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Research Article

Computation of Resolvability Parameters for Benzenoid Hammer Graph

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A representation of each vertex of a network into distance-based arbitrary tuple form, adding the condition of uniqueness of each vertex with reference to some settled vertices. Such settled vertices form a set known as resolving set. This idea was delivered in various problems of computer networking as well as in chemical graph theory. Due to its huge implications, many new variants were introduced such as edge resolving set, fault-tolerant version of edge, and vertex resolving set and its generalization named as partition resolving set. In this work, we addressed all these variants for a benzenoid chemical structure named a hammer graph. Moreover, we proved that all the above variants are independent of the size and order of this graph.

1. Introduction

Chemical structures are studied with various techniques in different fields of science. Particularly, mathematical chemistry provides various tools and techniques to study different chemical networks and structures. There are many ways for the deep study of chemical networks, also approaching towards the applications of these chemical networks through suggested tools. Not only in chemistry itself but also mathematical chemistry open blockages in different other fields, such as physical chemistry, where particularly for thermodynamics and topics related to compound energy, uses mathematical chemistry tools to undergo the study of chemicals. In mathematical chemistry, graph theory provides unique and useful variants and topics to study chemical structures and its topologies. Few papers related to the topic of this work are given in [1–5].

To characterize the structural properties of clusters, polymers, crystals, and molecules, chemical graph theory is a very reliable tool and also provides various methods. In chemical graph theory, a vertex can be a collection of atoms, orbitals, intermediates, a molecule, an electron, or an atom and many other objects which are solemnly depended on the situation, model, and topic of implication. Whereas, an edge is may be a connection between two atoms, intermolecular bonding, or any other forces such as Keesom forces.

In 1975 and 1976, three graph theoretical researchers and experts of computer networking gave an idea of studying a graph in terms of distance vector [6, 7]. They named this idea as resolving set, metric basis, or locating set independently, and names variy depend on the field. In this idea, few vertices are selected on the condition that the remaining vertices of entire structure, network, or graph have unique position. The selected vertices formed a set named as the locating set [8] in computer-related topics, the resolving set in chemical topics [9], and metric basis in terms of pure theoretical studies of graphs in mathematics [10].

To get benefit from this idea, lots of practical applications have been introduced as improved variants. Fault-tolerant version of the resolving set is proposed in [11]. It dealt the problem of when any of a vertex from the resolving set fails to deliver and no more behave as a member of the resolving set. Instead of vertices' position, if one can gain the unique position of entire graph's edges, then this variant is known as the edge resolving set and proposed in [12], while the edge version of the fault-tolerant resolving set was introduced in [13].

The applications of these ideas are cited. The resolving set has applications in chemical compounds, which are detailed in [14], in pharmaceutical research and evaluating drugs [15], in information technology such as robot navigation [16], other computer related topics [17–20], image processing [6], weighting problem [21], and some detailed applications are further cited in [22–24]. Different works on chemical structures and their resolvability are discussed in [25–27]. For the computational cost of these topics, we refer to see [28–30], in which authors concluded that all the parameters are NP-hard problems in terms of computing in generalize way.

Given here are some suggested articles as a literature review and closely related to the topics studied here. In [31], vertex-based resolvability is considered as a point of discussion and implemented on Harary graph. The same technique is implemented on kayak paddles graph in [32], on necklace graph in [33], and on some general mathematical structures in [34, 35]. For the next variant which is fault tolerant of the resolving set, we refer to see [36, 37], in which some general graphs are considered and some computer related networks for different topologies of networking are studied [38, 39]. For the articles related to the partition dimension or resolving set, we refer to see [40, 41]. The general mathematical topologies and graphs are studied in terms of the edge-based resolving set in [42, 43], and they raised few questions related to this variant. In [44], the authors reply back and answered some questions. Necklace graph is studied in [45], polycyclic hydrocarbon-related structure is discussed in [46], Peterson graph is generalized in [47], and another general k-multiwheel graph is studied in [48].

Mathematical notations, definitions, and parameters are given below.

Definition 1. The distance between two vertices, $x_1, x_2 \in N(G)$, is the minimum count of edges on the way from x_1 to x_2 or vice versa; usually, it is denoted by $d(x_1, x_2)$, with $N(G)$ is the set of vertices of a graph G. Setting $L = \{x_1, x_2, \ldots, x_g\}$ is an ordered subsets of $N(G)$ and $p(x|L) = (d(x, x_1), d(x, x_2), \ldots, d(x, x_g))$ is the g-tuple vector of distance for the vertex x. If each vertex $x \in N(G)$ have unique p-vector, then L is considered as the resolving set and the minimum count of members of L is known as metric dimension $(\dim(G))$. By eliminating any arbitrary vertex from L and the condition of unique p-vector remains true, then the set L becomes L_f which is known as fault-tolerant resolving set, and similarly, the least count of its members is known as fault-tolerant metric dimension $(\dim_f(G))$. If we evaluate edges or retrieve the unique p-vector for the edges of a graph, then the set becomes L_e and known as the edge resolving set $(L_e(G))$, and its member's minimum count is denoted by $(\dim_e(G))$. The fault-tolerant version for edges (similarly L_f) is denoted by $(R_{e,f})$ and called as fault-tolerant edge metric dimension $(\dim_{e,f}(G))$. Now, the whole $N(G)$ arranging into subsets say $L_p = \{L_{p1}, L_{p2}, \ldots, L_{pg}\}$ and testing for unique p-vector for each vertex. If we are able to find such L_p, then the set known as the partition resolving set and the order of L_p is called as partition dimension $(\mathrm{pd}(G))$.

Theorem 1 (see [14]). *If Γ is a simple, undirected connected graph, then* $\dim(\Gamma) = 1$ *iff* $\Gamma = P_n$.

