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Research Article

Long-Term Dynamic Behavior of a Higher-Order Coupled Kirchhoff Model with Nonlinear Strong Damping

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In this paper, we study the long-time dynamics problem of a class of higher-order Kirchhoff coupled systems with nonlinear strong damping. The existence and uniqueness of the solutions of these equations in different spaces are proved by prior estimation and Faedo–Galerkin method; secondly, the family of global attractors of these problems is proved by using the compactness theorem. The results of the Kirchhoff coupled group are promoted through research.

1. Introduction

In this paper, we mainly consider the dynamic behavior of the higher-order Kirchhoff-type coupled equations on the bounded smooth domain $\Omega \subset R^{\mathbb{N}}$:

$$\begin{cases} u_{tt} + N_1\left(\left\|\nabla^m u\right\|^2\right)(-\Delta)^m u_t + M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m} v\right\|^2\right)(-\Delta)^m u + g_1(u, v) = f_1(x), \\ v_{tt} + N_2\left(\left\|\nabla^{2m} v\right\|^2\right)(-\Delta)^{2m} v_t + M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m} v\right\|^2\right)(-\Delta)^{2m} v + g_2(u, v) = f_2(x), \end{cases} \tag{1}$$

with boundary conditions:

$$\begin{aligned} &u(x) = 0, \frac{\partial^i u}{\partial n^i} = 0, \quad i = 1, \ldots, m-1, \\ &v(x) = 0, \frac{\partial^j v}{\partial n^j} = 0, \quad j = 1, \ldots, 2m-1, m > 1, \end{aligned} \tag{2}$$

and initial conditions:

$$\begin{aligned} u(x,0) &= u_0(x), u_t(x,0) = u_1(x), v(x,0) \\ &= v_0(x), v_t(x,0) = v_1(x), x \in \Omega. \end{aligned} \tag{3}$$

This is a kind of important generalized quasilinear wave equations of higher order. The proposed equation in this paper originated from Kirchhoff's vibration problem of stretchable strings in 1883:

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \right\} \frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where $0 < x < L, t \geq 0, u = u(x,t)$ is the lateral displacement in space coordinate x and time coordinate t, E represents Young's modulus, ρ represents the mass density, h represents the cross-sectional area, L represents the length, and p_0 represents the axial tension of the accident. The research studies on the long-time behavior of various forms of Kirchhoff-type equations have attracted the attention of many scholars in recent decades and have also achieved rich results [1–8]. Igor Chueshov [1] studied the well-posedness and long-time dynamical behavior of the following Kirchhoff equation with a nonlinear strong damping term:

$$u_{tt} + \sigma\left(\|\nabla u\|^2\right)\Delta u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + f(u) = h(x). \tag{5}$$

Guoguang Lin, Penghui Lv, and Ruijin Lou [2] studied the global dynamics of the following generalized nonlinear Kirchhoff–Boussinesq equations with damping terms:

$$\begin{aligned} u_{tt} + \alpha u_t - \beta\Delta u_t + \Delta^2 u = \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) \\ + \Delta h(u) + f(x). \end{aligned} \tag{6}$$

This paper proved that the semigroup conformed to the squeezing property and then obtained the existence of the exponential attractor of the system; then, it used the spectral interval theory to prove that the system had an inertial manifold.

Marina Ghisi and Massimo Gobbino [3] studied the global existence and local existence of solutions to the following Kirchhoff model with strong damping:

$$u_{tt}(t) + 2\delta A^{\sigma}u_t(t) + M\left(\left|A^{1/2}u(t)\right|^2\right)\mathrm{Au}(t) = 0. \tag{7}$$

Mitsuhiro Nakao [4] proved the initial-boundary value problem of the quasilinear Kirchhoff-type wave equation with standard dissipation u_t:

$$u_{tt} - \left(1 + \|\nabla u(t)\|_2^2\right)\Delta u + u_t + g(x,u) = f(x). \tag{8}$$

With the deepening of research, scholars began to turn their research directions to the dynamics of the higher-order Kirchhoff equations. Ye Yaojun and Tao Xiangxing [9] studied the initial-boundary value problem of the following kind of higher-order Kirchhoff-type equation with nonlinear dissipation term:

$$u_{tt} + \Phi\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + a\left|u_t\right|^{q-2}u_t = b|u|^{r-2}u. \tag{9}$$

Lin Guoguang and Zhu Changqing [10] studied the initial and boundary value problems of the following nonlinear nonlocal higher-order Kirchhoff-type equations:

$$u_{tt} + M\left(\left\|D^m u\right\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g\left(x, u_t\right) = f(x). \tag{10}$$

The paper obtained the existence and uniqueness of the solution and obtained the existence of the family of global attractors of the problem through the compact method and obtained the finite Hausdorff and Fractal dimensions.

System coupling originates from physics and is a metric used to refer to the mutual dependence of two entities on each other. Coupled systems refer to the system coupling occurring when some conditions or parameters are appropriate, and the potential energy of the system can make different systems realize the combination of structure and function and then produce new functions. The Kirchhoff models are mathematical equations derived from a physical background, and it will be natural to consider their coupled systems. Later, scholars gradually consider the dynamics of the Kirchhoff equations under the coupled effect. For example, Wang Yu and Zhang Jianwen [11] studied the long-term dynamics problem of a class of coupled beam equations with strong damping under nonlinear boundary conditions. Guoguang Lin and Ming Zhang [12] have studied the initial-boundary value problem of the following Kirchhoff coupling group with strong damping and source term:

$$\begin{cases} u_{tt} - \beta\Delta u_t - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + g_1(u,v) = f_1(x), \\ v_{tt} - \beta\Delta v_t - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + g_2(u,v) = f_2(x). \end{cases} \tag{11}$$

In this paper, the finite Hausdorff dimension of the global attractor is obtained.

In recent years, Guoguang Lin et al. [13–15] focused on the dynamics of a class of higher-order Kirchhoff coupled equations and obtained a series of ideal results. More conclusions about higher-order Kirchhoff-type systems can also be found in [16–19].

