Research Article

On Fractional Newton-Type Method for Nonlinear Problems

Mine Aylin Bayrak,1 Ali Demir,1 and Ebru Ozbilge2

1Department of Mathematics, Kocaeli University, Izmit, Kocaeli, Turkey
2Department of Mathematics and Statistics, American University of the Middle East, Egaila, Kuwait

Correspondence should be addressed to Mine Aylin Bayrak; aylin@kocaeli.edu.tr

Received 30 June 2022; Revised 7 September 2022; Accepted 11 October 2022; Published 21 November 2022

1. Introduction

Newton–Raphson method is one of the most powerful techniques to locate the solutions of linear or nonlinear equations in numerous areas of science. Compare to the other methods, the convergence rate of this method is much better. In this method, the neighbourhood of the solution of the equations is determined by utilizing the tangent lines of the curves. However, this method has some drawbacks such as having no solution for some forms of equations in a situation that the initial guess or an iteration coincides with a loop which leads to divergence or oscillation of the method.

The theory and applications of the Newton–Raphson method were presented in [1]. Modified versions of Newton’s method were given in [2–4]. Newton’s method has been implemented for solving constrained or unconstrained minimization problems in [5–7]. A novel block Newton method was developed for the computation invariant pairs to represent eigenvalues and eigenvectors in [8]. A criterion was established for selecting the appropriate model and its applications in [9]. Various optimization methods such as the Newton–Raphson, bisection, gradient, and secant methods were reviewed and discussed in [10]. The optimization solution of the estimating function in the regression models was determined in [11, 12].

In the current study, we focus on developing a novel modification of the Newton–Raphson method by utilizing fractional derivatives and fractional Taylor series expansion which allows us to eliminate the shortcomings of this method. One advantage of this method is that this method can be applied to various fractional derivatives such as Riemann–Liouville and Caputo derivatives [13–22]. Convergence analysis of first- and second-order fractional Newton–Raphson (FNR) methods is provided, and some conditions on initial guess are obtained. Moreover, these conditions are generalized for higher-order FNR. The main advantages of first- and second-order FNR are that they are more effective and accurate compared to other existing methods. Moreover, the convergency of the obtained solutions is faster than the convergency of the solutions, obtained by other methods.

2. Preliminaries

This section is devoted to fundamental notions in fractional calculus [16–18].

Definition 1. $\beta^{th}$ ($\beta \geq 0$) order of Riemann–Liouville integral is given by [16]
\[ J^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)dt, \quad \beta > 0, x > 0, \quad (1) \]
\[ J^0 f(x) = f(x). \]

**Definition 2.** \( \beta \)th order fractional derivative in Caputo sense is given by [16]
\[
\begin{align*}
C^\beta f(x) &= \int_0^x (x-t)^{m-\beta} D^m f(t)dt,
\end{align*}
\]
\[ m - 1 < \beta < m, x > 0, \quad (2) \]
where \( D^m \) is the ordinary differential operator of order \( m \).

**Theorem 1.** [23, 24] Let us suppose that \( C^\beta f(x) \in C([a, b]) \) for \( j = 1, 2, \ldots, n \) where \( \beta \in (0, 1] \), and then, we have
\[
\begin{align*}
f(x) &= \sum_{j=0}^n \frac{C^\beta f(a)}{\Gamma((j+1)\beta)} (x-a)^{(j+1)\beta} + C^\beta f(\xi) \frac{(x-a)^{(n+1)\beta}}{\Gamma((n+1)\beta)},
\end{align*}
\]
with \( a \leq \xi \leq x \), for all \( x \in (a, b) \) where \( C^\beta f = C^\beta D^\beta, C^\beta D^\beta, \ldots, C^\beta D^\beta (n \text{ times}) \). Notice that the property of Riemann–Liouville derivative of constant is different than zero unlike the Caputo derivative.

### 3. Fractional Newton–Raphson (FNR) Method

In this section, we give the formulation of the first- and second-order FNR [25–28].