Theorem 2 (see [12]). *For integers* $n \geq 2$, $\dim_e(P_n) = 1$, $\dim_e(C_n) = 2$, *and* $\dim_e(K_n) = n - 1$. *Moreover,* $\dim_e(\Gamma) = 1$ *iff* Γ *is a path* P_n.

2. Results on the Vertex Edge-Based Resolvability and Their Variants for Benzenoid Hammer Structure

The graph shown in Figure 1 is a hammer structure with total $4n + 30$ number of vertices and $5n + 37$ count of total edges [49]. Hammer structure is basically a benzenoid structure and belongs to the benzenoid hydrocarbons family. It contains total $n + 8$ hexagons or cycle C_6 attached in systematic way to build a hammer-like structure. Maximum edges attached to a vertex is three and the least edges are two. Moreover, given below is the vertex and edge set of hammer structure:

$$\begin{aligned} N(H(n)) &= \{a_i, b_i\colon\ 1 \leq i \leq 16\} \cup \{c_i, c_i^*\colon\ 1 \leq i \leq 2n-1\}, B(H(n)) \\ &= \{a_i a_{i+1}, b_i b_{i+1}\colon\ 1 \leq i \leq 13\} \cup \{c_i c_{i+1}, c_i^* c_{i+1}^*\colon\ 1 \leq i \leq 2n-2\} \cup \{c_i c_i^*\colon\ 2 \leq i\,(\text{even}) \leq 2n-2\} \\ &\quad \{a_1 a_{14}, a_2 a_{15}, a_9 a_{16}, a_6 a_{15}, a_{13} a_{16}, a_{15} a_{16}, a_7 c_1, a_8 c_1^*, b_7 c_{2n-1}, b_8 c_{2n-1}^*, b_1 b_{14}, b_2 b_{15}, b_9 b_{16}, b_6 b_{15}, b_{13} b_{16}, b_{15} b_{16}\}. \end{aligned} \tag{1}$$

Lemma 1. *Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then, the least possible cardinality in its resolving set is two.*

Proof. There are in collective $4n + 30, 4n + 30$ vertices in the mathematical graph of benzene hammer having limits $n \geq 1$, and to evaluate the least possible cardinality of its resolving set, by assuming both, the formula is $C(4n + 30, 2) = (4n + 30)!/2 \times (4n + 28)!$. For any graph the least possible cardinality of its resolving set can be one as well but by Theorem 1, the path graph is the only graph having single member in its resolving set. As we know that the choosing resolving set for any graph is NP-hardness category of problems, that is why we cannot find the exact

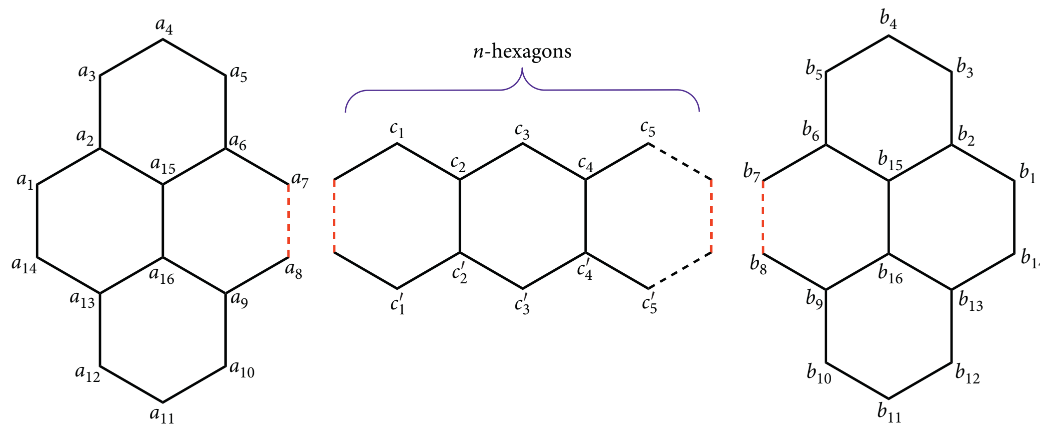

Figure 1: Benzenoid hammer with $n = 3$.

counts of resolving sets; therefore, we have a choice to make a single resolving set from $(4n + 30)!/2 \times (4n + 28)!$-possibilities. For this particular graph, we defined L as a resolving set and their members are $L = \{a_4, b_4\}$. By implementing Definition 1, we will prove our main claim that L is suitable for $H(n)$ or benzene hammer structure's resolving set. As the method defined in Definition 1, we will check all possible positions of each vertex of $H(n)$ respective to L considering as a resolving set.

Positions $p(a_i|L)$ in relation to L, for the nodes a_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(a_i|L) = \begin{cases} (|i-4|, 2(n+4)-i), & \text{if } i = 1, 2, \\ (|i-4|, 2(n+5)-i), & \text{if } i = 3, 4, \ldots, 7, \\ (|i-4|, 2(n-2)+i), & \text{if } i = 8, 9, \ldots, 11, \\ (18-i, 2(n+10)-i), & \text{if } i = 12, 13, \\ (18-i, 2(n+4)), & \text{if } i = 14, \\ (i-12, 2(n-5)+i), & \text{if } i = 15, 16. \end{cases} \tag{2}$$

Positions $p(b_i|L)$ in relation to L, for the nodes b_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(b_i|L) = \begin{cases} (2n+9-i, 2(n+4)-i), & \text{if } i = 1, 2, \\ (2n+11-i, 2(n+5)-i), & \text{if } i = 3, 4, \ldots, 7, \\ (2n+i-5, 2(n-2)+i), & \text{if } i = 8, 9, \ldots, 12, \\ (2n+i-7, 2(n-3)+i), & \text{if } i = 13, 14, \\ (2n+21-i, 2(n-5)+i), & \text{if } i = 15, 16. \end{cases} \tag{3}$$