For the higher-order Kirchhoff coupled problems, there are few articles at present, and the problem of higher-order beam-plate coupled with nonlinear strong damping has not been studied. The main difficulty lies in the estimation and processing of the harmonic term and the nonlinear damping term, and when proving the uniqueness, the nonlinear damping brings some difficulties. In this regard, this paper overcomes these difficulties and obtains the global solution of the problem and the family of global attractors.

2. Preparatory Knowledge

In this paper, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to represent the norm and inner product in $H = L^2(\Omega)$, respectively. In order to get a more ideal conclusion, given the following assumptions:

(i) $(A1).M(s)$ is a continuous function on the interval $[0,+\infty)$, $M(s) \in C^1(R^+)$, and $(1)M'(s) \geq 0$, $(2)M(0) \equiv M_0 > 0$.

(ii) $(A2)$. For any $u, v \in H$, let $J(u,v) = \int_\Omega [G_1(u,v) + G_2(u,v)]\mathrm{d}x$, where $G_1(u,v) = \int_0^u g_1(s,v)\mathrm{d}s, G_2(u,v) = \int_0^v g_2(u,s)\mathrm{d}s$, then for any $\mu \geq 0$, there exists $C_1 \geq 0, C_\mu \geq 0, C'_\mu \geq 0$, such that $G_1(u,v) + G_2(u,v) - C_1 J(u,v) + \mu(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2) \geq -C_\mu$, $J(u,v) + \mu(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2) \geq -C'_\mu$.

(iii) $(A3).g_j(u,v)(j=1,2) \in C^1(R)$, and $|g_j(u,v)| \leq C_2(1 + |u|^{p_j} + |v|^{q_j})$; $|g_{ju}(u,v)| \leq C_3(1 + |u|^{p_j - 1} + |v|^{q_j})$; $|g_{jv}(u,v)| \leq C_4(1 + |u|^{p_j} + |v|^{q_j - 1})$. Among them, when $\mathbb{N} = 1, 2$, $1 \leq p_j(q_j)$; when $3 \leq \mathbb{N} \leq 4m$, $1 \leq p_j(q_j) \leq 2m/\mathbb{N} - 2m$.

(iv) $(A4).N_j(s_j) \geq N_{j0}$, N_{j0} $(j = 1, 2)$ are positive constants, and there exists $\rho > 0$, so that $M(s_1 + s_2) - \rho N_1(s_1) - \rho N_2(s_2) > 0$.

Next, the research phase space of this paper is given as follows:

$$\begin{aligned} &V_0 = H, V_m = H^m(\Omega) \cap H_0^1(\Omega), X_0 = V_m(\Omega) \times H(\Omega) \times V_{2m} \times H(\Omega), \\ &X_k = V_{m+k}(\Omega) \times V_k(\Omega) \times V_{2m+2k} \times V_{2k}(\Omega), k = 1, 2, \ldots, m, \end{aligned} \tag{12}$$

and the norms of the corresponding spaces are as follows:

$$\left\|(u, y_1, v, y_2)\right\|_{X_k}^2 = \left\|\nabla^{m+k}u\right\|^2 + \left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2m+2k}v\right\|^2 + \left\|\nabla^{2k}y_2\right\|^2, \quad k = 0, 1, \ldots, m. \tag{13}$$

Meanwhile, there exists the general form of Poincare's inequality: $\lambda_1\|\nabla^r u\|^2 \leq \|\nabla^{r+1}u\|^2$, where λ_1 is the first eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary on Ω. In this paper, C_i is a constant, and $C(\cdot)$ is a constant that depends on the parameters in parentheses.

Lemma 1 (see [20]). *Let y: $R^+ \longrightarrow R^+$ be an absolutely continuous positive function on $[0, +\infty)$, which satisfies for some $\delta > 0$ the differential inequality*

$$\frac{d}{dt}y(t) + 2\delta y(t) \leq g(t)y(t) + K, \quad t > 0, \tag{14}$$

where $K \geq 0$; for $t \geq s \geq 0$, there exists $a \geq 0$, such that $\int_s^t g(\tau)d\tau \leq \delta(t - s) + a$, then

$$y(t) \leq e^a y(0)e^{-\delta t} + \frac{Ke^a}{\delta}, \quad t \geq 0. \tag{15}$$

Lemma 2 (see [10]). *Let X be a Banach space, and the continuous operator semigroup $\{S(t)\}_{t\geq 0}$ satisfies the following:*

(1) The semigroup $\{S(t)\}_{t\geq 0}$ is uniformly bounded in X, that is, $\forall R_0 > 0$; there exists a positive constant $C_0(R_0)$ such that when $\|u\|_X \leq R_0$,

$$\|S(t)u\|_X \leq C_0(R_0), (\forall t \in [0, +\infty)). \tag{16}$$

(2) There exists a bounded absorbing set B_0 in X, and for any bounded set $B \subset X$, there exists a moment t_0 such that

$$S(t)B \subset B_0(t \geq t_0). \tag{17}$$

(3) For $t > 0, S(t)$ is a fully continuous operator.

Then, the semigroup $\{S(t)\}_{t\geq 0}$ has a global attractor A in X, and

$$A = \omega(B_0) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} S(t)B_0}. \tag{18}$$

3. Main Conclusion

Let $\varepsilon > 0$ be small enough and $\lambda_1^m N_{10} - 2 - 4\varepsilon - 2\varepsilon^2 \geq 0, \lambda_1^{2m} N_{20} - 2 - 4\varepsilon - 2\varepsilon^2 \geq 0$.

Lemma 3. *Assume that assumptions $(A1)-(A4)$ hold if $f_j \in H(\Omega)$ and initial data $(u_0, u_1, v_0, v_1) \in X_0$; then, for $R_0 > 0$, there exist positive constants $C(R_0)$ and t_0, so that when $t \geq t_0$, (u, y_1, v, y_2) determined by problems (1)–(3) satisfies*

$$\begin{aligned} \left\|(u, y_1, v, y_2)\right\|_{X_0}^2 &= \left\|\nabla^m u\right\|^2 + \left\|y_1\right\|^2 \\ &+ \left\|\nabla^{2m}v\right\|^2 + \left\|y_2\right\|^2 \leq C(R_0), \end{aligned} \tag{19}$$

where $y_1 = u_t + \varepsilon u$ and $y_2 = v_t + \varepsilon v$.