#### 3.1. First-Order FNR

By taking the first two terms of fractional Taylor series expansion, we have

\[
\begin{align*}
x_1 &= x_0 + \left( \frac{\Gamma(2\beta + 1)}{2 f^{(2\beta)}(x_0)} \frac{-(f^{(\beta)}(x_0)\Gamma(\beta + 1)) + \sqrt{(f^{(\beta)}(x_0)\Gamma(\beta + 1))^2 - 4(f^{(2\beta)}(x_0)\Gamma(2\beta + 1))f(x_0)} \right)^{1/\beta}. \quad (10)
\end{align*}
\]

Repetition of this algorithm generates a sequence of \( x \) values \( x_0, x_1, x_2, \ldots \) which are formulated by the following recurrence relation:

\[
\begin{align*}
x_{n+1} &= x_n + \left( \frac{\Gamma(2\beta + 1)}{2 f^{(2\beta)}(x_n)} \frac{-(f^{(\beta)}(x_n)\Gamma(\beta + 1)) + \sqrt{(f^{(\beta)}(x_n)\Gamma(\beta + 1))^2 - 4(f^{(2\beta)}(x_n)\Gamma(2\beta + 1))f(x_n)} \right)^{1/\beta}. \quad (11)
\end{align*}
\]

#### 3.2. Second-Order FNR

By taking the first three terms of the fractional Taylor series expansion, we get

\[
\begin{align*}
y - f(x_0) &= f^{(\beta)}(x_0) \frac{(x-x_0)^\beta}{\Gamma(\beta + 1)} + f^{(2\beta)}(x_0) \frac{(x-x_0)^{2\beta}}{2 \Gamma(2\beta + 1)}.
\end{align*}
\]

To solve for the \( x \) intercept, we set \( y = 0 \) and rearrange the terms

\[
\begin{align*}
0 &= f(x_0) + \frac{f^{(\beta)}(x_0)}{\Gamma(\beta + 1)}(x_1 - x_0)^\beta.
\end{align*}
\]

Thus, we have

\[
\begin{align*}
x_1 &= x_0 + \left( \frac{-f(x_0)}{f^{(\beta)}(x_0) \Gamma(\beta + 1)} \right)^{1/\beta}.
\end{align*}
\]

Repetition of this algorithm generates a sequence of \( x \) values \( x_0, x_1, x_2, \ldots \) which leads to the following recurrence relation:

\[
\begin{align*}
x_{n+1} &= x_n + \left( \frac{-f(x_n)}{f^{(\beta)}(x_n) \Gamma(\beta + 1)} \right)^{1/\beta}, \quad f^{(\beta)}(x_n) \neq 0. \quad (7)
\end{align*}
\]

### 4. Convergence

In this section, the convergence analysis of first- and second-order FNR is given by the following theorems, respectively.

**Theorem 2.** Assume that \( f \) is twice continuously differentiable on an open interval \((a, b)\) and that there exists a point \( x^* \in (a, b) \) with \( f^{(\beta)}(x^*) \neq 0 \). Implementing the first-order FNR method, we have the following recurrence relation:
\[ x_{n+1} = x_n + \left( \frac{f(x_n)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta}, \quad n = 1, 2, \ldots \]  

(12)

Under the assumption that \( x_n \) converges to \( x^* \) as \( n \to \infty \), we have

\[ |x_{n+1} - x^*| \leq M|x_n - x^*|^2, \]

\[ M > \left( \frac{f^{(\beta)}(x^*)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta}, \]

(13)

for \( n \) sufficiently large. Thus, \( x_n \) converges to \( x^* \) quadratically.

Proof. Let \( e_n = x_n - x^* \), so that \( x_n - e_n = x^* \). Setting \( x = x_n \) and \( h = -e_n \) in fractional Taylor’s Theorem, we get

\[ f(x_n - e_n) = f(x_n) - \frac{e_n^\beta}{\Gamma(\beta + 1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta + 1)} f^{(2\beta)}(\xi_n), \]

(14)

for some \( \xi_n \in (x_n, x^*) \). Since \( f(x_n - e_n) = f(x^*) = 0 \), we have

\[ 0 = f(x_n) - \frac{x_n - x^*}{\Gamma(\beta + 1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta + 1)} f^{(2\beta)}(\xi_n). \]

(15)