Positions $p(c_i|L)$ and $p(c_i^*|L)$ in relation to L, for the nodes c_i and c_i^* with $i = 1, 2, \ldots, 2n-1$, are provided as

$$\begin{aligned} p(c_i|L) &= (i+3, 2n-i+3), \\ p(c_i^*|L) &= (i+4, 2n-i+4). \end{aligned} \tag{4}$$

The given positions $p(\cdot|L)$ of all $4n+30$-nodes of $H(n)$ structure of benzene hammer having limits $n \geq 1$, according to L, are distinct. It is concluding that the structure of benzene hammer or $H(n)$ resolves with only two member's resolving set. So, the least possible cardinality of the resolving set of $H(n)$ structure is two. □

Remark 1. Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then,

$$\dim(H(n)) = 2. \tag{5}$$

Proof. The concept of metric dimension, defined in Definition 1, is entirely dependent on the chosen vertices in a resolving set or say L. The vertices are selected in a manner that each vertex of structure of benzene hammer have unique or distinct representations which are represented by $p(\cdot|L)$. In Lemma 1, we have selected an appropriate resolving set for the benzene hammer structure with least possible cardinality. Such chosen resolving set is $L = \{a_4, b_4\}$ for $H(n)$ or structure of benzene hammer and with possible values of $n \geq 1$. As we have seen that Lemma 1 already proved $|L| = 2$, it is enough for the prove of this remark and $H(n)$, or the structure of benzene hammer have two metric dimension; this concludes the proof. □

Lemma 2. *Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then, the least possible cardinality in its fault-tolerant resolving set is four.*

Proof. There are in collective $4n + 30, 4n + 30$ vertices in the mathematical graph of benzene hammer having limits $n \geq 1$, and to evaluate the least possible cardinality of its fault-tolerant resolving set, by assuming both, the formula is $C(4n + 30, 4) = (4n + 30)!/2 \times (4n + 28)!$. As we know that the choosing fault-tolerant resolving set for any graph is NP-hardness category of problems, that is why we cannot find the exact counts of fault-tolerant resolving sets; therefore, we have a choice to make a single fault-tolerant resolving set from $(4n + 30)!/2 \times (4n + 28)!$-possibilities. For this particular graph, we defined L_f as a fault-tolerant resolving set, and their members are $L_f = \{a_4, b_4, a_{11}, b_{11}\}$. By implementing Definition 1, we will prove our main claim that L_f is suitable for $H(n)$ or benzene hammer structure's fault-tolerant resolving set. As the method defined in Definition 1, we will check all possible positions of each vertex of $H(n)$ respective to L_f considering as a fault-tolerant resolving set.

Positions $p(a_i|L_f)$ in relation to L_f, for the nodes a_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(a_i|L_f) = \begin{cases} (|i-4|, 2(n+4)-i, i+3, 2n+9-i), & \text{if } i = 1, 2, \\ (|i-4|, 2(n+5)-i, i+3, 2n+11-i), & \text{if } i = 3, 4, \\ (|i-4|, 2(n+5)-i, |i-11|, 2n+11-i), & \text{if } i = 5, 6, 7, \\ (|i-4|, 2(n-2)+i, |i-11|, 2n+i-5), & \text{if } i = 8, 9, \ldots, 11, \\ (18-i, 2(n+10)-i, |i-11|, 2n+i-5), & \text{if } i = 12, \\ (18-i, 2(n+10)-i, |i-11|, 2n+i-7), & \text{if } i = 13, \\ (18-i, 2(n+4), |i-11|, 2n+i-7), & \text{if } i = 14, \\ (i-12, 2(n-5)+i, 19-i, 2n-i+21), & \text{if } i = 15, 16. \end{cases} \tag{6}$$

Positions $p(b_i|L_f)$ in relation to L_f, for the nodes b_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(b_i|L_f) = \begin{cases} (2n+9-i, 2(n+4)-i, 2n+9-i, i+3), & E\, i = 1, 2, \\ (2n+11-i, 2(n+5)-i, 2n+11-i, i+3), & \text{if } i = 3, 4, \\ (2n+11-i, 2(n+5)-i, 2n+11-i, |i-1|), & \text{if } i = 5, 6, 7, \\ (2n+i-5, 2(n-2)+i, 2n-5+i, |i-1|), & \text{if } i = 8, 9, \ldots, 12, \\ (2n+i-7, 2(n-3)+i, 2n+i-7, |i-1|), & \text{if } i = 13, 14, \\ (2n+21-i, 2(n-5)+i, 2n+21-i, 19-i), & \text{if } i = 15, 16. \end{cases} \tag{7}$$

Positions $p(c_i|L_f)$ and $p(c_i^*|L_f)$ in relation to L_f, for the nodes c_i and c_i^* with $i = 1, 2, \ldots, 2n-1$, are provided as

$$\begin{aligned} p(c_i|L_f) &= (i+3, 2n-i+3, i+4, 2(n+2)-i), \\ p(c_i^*|L_f) &= (i+4, 2n-i+4, i+3, 2n+3-i). \end{aligned} \tag{8}$$

On the discussion provided above, it is proved that chosen L_f is a suitable candidate for a fault-tolerant resolving set and fulfills the definition having four least possible members in it. Now, for the approach of proving the optimized count of $|L_f|$, we have to rethink about $|L_f|$. To check whether the assertion $|L_f| = 3$ is true or not and finding another fault-tolerant resolving set with three members in it for the structure of benzene hammer, given below are some general samples or cases. In these samples, we tried to prove that only $|L_f| > 3$ is possible. □

Case 1. Consider the subset $L_f^* \subset \{a_i \colon i = 1, 2, \ldots, 16\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices breached the definition of fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \le r,\ s \le 16$.