Proof. Multiplying the first equation of 1 by y_1 in $H(\Omega)$ and the second one by y_2 in $H(\Omega)$, we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left\|y_1\right\|^2 + \left\|y_2\right\|^2 + \int_0^{\|\nabla^m u\|^2 + \|\nabla^{2m}v\|^2} M(\tau)d\tau + 2J(u, v)\right] \\ &+ \varepsilon M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m}v\right\|^2\right)\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m}v\right\|^2\right) - \varepsilon\left(\left\|y_1\right\|^2 + \left\|y_2\right\|^2\right) + \varepsilon^2\left((u, y_1) + (v, y_2)\right) \end{aligned}$$

$$+ N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m y_1\|^2 + N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} y_2\|^2 - \varepsilon N_1\left(\|\nabla^m u\|^2\right)(\nabla^m y_1, \nabla^m u)$$
$$- \varepsilon N_2\left(\|\nabla^{2m} v\|^2\right)\left(\nabla^{2m} y_2, \nabla^{2m} v\right) + \varepsilon(g_1(u,v),u) + \varepsilon(g_2(u,v),v) = (f_1, y_1) + (f_2, y_2). \tag{20}$$

Using Holder's inequality, Young's inequality, Poincare's inequality, and so on to process the items in (20),

$$-\varepsilon\left(\|y_1\|^2 + \|y_2\|^2\right) + \varepsilon^2((u, y_1) + (v, y_2)) \geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|y_1\|^2 + \|y_2\|^2\right) - \frac{\varepsilon^2}{2}\left(\|u\|^2 + \|v\|^2\right)$$
$$\geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\|y_1\|^2 + \|y_2\|^2\right) - \frac{\varepsilon^2}{2}\lambda_1^{-m}\|\nabla^m u\|^2 - \frac{\varepsilon^2}{2}\lambda_1^{-2m}\|\nabla^{2m} v\|^2, \tag{21}$$

$$N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m y_1\|^2 + N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} y_2\|^2 - \varepsilon N_1\left(\|\nabla^m u\|^2\right)(\nabla^m y_1, \nabla^m u)$$
$$- \varepsilon N_2\left(\|\nabla^{2m} v\|^2\right)\left(\nabla^{2m} y_2, \nabla^{2m} v\right) \geq \frac{1}{2}N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m y_1\|^2 + \frac{1}{2}N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} y_2\|^2$$
$$- \frac{\varepsilon^2}{2}N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2 - \frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} v\|^2 \geq \frac{1}{2}\lambda_1^m N_1\left(\|\nabla^m u\|^2\right)\|y_1\|^2$$
$$+ \frac{1}{2}\lambda_1^{2m} N_2\left(\|\nabla^{2m} v\|^2\right)\|y_2\|^2 - \frac{\varepsilon^2}{2}N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2 - \frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} v\|^2, \tag{22}$$

$$(f_1, y_1) + (f_2, y_2) \leq \|f_1\|\|y_1\| + \|f_2\|\|y_2\| \leq \frac{1}{2}\|y_1\|^2 + \frac{1}{2}\|y_2\|^2 + \frac{1}{2}\|f_1\|^2 + \frac{1}{2}\|f_2\|^2. \tag{23}$$

Substituting (21)–(23) into (20), we can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|y_1\|^2 + \|y_2\|^2 + \int_0^{\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2} M(\tau)\mathrm{d}\tau + 2J(u,v)\right] +$$
$$+ \left(\frac{1}{2}\lambda_1^m N_1\left(\|\nabla^m u\|^2\right) - \frac{1}{2} - \varepsilon - \frac{\varepsilon^2}{2}\right)\|y_1\|^2 + \left(\frac{1}{2}\lambda_1^{2m} N_2\left(\|\nabla^{2m} v\|^2\right) - \frac{1}{2} - \varepsilon - \frac{\varepsilon^2}{2}\right)\|y_2\|^2$$
$$+ \varepsilon M\left(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2\right)\left(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2\right) - \left(\frac{\varepsilon^2}{2}N_1\left(\|\nabla^m u\|^2\right) + \frac{\varepsilon^2}{2}\lambda_1^{-m}\right)\|\nabla^m u\|^2$$
$$- \left(\frac{\varepsilon^2}{2}N_2\left(\|\nabla^{2m} v\|^2\right) + \frac{\varepsilon^2}{2}\lambda_1^{-2m}\right)\|\nabla^{2m} v\|^2 \leq -\varepsilon(g_1(u,v),u) - \varepsilon(g_2(u,v),v)$$
$$+ \frac{1}{2}\|f_1\|^2 + \frac{1}{2}\|f_2\|^2. \tag{24}$$

By (A_1), we get

$$\varepsilon M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)\geq\frac{\varepsilon}{4}\int_0^{\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2} M(\tau)\mathrm{d}\tau + \frac{3\varepsilon}{4}M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right), \tag{25}$$

and according to (A_2), we get

$$-\varepsilon(g_1(u,v),u)-\varepsilon(g_2(u,v),v)\leq -\varepsilon C_1 J(u,v) + \varepsilon\mu\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)+\varepsilon C_\mu. \tag{26}$$

Combining (25) and (26), (24) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|y_1\right\|^2+\left\|y_2\right\|^2+\int_0^{\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2} M(\tau)\mathrm{d}\tau+2J(u,v)\right]+\\&+\left(\lambda_1^m N_1\left(\left\|\nabla^m u\right\|^2\right)-1-2\varepsilon-\varepsilon^2\right)\left\|y_1\right\|^2+\left(\lambda_1^{2m} N_2\left(\left\|\nabla^{2m} v\right\|^2\right)-1-2\varepsilon-\varepsilon^2\right)\left\|y_2\right\|^2\\&+\frac{\varepsilon}{2}\int_0^{\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2} M(\tau)\mathrm{d}\tau+2\varepsilon C_1 J(u,v)+\left(\frac{3\varepsilon}{2}M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)-2\varepsilon\mu\right.\\&\left.-\varepsilon^2 N_1\left(\left\|\nabla^m u\right\|^2\right)-\varepsilon^2\lambda_1^{-m}\right)\left\|\nabla^m u\right\|^2+\left(\frac{3\varepsilon}{2}M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)-2\varepsilon\mu-\varepsilon^2 N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\right.\\&\left.-\varepsilon^2\lambda_1^{-2m}\right)\left\|\nabla^{2m} v\right\|^2\leq 2\varepsilon C_\mu+\left\|f_1\right\|^2+\left\|f_2\right\|^2.\end{aligned} \tag{27}$$

