30), (34) gives

\[ a_3 = \frac{\left(\mathcal{G}^\delta_1(t)\right)^2}{8\lambda^2 (y - 2)^2} (u_1^2 + \nu_1^2) + \frac{\mathcal{G}^\delta_1(t)}{2(3\lambda - 1)(3 - 2y)} (u_1^2 - \nu_1^2). \tag{35} \]

And we obtain

\[ |a_3| \leq \frac{\delta t^2}{\lambda^2 + \frac{2\|\delta t\|}{3\lambda - 1}}. \tag{36} \]

If $y = 0$, then the family $\mathcal{S}_\lambda (\lambda, y, t, \delta) = \mathcal{S}_\lambda (\lambda, t, \delta)$, the class of $\lambda$-pseudo biconvex functions with respect to symmetrical points.

**Corollary 1.** For $\lambda \geq 1$, $t \in ((1/2), 1$, and $\delta$ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathcal{S}_\lambda (\lambda, 1, t, \delta)$. Then,

\[ |a_3| \leq \frac{|\delta t\sqrt{2}\|\delta t\|}{\sqrt{\delta t^2 (\lambda - 1) - 2\lambda^2(2t^2 - 1)}}. \tag{37} \]

And we obtain

\[ |a_3| \leq \frac{|\delta t\sqrt{2}\|\delta t\|}{\sqrt{\delta t^2 (\lambda - 3) - 4\lambda^2(2t^2 - 1)}}. \tag{38} \]

If $y = 0$, then the family $\mathcal{S}_\lambda (\lambda, y, t, \delta) = \mathcal{S}_\lambda (\lambda, t, \delta)$, the class of $\lambda$-pseudo biconvex functions with respect to symmetrical points.

**Corollary 2.** For $\lambda \geq 1$, $t \in ((1/2), 1$, and $\delta$ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathcal{S}_\lambda (\lambda, t, \delta)$. Then,

\[ |a_3| \leq \frac{|\delta t\sqrt{2}\|\delta t\|}{\sqrt{\delta t^2 (\lambda - 1) - 2\lambda^2(2t^2 - 1)}}. \tag{39} \]

And we obtain.
If \( \gamma = 0 \), Theorem 1 gives the following corollary by Wanas [31] for the well-known family \( T_\kappa^+ (\lambda, \gamma, t) \).

**Corollary 3** (see [31]). For \( \lambda \geq 1, 0 \leq \gamma \leq 1 \), and \( t \in ((1/2), 1] \), let \( f \in \mathcal{A} \) be in the family \( T_\kappa^+ (\lambda, \gamma, t) \). Then,

\[
|a_2| \leq \frac{t \sqrt{2t}}{\sqrt{((\lambda + 2\gamma - 3)t^2 - \lambda^2 (\gamma - 2)^2(2t^2 - 1))}}
\]  
(41)

And we obtain

\[
|a_3| \leq \frac{\lambda^2 (\gamma - 2)^2}{(3\lambda - 1)(3 - 2\gamma)}
\]  
(42)

If \( \delta = \lambda = 1 \), Theorem 1 the following corollary by Wanas [30] for the well-known family \( \mathcal{D}_2^\kappa (\gamma, t) \).

**Corollary 4** (see [30]). For \( 0 \leq \gamma \leq 1 \) and \( t \in ((1/2), 1] \), let \( f \in \mathcal{A} \) be in the family \( \mathcal{D}_2^\kappa (\gamma, t) \). Then,

\[
|a_2| \leq \frac{t \sqrt{2t}}{\sqrt{[2(\gamma - 1)t^2 - (\gamma - 2)^2(2t^2 - 1)]}}
\]  
(43)

And we obtain

\[
|a_3| \leq \frac{\Delta^2 t^2 (\lambda + 2\gamma - 3) - \Delta\lambda^2 (\gamma - 2)^2(2t^2 - 1)}{\Delta^2 t^2 (3\lambda - 1)(3 - 2\gamma)}
\]  
(47)

**Proof.** In the light of (32) and (34), we deduce that

\[
a_3 - \mu a_2^2 = (1 - \mu) \left[ \frac{2t\delta}{(3\lambda - 1)(3 - 2\gamma)} \right] \leq \left\{ \begin{array}{ll}
\frac{2t|\delta|}{(3\lambda - 1)(3 - 2\gamma)} & \text{for } |\mu - 1| < \frac{\Delta^2 t^2 (\lambda + 2\gamma - 3) - \Delta\lambda^2 (\gamma - 2)^2(2t^2 - 1)}{\Delta^2 t^2 (3\lambda - 1)(3 - 2\gamma)} \\
\frac{4t^3|\delta|^3}{\Delta^2 t^2 (\lambda + 2\gamma - 3) - \Delta\lambda^2 (\gamma - 2)^2(2t^2 - 1)} & \text{for } |\mu - 1| \geq \frac{\Delta^2 t^2 (\lambda + 2\gamma - 3) - \Delta\lambda^2 (\gamma - 2)^2(2t^2 - 1)}{\Delta^2 t^2 (3\lambda - 1)(3 - 2\gamma)}
\end{array} \right.
\]  
(48)
where

\[
\psi(\mu) = \frac{(\mathcal{G}_1^\delta(t))^2(1-\mu)}{2\left(2\lambda^2(y-2)^2 + \lambda + 2y - 3\right)(\mathcal{G}_1^\delta(t))^2 - 4\lambda^2(y-2)^2\mathcal{G}_2^\delta(t)}
\] (49)

According to (16), we deduce that

\[
|a_3 - \mu a_2^2| \leq \frac{2t|\delta|}{(3\lambda - 1)(3 - 2\gamma)}, 0 \leq |\psi(\mu)| \leq \frac{1}{2(3\lambda - 1)(3 - 2\gamma)}
\] (50)

After some computations, we obtain

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{2t|\delta|}{(3\lambda - 1)(3 - 2\gamma)}, & \text{for } |\mu - 1| \leq \frac{\delta^2 t^2 (\lambda + 2y - 3) - \delta \lambda^2 (y - 2)^2 (2t^2 - 1)}{\delta^2 t^2 (3\lambda - 1)(3 - 2\gamma)} \\
4t^2|\delta||\mu - 1|, & \text{for } |\mu - 1| \geq \frac{\delta^2 t^2 (\lambda + 2y - 3) - \delta \lambda^2 (y - 2)^2 (2t^2 - 1)}{\delta^2 t^2 (3\lambda - 1)(3 - 2\gamma)}
\end{cases}
\] (51)

In particular, putting \( \mu = 1 \) in Theorem 2, we conclude the following result.

\[
|a_3 - \mu a_2^2| \leq \frac{2t|\delta|}{(3\lambda - 1)(3 - 2\gamma)}
\] (52)

3. Conclusion

In the present work, we obtain a family \( \mathbb{S}_{2\delta}(\lambda, \gamma, t, \delta) \) of \( \lambda \)-pseudo bistarlike and \( \lambda \)-pseudo biconvex functions with respect to symmetrical points defined by Gegenbauer polynomials. We generated Taylor–Maclaurin coefficient inequalities of functions belonging to this family and viewed the famous Fekete–Szegő problem. Furthermore, by specifying parameters, consequences of this family are mentioned.

Data Availability

The data are available upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