Case 2. Consider the subset $L_f^* \subset \{b_i \colon i = 1, 2, \ldots, 16\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \le r$ and $s \le 16$.

Case 3. Consider the subset $L_f^* \subset \{c_i \colon i = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \le r$ and $s \le 16$.

Case 4. Consider the subset $L_f^* \subset \{a_i, b_j \colon i,\ j = 1, 2, \ldots, 16\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \le r$ and $s \le 16$.

Case 5. Consider the subset $L_f^* \subset \{a_i, c_j: i = 1, 2, \ldots, 16, \ j = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \leq r$ and $s \leq 16$.

Case 6. Consider the subset $L_f^* \subset \{a_i, c_j^*: i = 1, 2, \ldots, 16, \ j = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(b_r|L_f^*) = p(b_s|L_f^*)$, with $1 \leq r$ and $s \leq 16$.

Case 7. Consider the subset $L_f^* \subset \{b_i, c_j: i = 1, 2, \ldots, 16, \ j = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(c_r|L_f^*) = p(c_s^*|L_f^*)$, with $1 \leq r$ and $s \leq 2m-1$.

Case 8. Consider the subset $L_f^* \subset \{b_i, c_j^*: i = 1, 2, \ldots, 16, \ j = 1, 2, \ldots, 2n-1\}$, also assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality that is $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of the fault-tolerant resolving set and our assumption, and the reason is $p(c_r|L_f^*) = p(c_s^*|L_f^*)$, with $1 \leq r$ and $s \leq 2m-1$.

Case 9. Consider the subset $L_f^* \subset \{c_i, c_j^*: i, j = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_f^*| = 3$. This sample resulted in the same positions of two vertices and breached the definition of fault-tolerant resolving set and our assumption, and the reason is $p(a_r|L_f^*) = p(a_s|L_f^*)$, with $1 \leq r$ and $s \leq 16$.

The given positions $p(\cdot|L_f)$ of all $4n+30$-nodes of $H(n)$ structure of benzene hammer having limits $n \geq 1$, according to L_f, are distinct. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only two members' resolving set. So, the least possible cardinality of the resolving set of $H(n)$ structure is two. It also fulfills the definition of eliminating any of arbitrary nodes in the chosen fault-tolerant resolving set, and it will still resolve the nodes of structure. The assertion $|L_f| = 3$ for the fault-tolerant resolving set L_f is not true, concluding having the same position of two nodes of structure. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only four member's fault-tolerant resolving sets. So, the least possible cardinality of the fault-tolerant resolving set of $H(n)$ structure is four.

Remark 2. Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then,

$$\dim_f(H(n)) = 4. \tag{9}$$

Proof. The concept of fault-tolerant metric dimension, defined in Definition 1, is entirely dependent on the chosen vertices in a fault-tolerant resolving set or say L_f. The vertices are selected in a manner that each vertex of structure of benzene hammer have unique or distinct representations which are represented by $p(\cdot|L_f)$. In Lemma 2, we have selected an appropriate fault-tolerant resolving set for the benzene hammer structure with least possible cardinality. Such chosen fault-tolerant resolving set is $L_f = \{a_4, b_4, a_{11}, b_{11}\}$ for $H(n)$ or structure of benzene hammer with possible values of $n \geq 1$. As we have seen that Lemma 2 is already proved $|L_f| = 4$, it is enough for the prove of this remark and $H(n)$ or the structure of benzene hammer has four fault-tolerant metric dimension; this concludes the proof. □

Lemma 3. *Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then, the least possible cardinality in its edge resolving set is two.*

Proof. There are in collective $4n+30$, $4n+30$ vertices in the mathematical graph of benzene hammer having limits $n \geq 1$, and to evaluate the least possible cardinality of its edge resolving set, by assuming both, the formula is $C(4n+30, 2) = (4n+30)!/2 \times (4n+28)!$. For any graph, the least possible cardinality of its edge resolving set can be one as well, but by Theorem 1, the path graph is the only graph having single member in its edge resolving set. As we know that choosing edge resolving set for any graph is NP-hardness category of problems, that is why we cannot find the exact counts of edge resolving sets; therefore, we have a choice to make a single-edge resolving set from $(4n+30)!/2 \times (4n+28)!$-possibilities. For this particular graph, we defined L_e as an edge resolving set, and their members are $L_e = \{a_4, b_4\}$. By implementing Definition 1, we will prove our main claim that L_e is suitable for $H(n)$ or benzene hammer structure's edge resolving set. As the method defined in Definition 1, we will check all possible positions of each edge of $H(n)$ respective to L_e considering as an edge resolving set.