Let $H_1(t)=\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^m u\|^2+\|\nabla^{2m} v\|^2} M(\tau)\,\mathrm{d}\tau+2J(u,v)$ and $\sigma_1=\min\{\lambda_1^m N_{j0}-1-2\varepsilon-\varepsilon^2,\varepsilon/2,\varepsilon C_1\}$, we infer from (27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\sigma_1 H_1(t)\leq 2\varepsilon C_\mu+\left\|f_1\right\|^2+\left\|f_2\right\|^2. \tag{28}$$

Using Gronwall's inequality, we get

$$H_1(t)\leq H_1(0)\mathrm{e}^{-\sigma_1 t}+\frac{2\varepsilon C_\mu+\left\|f_1\right\|^2+\left\|f_2\right\|^2}{\sigma_1}. \tag{29}$$

According to $(A_1)(A_2)$, taking $\mu=M_0/4$, we have

$$H_1(t)\geq\left\|y_1\right\|^2+\left\|y_2\right\|^2+M_0\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)+2J(u,v)\geq\left\|y_1\right\|^2+\left\|y_2\right\|^2 +\frac{M_0}{2}\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)-2C_\mu'\geq C_5\left(\left\|y_1\right\|^2+\left\|y_2\right\|^2+\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m} v\right\|^2\right)-2C_\mu', \tag{30}$$

where $C_5=\min\{1,M_0/2\}$.

Then,

$$\left\|(u,y_1,v,y_2)\right\|_{X_0}^2=\left\|\nabla^m u\right\|^2+\left\|y_1\right\|^2+\left\|\nabla^{2m} v\right\|^2+\left\|y_2\right\|^2\leq\frac{\left(H_1(t)+2C_\mu'\right)}{C_5} \leq\frac{H_1(0)\mathrm{e}^{-\sigma_1 t}+2C_\mu'}{C_5}+\frac{2\varepsilon C_\mu+\left\|f_1\right\|^2+\left\|f_2\right\|^2}{\sigma_1 C_5}, \tag{31}$$

and thus

$$\overline{\lim_{t\longrightarrow\infty}}\left\|\left(u, y_1, v, y_2\right)\right\|_{X_0}^2 \leq \frac{2C_\mu'}{C_5} + \frac{2\varepsilon C_\mu + \left\|f_1\right\|^2 + \left\|f_2\right\|^2}{\sigma_1 C_5} = R_0. \tag{32}$$

Therefore, there exist positive constants $C(R_0)$ and t_0, such that when $t \geq t_0$, we get

$$\begin{aligned}\left\|\left(u, y_1, v, y_2\right)\right\|_{X_0}^2 &= \left\|\nabla^m u\right\|^2 + \left\|y_1\right\|^2 \\ &\quad + \left\|\nabla^{2m} v\right\|^2 + \left\|y_2\right\|^2 \leq C\left(R_0\right).\end{aligned} \tag{33}$$

The proof of Lemma 3 is completed. □

Lemma 4. *Assume that assumptions $(A1)-(A4)$ hold, if $f_1 \in V_k, f_2 \in V_{2k}$ and initial data $(u_0, u_1, v_0, v_1) \in X_k$, $k = 1, 2, \ldots, m$; then, for $R_k > 0$, there exist positive constants $C(R_k)$ and t_k, so that when $t \geq t_k$, (u, y_1, v, y_2) determined by problems (1)–(3) satisfies*

$$\begin{aligned}\left\|\left(u, y_1, v, y_2\right)\right\|_{X_k}^2 &= \left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^k y_1\right\|^2 \\ &\quad + \left\|\nabla^{2m+2k} v\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 \leq C\left(R_k\right),\end{aligned} \tag{34}$$

where $y_1 = u_t + \varepsilon u$ and $y_2 = v_t + \varepsilon v$.

Proof. Multiplying the first equation of 1 by $(-\Delta)^k y_1$ in $H(\Omega)$ and the second one by $(-\Delta)^{2k} y_2, k = 1, 2, \ldots, m$ in $H(\Omega)$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 + M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m} v\right\|^2\right)\left(\left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2\right)\right] \\ &\quad + \varepsilon M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m} v\right\|^2\right)\left(\left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2\right) - \varepsilon\left(\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2\right) \\ &\quad + \varepsilon^2\left(\left(\nabla^k u, \nabla^k y_1\right) + \left(\nabla^{2k} v, \nabla^{2k} y_2\right)\right) + N_1\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^{m+k} y_1\right\|^2 + N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\left\|\nabla^{2m+2k} y_2\right\|^2 \\ &\quad - \varepsilon N_1\left(\left\|\nabla^m u\right\|^2\right)\left(\nabla^{m+k} y_1, \nabla^{m+k} u\right) - \varepsilon N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\left(\nabla^{2m+2k} y_2, \nabla^{2m+2k} v\right) + \left(g_1(u, v), (-\Delta)^k y_1\right) \\ &\quad + \left(g_2(u, v), (-\Delta)^{2k} y_2\right) = \frac{\left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t} M\left(\left\|\nabla^m u\right\|^2 + \left\|\nabla^{2m} v\right\|^2\right) \\ &\quad + \left(f_1, (-\Delta)^k y_1\right) + \left(f_2, (-\Delta)^{2k} y_2\right).\end{aligned} \tag{35}$$