Positions $p(a_i a_{i+1}|L_e)$ in relation to L_e, for the edges $a_i a_{i+1}$ with $i = 1, 2, \ldots, 13$, are provided as

$$p(a_i a_{i+1}|L_e) = \begin{cases} (3-i, 2(n+3)), & \text{if } i = 1, 2, \\ (3-i, 2n+9-i), & \text{if } i = 3, \\ (i-4, 2n+9-i), & \text{if } i = 4, 5, 6, \\ (i-4, 2(n-2)+i), & \text{if } i = 7, 8, 9, 10, \\ (17-i, 2(n-2)+i), & \text{if } i = 11, \\ (17-i, 2n+7), & \text{if } i = 12, 13. \end{cases} \quad (10)$$

Positions $p(b_i b_{i+1}|L_e)$ in relation to L_e, for the edges $b_i b_{i+1}$ with $i = 1, 2, \ldots, 13$, are provided as

$$p(b_i b_{i+1}|L_e) = \begin{cases} (2(n+3), 3-i), & \text{if } i = 1, 2, \\ (2n+9-i, 3-i), & \text{if } i = 3, \\ (2n+9-i, i-4), & \text{if } i = 4, 5, 6, \\ (2(n-2)+i, i-4), & \text{if } i = 7, 8, 9, 10, \\ (2(n-2)+i, 17-i), & \text{if } i = 11, \\ (2n+7, 17-i), & \text{if } i = 12, 13. \end{cases} \quad (11)$$

Positions $p(c_i c_{i+1}|L_e)$ and $p(c_i^* c_{i+1}^*|L_e)$ in relation to L_e, for the edges $c_i c_{i+1}$ and $c_i^* c_{i+1}^*$ with $i = 1, 2, \ldots, 2n-2$, are provided as

$$\begin{aligned} p(c_i c_{i+1}|L_e) &= (i+3, 2n-i+2), \\ p(c_i^* c_{i+1}^*|L_e) &= (i+4, 2n-i+3). \end{aligned} \quad (12)$$

Positions $p(c_i c_i^*|L_e)$ in relation to L_e, for the edges $c_i c_i^*$ with $i = 2, 4, \ldots, 2n-2$, are provided as

$$p(c_i c_i^*|L_e) = (i+3, 2n-i+3). \quad (13)$$

Positions of the joint edges in relation to L_e are provided as

$$\begin{aligned} p(a_1 a_{14}|L_e) &= (3, 2n+7), \\ p(a_2 a_{15}|L_e) &= (2, 2n+5), \\ p(a_9 a_{16}|L_e) &= (4, 2n+5), \\ p(a_6 a_{15}|L_e) &= (2, 2n+4), \\ p(a_{13} a_{16}|L_e) &= (4, 2n+6), \\ p(a_{15} a_{16}|L_e) &= (3, 2n+5), \\ p(a_7 c_1|L_e) &= (3, 2n+2), \\ p(a_8 c_1^*|L_e) &= (4, 2n+3), \\ p(b_7 c_{2n-1}|L_e) &= (2n+2, 3), \\ p(b_8 c_{2n-1}^*|L_e) &= (2n+3, 4), \\ p(b_1 b_{14}|L_e) &= (2n+7, 3), \\ p(b_2 b_{15}|L_e) &= (2n+5, 2), \\ p(b_9 b_{16}|L_e) &= (2n+5, 4), \\ p(b_6 b_{15}|L_e) &= (2n+4, 2), \\ p(b_{13} b_{16}|L_e) &= (2n+6, 4), \\ p(b_{15} b_{16}|L_e) &= (2n+5, 3). \end{aligned} \quad (14)$$

The given positions $p(\cdot|L_e)$ of all $5n+37$-bonds of $H(n)$ structure of benzene hammer having limits $n \geq 1$, according to L_e, are distinct. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only two member's edge resolving set. So, the least possible cardinality of edge resolving set of $H(n)$ structure is two. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only two members of edge resolving set. So, the least possible cardinality of edge resolving set of $H(n)$ structure is two. □

Remark 3. Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then,

$$\dim_e(H(n)) = 2. \tag{15}$$

Proof. The concept of edge metric dimension, defined in Definition 1, is entirely dependent on the chosen vertices in an edge resolving set or say L_e. The vertices are selected in a manner that each edge of structure of benzene hammer have unique or distinct representations which are represented by $p(\cdot|L_e)$. In Lemma 3, we have selected an appropriate edge resolving set for the benzene hammer structure with least possible cardinality. Such chosen edge resolving set is $L_e = \{a_4, b_4\}$ for $H(n)$ or structure of benzene hammer with possible values of $n \geq 1$. As we seen that Lemma 3 already proved $|L_e| = 2$, it is enough for the proof of this remark and $H(n)$, or the structure of benzene hammer has two edge metric dimensions; this concludes the proof. □

Lemma 4. *Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then, the least possible cardinality in its fault-tolerant edge resolving set is four.*

Proof. There are in collective $4n + 30$, $4n + 30$ vertices in the mathematical graph of benzene hammer having limits $n \geq 1$, and to evaluate the least possible cardinality of its fault-tolerant edge resolving set, by assuming both, the formula is $C(4n + 30, 2) = (4n + 30)!/2 \times (4n + 28)!$. As we know the choosing fault-tolerant edge resolving set for any graph is NP-hardness category of problems, that is why we cannot find the exact counts of fault-tolerant edge resolving sets; therefore, we have a choice to make a single fault-tolerant edge resolving set from $(4n + 30)!/2 \times (4n + 28)!$-possibilities. For this particular graph, we defined $L_{e,f}$ as a fault-tolerant edge resolving set, and their members are $L_{e,f} = \{a_4, b_4, a_{11}, b_{11}\}$. By implementing Definition 1, we will prove our main claim that $L_{e,f}$ is suitable for $H(n)$ or benzene hammer structure's fault-tolerant edge resolving set. As the method defined in Definition 1, we will check all possible positions of each edge of $H(n)$ respective to $L_{e,f}$ considering as a fault-tolerant edge resolving set.

Positions $p(a_i a_{i+1}|L_{e,f})$ in relation to $L_{e,f}$, for the edges $a_i a_{i+1}$ with $i = 1, 2, \ldots, 13$, are provided as

$$p\left(a_i a_{i+1}|L_{e,f}\right) = \begin{cases} (3-i, 2(n+3), i+3, 2n+7), & \text{if } i = 1, 2, \\ (3-i, 2n+9-i, i+3, 2n+10-i), & \text{if } i = 3, \\ (i-4, 2n+9-i, 10-i, 2n+10-i), & \text{if } i = 4, 5, 6, \\ (i-4, 2(n-2)+i, 10-i, 2n+10-i), & \text{if } i = 7, \\ (i-4, 2(n-2)+i, 10-i, 2n-5+i), & \text{if } i = 8, 9, 10, \\ (17-i, 2(n-2)+i, i-11, 2n-5+i), & \text{if } i = 11, \\ (17-i, 2n+7, i-11, 2n+6), & \text{if } i = 12, 13. \end{cases} \tag{16}$$

Positions $p(b_i b_{i+1}|L_{e,f})$ in relation to $L_{e,f}$, for the edges $b_i b_{i+1}$ with $i = 1, 2, \ldots, 13$, are provided as

$$p\left(b_i b_{i+1}|L_{e,f}\right) = \begin{cases} (2(n+3), 3-i, 2n+7, i+3), & \text{if } i = 1, 2, \\ (2n+9-i, 3-i, 2n+10-i, i+3), & \text{if } i = 3, \\ (2n+9-i, i-4, 2n+10-i, 10-i), & \text{if } i = 4, 5, 6, \\ (2(n-2)+i, i-4, 2n+10-i, 10-i), & \text{if } i = 7, \\ (2(n-2)+i, i-4, 2n-5+i, 10-i), & \text{if } i = 8, 9, 10, \\ (2(n-2)+i, 17-i, 2n-5+i, i-11), & \text{if } i = 11, \\ (2n+7, 17-i, 2n+6, i-11), & \text{if } i = 12, 13. \end{cases} \tag{17}$$

Positions $p(c_i c_{i+1}|L_{e,f})$ and $p(c_i^* c_{i+1}^*|L_{e,f})$ in relation to $L_{e,f}$, for the edges $c_i c_{i+1}$ and $c_i^* c_{i+1}^*$ with $i = 1, 2, \ldots, 2n-2$, are provided as

$$\begin{aligned} p\left(c_i c_{i+1}|L_{e,f}\right) &= (i+3, 2n-i+2, i+4, 2n+3-i), \\ p\left(c_i^* c_{i+1}^*|L_{e,f}\right) &= (i+4, 2n-i+3, i+3, 2n+2-i). \end{aligned} \tag{18}$$

Positions $p(c_i c_i^* | L_{e,f})$ in relation to $L_{e,f}$, for the edges $c_i c_i^*$ with $i = 2, 4, \ldots, 2n-2$, are provided as

$$p\left(c_i c_i^* | L_{e,f}\right) = (i+3, 2n-i+3, i+3, 2n+3-i). \tag{19}$$

Positions of the joint edges in relation to $L_{e,f}$ are provided as

$$\begin{aligned}
p\left(a_1 a_{14} | L_{e,f}\right) &= (3, 2n+7, 3, 2n+7),\\
p\left(a_2 a_{15} | L_{e,f}\right) &= (2, 2n+5, 4, 2n+6),\\
p\left(a_9 a_{16} | L_{e,f}\right) &= (4, 2n+5, 2, 2n+4),\\
p\left(a_6 a_{15} | L_{e,f}\right) &= (2, 2n+4, 4, 2n+5),\\
p\left(a_{13} a_{16} | L_{e,f}\right) &= (4, 2n+6, 2, 2n+5),\\
p\left(a_{15} a_{16} | L_{e,f}\right) &= (3, 2n+5, 3, 2n+5),\\
p\left(a_7 c_1 | L_{e,f}\right) &= (3, 2n+2, 4, 2n+3),\\
p\left(a_8 c_1^* | L_{e,f}\right) &= (4, 2n+3, 3, 2n+4),\\
p\left(b_7 c_{2n-1} | L_{e,f}\right) &= (2n+2, 3, 2n+3, 4),\\
p\left(b_8 c_{2n-1}^* | L_{e,f}\right) &= (2n+3, 4, 2n+2, 3),\\
p\left(b_1 b_{14} | L_{e,f}\right) &= (2n+7, 3, 2n+7, 3),\\
p\left(b_2 b_{15} | L_{e,f}\right) &= (2n+5, 2, 2n+6, 4),\\
p\left(b_9 b_{16} | L_{e,f}\right) &= (2n+5, 4, 2n+4, 2),\\
p\left(b_6 b_{15} | L_{e,f}\right) &= (2n+4, 2, 2n+5, 4),\\
p\left(b_{13} b_{16} | L_{e,f}\right) &= (2n+6, 4, 2n+5, 2),\\
p\left(b_{15} b_{16} | L_{e,f}\right) &= (2n+5, 3, 2n+5, 3).
\end{aligned} \tag{20}$$

On the discussion provided above, it is proved that chosen $L_{e,f}$ is a suitable candidate for a fault-tolerant edge resolving set and fulfills the definition having four least possible members in it. Now, for the approach of proving the optimized count of $|L_{e,f}|$, we have to rethink about $|L_{e,f}|$. To check whether the assertion $|L_{e,f}| = 3$ is true or not and finding another fault-tolerant edge resolving set with three members in it for the structure of benzene hammer, given below are some general samples or cases. In these samples, we tried to prove that only $|L_{e,f}| > 3$ is possible. □

Case 10. Consider the subset $L_{e,f}^* \subset \{a_i \colon i = 1, 2, \ldots, 16\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(c_r c_{r+1} | L_{e,f}^*) = p(c_r c_{r+1}^* | L_{e,f}^*)$, with $1 \le r \le 2n-2$ and $2 \le s\,(\text{even}) \le 2n-2$.