Using Holder's inequality, Young's inequality, Poincare's inequality, and so on to process the items in (35),

$$\begin{aligned}&-\varepsilon\left(\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2\right) + \varepsilon^2\left(\left(\nabla^k u, \nabla^k y_1\right) + \left(\nabla^{2k} v, \nabla^{2k} y_2\right)\right) \\ &\quad \geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2\right) - \frac{\varepsilon^2}{2}\left(\left\|\nabla^k u\right\|^2 + \left\|\nabla^{2k} v\right\|^2\right) \\ &\quad \geq \left(-\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2\right) - \frac{\varepsilon^2}{2}\lambda_1^{-m}\left\|\nabla^{m+k} u\right\|^2 - \frac{\varepsilon^2}{2}\lambda_1^{-2m}\left\|\nabla^{2m+2k} v\right\|^2, \\ &N_1\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^{m+k} u\right\|^2 + N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\left\|\nabla^{2m+2k} v\right\|^2 - \varepsilon N_1\left(\left\|\nabla^m u\right\|^2\right)\left(\nabla^{m+k} y_1, \nabla^{m+k} u\right) \\ &\quad - \varepsilon N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\left(\nabla^{2m+2k} y_2, \nabla^{2m+2k} v\right) \geq \frac{1}{2} N_1\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^{m+k} y_1\right\|^2 + \frac{1}{2} N_2\left(\left\|\nabla^{2m} v\right\|^2\right) \cdot \\ &\left\|\nabla^{2m+2k} y_2\right\|^2 - \frac{\varepsilon^2}{2} N_1\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^{m+k} u\right\|^2 - \frac{\varepsilon^2}{2} N_2\left(\left\|\nabla^{2m} v\right\|^2\right)\left\|\nabla^{2m+2k} v\right\|^2,\end{aligned}$$

$$
\begin{aligned}
&\left(g_1(u,v),(-\Delta)^k y_1\right)+\left(g_2(u,v),(-\Delta)^{2k} y_2\right)\le \|g_1(u,v)\|\left\|\nabla^{2k}y_1\right\|+\|g_2(u,v)\|\left\|\nabla^{4k}y_2\right\|\\
&\le \frac{N_{10}}{4}\left\|\nabla^{m+k}y_1\right\|^2+\frac{\lambda_1^{k-m}}{N_{10}}\|g_1(u,v)\|^2+\frac{N_{20}}{4}\left\|\nabla^{2m+2k}y_2\right\|^2+\frac{\lambda_1^{2k-2m}}{N_{20}}\|g_2(u,v)\|^2,\\
&\left(f_1,(-\Delta)^k y_1\right)+\left(f_2,(-\Delta)^{2k}y_2\right)\le\left\|\nabla^k f_1\right\|\left\|\nabla^k y_1\right\|+\left\|\nabla^{2k}f_2\right\|\left\|\nabla^{2k}y_2\right\|\\
&\le\frac{1}{2}\left\|\nabla^k y_1\right\|^2+\frac{1}{2}\left\|\nabla^{2k}y_2\right\|^2+\frac{1}{2}\left\|\nabla^k f_1\right\|^2+\frac{1}{2}\left\|\nabla^{2k}f_2\right\|^2,
\end{aligned}
\tag{36}
$$

According to (A_3), we get

$$
\begin{aligned}
\|g_1(u,v)\|^2&=\int_\Omega |g_1(u,v)|^2\mathrm{d}x\le\int_\Omega\left|C_2\left(1+|u|^{p_1}+|v|^{q_1}\right)\right|^2\mathrm{d}x\\
&\le C_6\int_\Omega\left(1+|u|^{2p_1}+|v|^{2q_1}\right)\mathrm{d}x\le C_7\left(1+\|u\|_{2p_1}^{2p_1}+\|v\|_{2q_1}^{2q_1}\right),
\end{aligned}
\tag{37}
$$

and $\|g_2(u,v)\|^2\le C_8\left(1+\|u\|_{2p_2}^{2p_2}+\|v\|_{2q_2}^{2q_2}\right)$. Furthermore, using the Gagliardo–Nirenberg inequality, we conclude

$$
\begin{cases}
\|u\|_{2p_j}^{2p_j}\le C_{9j}\left\|\nabla^m u\right\|^{\mathbb{N}\left(p_j-1\right)/m}\|u\|^{2mp_j-\mathbb{N}\left(p_j-1\right)/m},\\
\|v\|_{2q_j}^{2q_j}\le C_{10j}\left\|\nabla^{2m}v\right\|^{\mathbb{N}\left(q_j-1\right)/2m}\|v\|^{4mq_j-\mathbb{N}\left(q_j-1\right)/2m},
\end{cases}
\tag{38}
$$

then

$$
\left\|g_1(u,v)\right\|^2+\left\|g_2(u,v)\right\|^2\le C(R_0).
\tag{39}
$$

Combining (36)–(38) and (39), (28) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\nabla^k y_1\right\|^2+\left\|\nabla^{2k}y_2\right\|^2+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)\right]\\
&+\frac{\left(2N_1\left(\left\|\nabla^m u\right\|^2\right)-N_{10}\right)\lambda_1^m-2-4\varepsilon-2\varepsilon^2}{4}\left\|\nabla^k y_1\right\|^2+\frac{\left(2N_2\left(\left\|\nabla^{2m}v\right\|^2\right)-N_{20}\right)\lambda_1^{2m}-2-4\varepsilon-2\varepsilon^2}{4}\cdot\\
&\left\|\nabla^{2k}y_2\right\|^2+\left(\varepsilon M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)-\frac{\varepsilon^2}{2}N_1\left(\left\|\nabla^m u\right\|^2\right)-\frac{\varepsilon^2}{2}\lambda_1^{-m}\right)\left\|\nabla^{m+k}u\right\|^2\\
&+\left(\varepsilon M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)-\frac{\varepsilon^2}{2}N_2\left(\left\|\nabla^{2m}v\right\|^2\right)-\frac{\varepsilon^2}{2}\lambda_1^{-2m}\right)\left\|\nabla^{2m+2k}v\right\|^2\\
&\le\frac{\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)\\
&+\frac{1}{2}\left\|\nabla^k f_1\right\|^2+\frac{1}{2}\left\|\nabla^{2k}f_2\right\|^2+C(R_0,\lambda_1)\le\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)M'\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)\\
&\cdot\left(\left(\nabla^m u,\nabla^m u_t\right)+\left(\nabla^{2m}v,\nabla^{2m}v_t\right)\right)+\frac{1}{2}\left\|\nabla^k f_1\right\|^2+\frac{1}{2}\left\|\nabla^{2k}f_2\right\|^2+C(R_0,\lambda_1)\\
&\le C_8\left(\left\|\nabla^m u_t\right\|+\left\|\nabla^{2m}v_t\right\|\right)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)+\frac{1}{2}\left\|\nabla^k f_1\right\|^2\\
&+\frac{1}{2}\left\|\nabla^{2k}f_2\right\|^2+C(R_0,\lambda_1).
\end{aligned}
\tag{40}
$$