Case 11. Consider the subset $L_{e,f}^* \subset \{b_i \colon i = 1, 2, \ldots, 16\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of the fault-tolerant edge resolving set and our assumption, and the reason is $p(c_r c_{r+1} | L_{e,f}^*) = p(c_r c_{r+1}^* | L_{e,f}^*)$, with $1 \le r \le 2n-2$ and $2 \le s\,(\text{even}) \le 2n-2$.

Case 12. Consider the subset $L_{e,f}^* \subset \{c_i \colon i = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(a_r a_{r+1} | L_{e,f}^*) = p(a_s a_{s+1} | L_{e,f}^*)$, with $1 \le r,\ s \le 13$.

Case 13. Consider the subset $L_{e,f}^* \subset \left\{a_i, b_j \colon i, j = 1, 2, \ldots, 16\right\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(b_r b_{r+1} | L_{e,f}^*) = p(b_s b_{s+1} | L_{e,f}^*)$, with $1 \le r,\ s \le 13$.

Case 14. Consider the subset $L_{e,f}^* \subset \left\{a_i, c_j \colon i = 1, 2, \ldots, 16,\ j = 1, 2, \ldots, 2n-1\right\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(c_r c_{r+1} | L_{e,f}^*) = p(c_r c_{r+1}^* | L_{e,f}^*)$, with $1 \le r \le 2n-2$ and $2 \le s\,(\text{even}) \le 2n-2$.

Case 15. Consider the subset $L_{e,f}^* \subset \left\{a_i, c_j^* \colon i = 1, 2, \ldots, 16,\ j = 1, 2, \ldots, 2n-1\right\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(b_r b_{r+1} | L_{e,f}^*) = p(b_s b_{s+1} | L_{e,f}^*)$, with $1 \le r,\ s \le 13$.

Case 16. Consider the subset $L_{e,f}^* \subset \left\{b_i, c_j \colon i = 1, 2, \ldots, 16,\ j = 1, 2, \ldots, 2n-1\right\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of fault-tolerant edge resolving set and our assumption, and the reason is $p(c_r c_{r+1} | L_{e,f}^*) = p(c_r c_{r+1}^* | L_{e,f}^*)$, with $1 \le r \le 2n-2$ and $2 \le s\,(\text{even}) \le 2n-2$.

Case 17. Consider the subset $L_{e,f}^* \subset \left\{b_i, c_j^* \colon i = 1, 2, \ldots, 16,\ j = 1, 2, \ldots, 2n-1\right\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L_{e,f}^*| = 3$. This sample resulted in the same positions of two edges and breached the definition of the fault-tolerant edge resolving set and our assumption, and the reason is

$p(c_r c_{r+1}|L^*_{e,f}) = p(b_s b_{s+1}|L^*_{e,f})$, with $1 \le r \le 2n-2$, and $1 \le s \le 13$.

Case 18. Consider the subset $L^*_{e,f} \subset \{c_i, c^*_j \colon i, j = 1, 2, \ldots, 2n-1\}$; also, assume eliminating any arbitrary vertex from this assumed subset according to the requisite of definition and the restrictions on the cardinality, that is, $|L^*_{e,f}| = 3$. This sample resulted in the same positions of two edges and breached the definition of the fault-tolerant edge resolving set and our assumption, and the reason is $p(a_r a_{r+1}|L^*_{e,f}) = p(a_s a_{s+1}|L^*_{e,f})$, with $1 \le r, s \le 13$.

The given positions $p(\cdot|L^*_{e,f})$ of all $5n+37$-bonds of $H(n)$ structure of benzene hammer having limits $n \ge 1$, according to $L_{e,f}$, are distinct. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only four members' fault-tolerant edge resolving set. So, the least possible cardinality of the fault-tolerant edge resolving set of $H(n)$ structure is four. It also fulfills the definition of eliminating any of arbitrary nodes in the chosen fault-tolerant edge resolving set, and it will still resolve the edges of structure. The assertion $|L_{e,f}| = 3$ for the fault-tolerant edge resolving set $L_{e,f}$ is not true concluding having the same position of two nodes of structure. It is concluded that the structure of benzene hammer or $H(n)$ resolves with only four member's of the fault-tolerant edge resolving set. So, the least possible cardinality of the fault-tolerant edge resolving set of $H(n)$ structure is four.

Remark 4. Let $H(n)$ be a structure of benzene hammer for $n \ge 1$. Then,

$$\dim_{e,f}(H(n)) = 4. \tag{21}$$

Proof. The concept of fault-tolerant edge metric dimension, defined in Definition 1, is entirely depend on the chosen vertices in a fault-tolerant edge resolving set or say $L_{e,f}$. The vertices are selected in a manner that each edge of structure of benzene hammer have unique or distinct representations which are represented by $p(\cdot|L_{e,f})$. In Lemma 4, we have selected an appropriate fault-tolerant edge resolving set for the benzene hammer structure with least possible cardinality. Such chosen fault-tolerant edge resolving set is $L_{e,f} = \{a_4, b_4, a_{11}, b_{11}\}$ for $H(n)$ or structure of benzene hammer with possible values of $n \ge 1$. As we have seen that Lemma 4 already proved $|L_{e,f}| = 4$, it is enough for the prove of this remark and $H(n)$ or the structure of benzene hammer has four fault-tolerant edge metric dimension; this concludes the proof. □

Lemma 5. *Let $H(n)$ be a structure of benzene hammer for $n \ge 1$. Then, the minimum subsets of its partition resolving set is three.*

Proof. There are in collective $4n+30$, $4n+30$ vertices in the mathematical graph of benzene hammer having limits $n \ge 1$, and to evaluate the possible combinations given by Bell number which is (HTML translation failed), $S(4n+30, \alpha)$ is the Stirling number of second kind [50]. Bell$(4n+30)$ is the possible number of choosing partition resolving set for $H(n)$, but the best and suited ones are presented here and defined as $L_p = \{L_{p_1}, L_{p_2}, L_{p_3}\}$, with $L_{p_1} = \{a_4\}$, $L_{p_2} = \{b_4\}$, and $L_{p_3} = N(H(n)) \backslash \{a_4, b_4\}$.

For any graph, the least possible cardinality of its partition resolving set can be two as well, but by Theorem 1, the path graph is the only graph having two members in its partition resolving set. As we know the choosing partition resolving set for any graph is NP-hardness category of problems, that is why we cannot find the exact counts of the partition resolving set; therefore, we have a choice to make a single partition resolving set from Bell$(4n+30) = \sum_{\alpha=0}^{4n+30} S(4n+30, \alpha)$-possibilities. For this particular graph, we defined L_p as a partition resolving set, and their members are $L_p = \{L_{p_1}, L_{p_2}, L_{p_3}\}$, with $L_{p_1} = \{a_4\}$, $L_{p_2} = \{b_4\}$, and $L_{p_3} = N(H(n)) \backslash \{a_4, b_4\}$. By implementing Definition 1, we will prove our main claim that L_p is suitable for $H(n)$ or benzene hammer structure's partition resolving set. As the method defined in Definition 1, we will check all possible positions of each vertex of $H(n)$ respective to L_p considering as a partition resolving set.

Positions $p(a_i|L_p)$ in relation to L_p, for the nodes a_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(a_i|L_p) = \begin{cases} (|i-4|, 2(n+4)-i, 0), & \text{if } i = 1, 2, \\ (|i-4|, 2(n+5)-i, z_1), & \text{if } i = 3, 4, \ldots, 7, \\ (|i-4|, 2(n-2)+i, 0), & \text{if } i = 8, 9, \ldots, 11, \\ (18-i, 2(n+10)-i, 0), & \text{if } i = 12, 13, \\ (18-i, 2(n+4), 0), & \text{if } i = 14, \\ (i-12, 2(n-5)+i, 0), & \text{if } i = 15, 16, \end{cases} \tag{22}$$

where $z_1 = \begin{cases} 1, & \text{if } i = 4, \\ 0, & \text{otherwise.} \end{cases}$

Positions $p(b_i|L_p)$ in relation to L_p, for the nodes b_i with $i = 1, 2, \ldots, 16$, are provided as

$$p(b_i|L_p) = \begin{cases} (2n+9-i, 2(n+4)-i, 0), & \text{if } i = 1, 2, \\ (2n+11-i, 2(n+5)-i, z_1), & \text{if } i = 3, 4, \ldots, 7, \\ (2n+i-5, 2(n-2)+i, 0), & \text{if } i = 8, 9, \ldots, 12, \\ (2n+i-7, 2(n-3)+i, 0), & \text{if } i = 13, 14, \\ (2n+21-i, 2(n-5)+i, 0), & \text{if } i = 15, 16. \end{cases} \tag{23}$$

Positions $p(c_i|L_p)$ and $p(c^*_i|L_p)$ in relation to L_p, for the nodes c_i and c^*_i with $i = 1, 2, \ldots, 2n-1$, are provided as

$$\begin{aligned} p(c_i|L_p) &= (i+3, 2n-i+3, 0), \\ p(c^*_i|L_p) &= (i+4, 2n-i+4, 0). \end{aligned} \tag{24}$$

The given positions $p(\cdot|L_p)$ of all $4n+30$-nodes of $H(n)$ structure of benzene hammer having limits $n \ge 1$, according to L_p, are distinct. It is concluded that the structure of benzene hammer or $H(n)$ resolves by making only three members' partition resolving set. So, the least possible cardinality of the partition resolving set of $H(n)$ structure is three. It is concluded that the structure of benzene hammer or $H(n)$ resolves by making only three member's partition resolving set. So, the least possible cardinality of the partition resolving set of $H(n)$ structure is three. □

Remark 5. Let $H(n)$ be a structure of benzene hammer for $n \geq 1$. Then,

$$\mathrm{pd}(H(n)) = 3. \tag{25}$$

Proof. The concept of partition dimension, defined in Definition 1, is entirely dependent on the chosen vertices in a partition resolving set or say L_p. The vertices are selected in a manner that each vertex of structure of benzene hammer have unique or distinct representations which are represented by $p(\cdot|L_p)$. In Lemma 5, we have selected an appropriate partition resolving set for the benzene hammer structure with least possible cardinality. Such chosen partition resolving set is $L_p = \{L_{p_1}, L_{p_2}, L_{p_3}\}$, with $L_{p_1} = \{a_4\}$, $L_{p_2} = \{b_4\}$, and $L_{p_3} = N(H(n))\backslash\{a_4, b_4\}$, for $H(n)$, or structure of benzene hammer and with possible values of $n \geq 1$. As we have seen that Lemma 5 already proved $|L_p| = 3$, it is enough for the proof of this remark and $H(n)$ or the structure of benzene hammer has three partition dimensions; this concludes the proof. □

3. Conclusion

Under the field of graph theoretical chemistry, lots of methods have been developed to solve problems related to complex networks. Also, lots of new tools are introduced to create network and study them numerically, so they can further be studied under the field of computer-based algorithms. Study of resolvability parameters is a cluster of various techniques in which networks are developed and represented numerically. This can be only possible if the resolving set develops a graph's numerical form when each vertex has its unique position, location, or representation. In this work, we consider a benzenoid chemical structure named as hammer graph and developed various types of resolvability parameters and proved that all these parameters are free from the order and size of a hammer graph.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

There are no conflicts of interest with this article.

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