Let $H_2(t) = \|\nabla^k y_1\|^2 + \|\nabla^{2k} y_2\|^2 + M(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2)(\|\nabla^{m+k} u\|^2 + \|\nabla^{2m+2k} v\|^2)$ and $\sigma_2 = \min\{\lambda_1^m N_{j0} - 2 - 4\varepsilon - 2\varepsilon^2/2, \varepsilon/2\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + \sigma_2 H_2(t) \leq C_9\left(\|\nabla^m u_t\| + \|\nabla^{2m} v_t\|\right)H_2(t) + \left\|\nabla^k f_1\right\|^2 + \left\|\nabla^{2k} f_2\right\|^2 + C(R_0, \lambda_1). \tag{41}$$

Taking the scalar product in $H(\Omega)$ of (1) with u_t, v_t, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u_t\|^2 + \|v_t\|^2 + \int_0^{\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2} M(\tau)\mathrm{d}\tau + 2J(u,v) - 2(f_1,u) - 2(f_2,v)\right]\\&\quad + N_1\left(\|\nabla^m u\|^2\right)\|\nabla^m u_t\|^2 + N_2\left(\|\nabla^{2m} v\|^2\right)\|\nabla^{2m} v_t\|^2 = 0,\end{aligned} \tag{42}$$

and integrating (42) in dt on $(0,t)$, we get

$$\begin{aligned}&\int_0^t\left(\|\nabla^m u_t(\tau)\|^2 + \|\nabla^{2m} v_t(\tau)\|^2\right)\mathrm{d}\tau \leq \frac{1}{\min\{N_{10}, N_{20}\}}\int_0^t\left(N_1\left(\|\nabla^m u(\tau)\|^2\right)\|\nabla^m u_t(\tau)\|^2 +\right.\\&\left.N_2\left(\|\nabla^{2m} v(\tau)\|^2\right)\|\nabla^{2m} v_t(\tau)\|^2\right)\mathrm{d}\tau \leq \frac{1}{\min\{N_{10}, N_{20}\}}\left(\|u_1\|^2 + \|v_1\|^2\right.\\&\qquad\left. + \int_0^{\|\nabla^m u_0\|^2 + \|\nabla^{2m} v_0\|^2} M(\tau)\mathrm{d}\tau + 2J(u_0,v_0) - 2(f_1,u_0) - 2(f_2,v_0)\right) \leq C_{10},\end{aligned} \tag{43}$$

and then we have

$$C_9\int_s^t\left(\left(\|\nabla^m u_t(\tau)\| + \|\nabla^{2m} v_t(\tau)\|\right)\right)\mathrm{d}\tau \leq \frac{\sigma_2}{2}(t-s) + a, \quad \text{for } t > s \geq 0 \text{ and some } a > 0. \tag{44}$$

Combining (41) and (44), according to Lemma 1, we know

$$H_2(t) \leq C_{11}H_2(0)\mathrm{e}^{-\sigma_2/2t} + C_{12}. \tag{45}$$

By (A_1), we have

$$\begin{aligned}H_2(t) &\geq \left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 + M\left(\|\nabla^m u\|^2 + \|\nabla^{2m} v\|^2\right)\left(\left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2\right)\\&\geq C_{13}\left(\left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 + \|\nabla^{m+k} u\|^2 + \|\nabla^{2m+2k} v\|^2\right),\end{aligned} \tag{46}$$

then

$$\|(u, y_1, v, y_2)\|_{X_k}^2 = \left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 + \left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2 \le \frac{C_{11} H_2(0) e^{-\sigma_2/2t} + C_{12}}{C_{13}}, \tag{47}$$

that is,

$$\overline{\lim_{t\longrightarrow\infty}} \|(u, y_1, v, y_2)\|_{X_k}^2 \le R_k. \tag{48}$$

Therefore, there exist positive constants $C(R_k)$ and t_k, such that when $t \ge t_k$, (u, y_1, v, y_2) satisfies

$$\|(u, y_1, v, y_2)\|_{X_k}^2 = \left\|\nabla^k y_1\right\|^2 + \left\|\nabla^{2k} y_2\right\|^2 + \left\|\nabla^{m+k} u\right\|^2 + \left\|\nabla^{2m+2k} v\right\|^2 \le C(R_k), \quad k = 1, 2, \ldots, m. \tag{49}$$

The proof of Lemma 4 is completed. □

Theorem 1. *Assume that assumptions $(A1)-(A4)$ hold if $f_1 \in V_k, f_2 \in V_{2k}$ and initial data $(u_0, u_1, v_0, v_1) \in X_k$, $k = 0, 1, \ldots, m$, then problems (1)– (3) have a unique solution (u, v) satisfying*

$$\begin{aligned} u &\in L^\infty(0, \infty; V_{m+k}); \\ u_t &\in L^\infty(0, \infty; H) \cap L^2(0, T; V_k); \\ v &\in L^\infty(0, \infty; V_{2m+2k}); \\ v_t &\in L^\infty(0, \infty; H) \cap L^2(0, T; V_{2k}). \end{aligned} \tag{50}$$

Proof. According to [10] and the Faedo–Galerkin method, it is easy to obtain that (1)– (3) have global solutions combining with Lemma 3 and Lemma 4.

The uniqueness of the solution is as follows:

Let u^1, v^1 and u^2, v^2 be two solutions of problems (1)–(3) with the same initial value. Write $w = u^1 - u^2, z = v^1 - v^2$, then (w, z) satisfies

$$\begin{cases} w_{tt} + \dfrac{1}{2}\sigma_{12}(t)(-\Delta)^m w_t + \dfrac{1}{2}\Phi_{12}(t)(-\Delta)^m w + G_1\left(u^1, u^2, v^1, v^2; t\right) = 0, \\ z_{tt} + \dfrac{1}{2}\sigma_{34}(t)(-\Delta)^{2m} z_t + \dfrac{1}{2}\Phi_{12}(t)(-\Delta)^{2m} z + G_2\left(u^1, u^2, v^1, v^2; t\right) = 0, \end{cases} \tag{51}$$

where

$$\begin{aligned} &\sigma_{12} = \sigma_1(t) + \sigma_2(t), \Phi_{12}(t) = \Phi_1(t) + \Phi_2(t), \sigma_i(t) = N_1\left(\left\|\nabla^m u^i\right\|^2\right), \\ &\Phi_i(t) = M\left(\left\|\nabla^m u^i\right\|^2 + \left\|\nabla^{2m} v^i\right\|^2\right), \quad i = 1, 2, \sigma_{34} = \sigma_3(t) + \sigma_4(t), \\ &\sigma_j(t) = N_2\left(\left\|\nabla^{2m} v^j\right\|^2\right), j = 3, 4, G_1\left(u^1, u^2, v^1, v^2; t\right) = \frac{1}{2}\left\{\left[\sigma_1(t) - \sigma_2(t)\right](-\Delta)^m\left(u_t^1 + u_t^2\right)\right. \\ &\quad \left. + \left[\Phi_1(t) - \Phi_2(t)\right](-\Delta)^m\left(u^1 + u^2\right)\right\} + g_1(u_1, v_1) - g_1(u_2, v_2), \\ &G_2\left(u^1, u^2, v^1, v^2; t\right) = \frac{1}{2}\left\{\left[\sigma_3(t) - \sigma_4(t)\right](-\Delta)^{2m}\left(v_t^1 + v_t^2\right) + \left[\Phi_1(t) - \Phi_2(t)\right](-\Delta)^{2m}\left(v^1 + v^2\right)\right\} + g_2(u_1, v_1) - g_2(u_2, v_2). \end{aligned} \tag{52}$$

From Lemma 3, it can be known that $\sigma_{12}' \le C(R_0)(\|\nabla^m u_t^1\| + \|\nabla^m u_t^2\|)$ and $\sigma_{34}' \le C(R_0)(\|\nabla^{2m} v_t^1\| + \|\nabla^{2m} v_t^2\|)$.

Taking the scalar product in $H(\Omega)$ of (51) with w_t, z_t, we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right)\right]+\frac{1}{2}\sigma_{12}(t)\|\nabla^m w_t\|^2+\frac{1}{2}\sigma_{34}(t)\|\nabla^{2m}z_t\|^2\\&\quad+\left(G_1\left(u^1,u^2,v^1,v^2;t\right),w_t\right)+\left(G_2\left(u^1,u^2,v^1,v^2;t\right),z_t\right)=0.\end{aligned}\tag{53}$$

From Lemma 3 and $(A1)$, we know $M_0\le M\le C(R_0,H_1(0))\equiv M_1$; when $\mathrm{d}/\mathrm{d}t\,(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2)\ge 0$, we write $\Phi_0=2M_0$; otherwise, $\Phi_0=2M_1$.

Let $(G_1(u^1,u^2,v^1,v^2;t),w_t)=G_{11}+G_{12}+G_{13}$, $(G_2(u^1,u^2,v^1,v^2;t),z_t)=G_{21}+G_{22}+G_{23}$, we have

$$\begin{aligned}G_{11}&=\frac{1}{2}(\sigma_1(t)-\sigma_2(t))\left(\nabla^m\left(u_t^1+u_t^2\right),\nabla^m w_t\right)\le C(R_0)\left(\|\nabla^m u_t^1\|+\|\nabla^m u_t^2\|\right)\|\nabla^m w\|\|\nabla^m w_t\|\\&\le\frac{\sigma_{120}}{8}\|\nabla^m w_t\|^2+\frac{2C(R_0)}{\sigma_{120}}\left(\|\nabla^m u_t^1\|^2+\|\nabla^m u_t^2\|^2\right)\|\nabla^m w\|^2,\end{aligned}\tag{54}$$

$$\begin{aligned}G_{12}&=\frac{1}{2}(\Phi_1(t)-\Phi_2(t))\left(\nabla^m\left(u^1+u^2\right),\nabla^m w_t\right)\le C(R_0)\left(\|\nabla^m w\|+\|\nabla^{2m}z\|\right)\|\nabla^m w_t\|\\&\le\frac{\sigma_{120}}{8}\|\nabla^m w_t\|^2+\frac{2C(R_0)}{\sigma_{120}}\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right),\end{aligned}\tag{55}$$

$$G_{13}=(g_1(u_1,v_1)-g_1(u_2,v_2),w_t)\le C(R_0)\left(\|w_t\|^2+\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right),\tag{56}$$

$$\begin{aligned}G_{21}&=\frac{1}{2}(\sigma_3(t)-\sigma_4(t))\left(\nabla^{2m}\left(v_t^1+v_t^2\right),\nabla^{2m}z_t\right)\le C(R_0)\left(\|\nabla^{2m}v_t^1\|+\|\nabla^{2m}v_t^2\|\right)\|\nabla^{2m}z\|\|\nabla^{2m}z_t\|\\&\le\frac{\sigma_{340}}{8}\|\nabla^{2m}z_t\|^2+\frac{2C(R_0)}{\sigma_{340}}\left(\|\nabla^{2m}v_t^1\|^2+\|\nabla^{2m}v_t^2\|^2\right)\|\nabla^{2m}z\|^2,\end{aligned}\tag{57}$$

$$\begin{aligned}G_{22}&=\frac{1}{2}(\Phi_1(t)-\Phi_2(t))\left(\nabla^{2m}\left(v^1+v^2\right),\nabla^{2m}z_t\right)\le C(R_0)\left(\|\nabla^m w\|+\|\nabla^{2m}z\|\right)\|\nabla^{2m}z_t\|\\&\le\frac{\sigma_{340}}{8}\|\nabla^{2m}z_t\|^2+\frac{2C(R_0)}{\sigma_{340}}\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right),\end{aligned}\tag{58}$$

$$G_{23}=(g_2(u_1,v_1)-g_2(u_2,v_2),z_t)\le C(R_0)\left(\|z_t\|^2+\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right),\tag{59}$$

where $\sigma_{120}=2N_{10}$ and $\sigma_{340}=2N_{20}$.

Combining (53)–(59), (52) can be changed:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right)\right]\\&\quad\le C_{14}\left(1+\|\nabla^{2m}v_t^1\|^2+\|\nabla^{2m}v_t^2\|^2\right)\left[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right)\right],\end{aligned}\tag{60}$$

then

$$\begin{aligned}&\left[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot\left(\|\nabla^m w\|^2+\|\nabla^{2m}z\|^2\right)\right]\le\left[\|w_1\|^2+\|z_1\|^2\right.\\&\quad\left.+\frac{1}{4}\Phi_0\cdot\left(\|\nabla^m w_0\|^2+\|\nabla^{2m}z_0\|^2\right)\right]\exp\left(\int_0^t C_{14}\left(1+\|\nabla^{2m}v_t^1\|^2+\|\nabla^{2m}v_t^2\|^2\right)\mathrm{d}s\right).\end{aligned}\tag{61}$$

The uniqueness of the solution is proved.

The proof of Theorem 1 is completed.

According to Theorem 1, we define a nonlinear operator $S(t)$ on space X_0: $(u_0, u_1, v_0, v_1) \longrightarrow (u, u_t, v, v_t), \forall t \geq 0$. It can be known that $S(t)$ is a nonlinear C_0−semigroup defined on X_0. □

Theorem 2. *Assume that assumptions (A1)–(A4) hold, if* $f_1 \in V_k, f_2 \in V_{2k}$ *and initial data* $(u_0, u_1, v_0, v_1) \in X_k, k = 1, 2, \ldots, m$, *then problems (1)–(3) have a family of global attractors* $\mathscr{A}$ *in* X_0:

$$\mathscr{A} = \{A_k\}, A_k = \omega(B_{0k}) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t) B_{0k}}, \quad k = 1, 2, \ldots, m, \tag{62}$$

where $B_{0k} = \{(u, u_t, v, v_t) \in X_k\colon \|(u, u_t, v, v_t)\|_{X_k}^2 = \|\nabla^{m+k} u\|^2 + \|\nabla^k u_t\|^2 + \|\nabla^{2m+k} v\|^2 + \|\nabla^{2k} v_t\|^2 \leq C(R_0) + C(R_{0k})\}$. *Obviously,* B_{0k} *are bounded absorbing sets in* X_0 *and compact in* X_0.

(1) $S(t) A_k = A_k$, $(\forall t \geq 0)$,

(2) A_k *attracts all bounded sets in* X_0, *that is,* $\forall B_k \subset X_0$ *are bounded sets in* X_0, *and*

$$\mathrm{dist}(S(t) B_k, A_k) = \sup_{x \in B_k} \inf_{y \in A_k} \|S(t)x - y\|_{X_0} \longrightarrow 0 \, (t \longrightarrow \infty), \tag{63}$$

$\{S(t)\}_{t \geq 0}$ *is the solution semigroup generated by problems (1)–(3).*

Proof. From Lemma 3, $\forall R_0 > 0$, $\|(u_0, u_1, v_0, v_1)\|_{X_0} \leq R_0$, such that

$$\begin{aligned} \|S(t)(u_0, u_1, v_0, v_1)\|_{X_0}^2 &= \|u\|_{V_m}^2 + \|u_t\|_{V_0}^2 \\ &\quad + \|v\|_{V_{2m}}^2 + \|v_t\|_{V_0}^2 \leq C(R_0), \end{aligned} \tag{64}$$

which shows that $\{S(t)\}_{t \geq 0}$ are uniformly bounded in X_0.

Further, $B_{0k} = \{(u, u_t, v, v_t) \in X_k\colon \|(u, u_t, v, v_t)\|_{X_k}^2 = \|\nabla^{m+k} u\|^2 + \|\nabla^k u_t\|^2 + \|\nabla^{2m+k} v\|^2 + \|\nabla^{2k} v_t\|^2 \leq C(R_0) + C(R_{0k})\}$ are bounded absorbing sets of the semigroup $\{S(t)\}_{t \geq 0}$ in X_0 because $\hookrightarrow$ are compactly embedding, that is, the bounded sets in X_k are compact sets in X_0, so the solution semigroup $\{S(t)\}_{t \geq 0}$ is a fully continuous operator.

To sum up, we get the family of global attractors $\mathscr{A} = \{A_k\}$ of the solution semigroup $\{S(t)\}_{t \geq 0}$ in X_0, and

$$A_k = \omega(B_{0k}) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t) B_{0k}}, A_k \subset X_0, \quad k = 1, 2, \ldots, m. \tag{65}$$

The proof of Theorem 2 is completed. □

Note 1. Lemma 4 and Theorem 2 show that the bounded absorbing sets $B_{0k} = \{(u, u_t, v, v_t) \in X_k\colon \|(u, u_t, v, v_t)\|_{X_k}^2 = \|\nabla^{m+k} u\|^2 + \|\nabla^k u_t\|^2 + \|\nabla^{2m+k} v\|^2 + \|\nabla^{2k} v_t\|^2 \leq C(R_0) + C(R_{0k})\}, (k = 1, 2, \ldots, m)$ are compact bounded absorbing sets in X_0. Therefore, using condition 3 in Lemma 2, the operator semigroup $S(t)$ only needs to be a continuous operator; from Theorem 1, it can be seen that the semigroup $S(t)$ is already a continuous semigroup; from this angle, the family of global attractors $\mathscr{A} = \{A_k\}$ of problems (1)–(3) in X_0 can also be obtained.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to the writing of this paper. All authors read and approved the final manuscript.

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