Having the condition that \( \beta^\text{th} \) derivative of \( f \) is continuous with \( f^{(\beta)}(x_n) \neq 0 \) as long as \( x_n \) is close enough to \( x^* \) allows us to divide by \( (f^{(\beta)}(x_n)/\Gamma(\beta + 1)) \) which leads to the following:

\[ 0 = f(x_n) f^{(\beta)}(x_n) - (x_n - x^*)^\beta + \frac{e_n^{2\beta}}{f^{(2\beta)}(\xi_n)} \Gamma(\beta + 1) \]

(16)

As a result, the formulation of first-order FNR gives the following:

\[ (x_{n+1} - x^*)^\beta = e_n^{2\beta} f^{(2\beta)}(\xi_n) \Gamma(\beta + 1) \]

(17)

After rearrangement, we have

\[ |x_{n+1} - x^*| = e_n^{2\beta} \left( \frac{f^{(2\beta)}(\xi_n)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta}. \]

(18)

Finally,

\[ |x_{n+1} - x^*| = \left( \frac{f^{(2\beta)}(\xi_n)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta} |x_n - x^*|^2. \]

(19)

By continuity, \( f^{(\beta)}(x_n) \) converges to \( f^{(\beta)}(x^*) \). Convergence of \( \xi_n \in (x_n, x^*) \) to \( x^* \) leads to convergence of \( f^{(2\beta)}(\xi_n) \) to \( f^{(2\beta)}(x^*) \). As a result, we have

\[ |x_{n+1} - x^*| \leq M|x_n - x^*|^2, \]

(20)

if

\[ M > \left( \frac{f^{(2\beta)}(x^*)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta}, \]

(21)

or

\[ M\Gamma(2\beta + 1) \Gamma(\beta + 1) > \left( \frac{f^{(2\beta)}(x^*)}{f^{(\beta)}(x^*)} \right). \]

for sufficiently large \( n \).

\[ \square \]

**Theorem 3.** Assume that \( f \) is three times continuously differentiable on an open interval \((a, b)\) and that there exists \( x^* \in (a, b) \) with \( f^{(2\beta)}(x^*) \neq 0 \). Implementing the first-order FNR method, we have the following recurrence relation:

\[ x_{n+1} = x_n + \left( \frac{f(x_n)}{f^{(\beta)}(x_n)} \Gamma(\beta + 1) \right)^{1/\beta}, \]

(22)

for sufficiently large \( n \). Thus, \( x_n \) converges to \( x^* \) quadratically.

Proof. Let \( e_n = x_n - x^* \), so that \( x_n - e_n = x^* \). Setting \( x = x_n \) and \( h = -e_n \) in fractional Taylor’s Theorem, we get

\[ f(x_n - e_n) = f(x_n) - \frac{e_n^\beta}{\Gamma(\beta + 1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta + 1)} f^{(2\beta)}(\xi_n), \]

(23)

\[ f^{(\beta)}(x_n) = \frac{f(x_n)}{\Gamma(\beta + 1)} f^{(\beta)}(x_n), \]

\[ f^{(2\beta)}(\xi_n) = \frac{f(x_n)}{\Gamma(2\beta + 1)} f^{(2\beta)}(\xi_n). \]

(24)
for some \( \bar{x}_n \in (x_n, x^*). \) Since \( f(x_n - \bar{x}_n) = f(x^*) = 0, \) we have

\[
0 = f(x_n) - \frac{(x_n - x^*) \beta}{\Gamma(\beta + 1)} f^{(\beta)}(x_n) + \frac{(x_n - x^*)^{3\beta}}{\Gamma(3\beta + 1)} f^{(3\beta)}(x_n) - \frac{(x_n - x^*)^{3\beta}}{\Gamma(3\beta + 1)} f^{(3\beta)}(\bar{x}_n).
\]

Having the condition that \( \beta \)th derivative of \( f \) is continuous with as long as \( x_n \) is close enough to \( x^* \) allows us to divide by \( f^{(\beta)}(x_n) / \Gamma(2\beta) \) which leads to the following:

\[
0 = \frac{\Gamma(2\beta + 1) f(x_n)}{f^{(2\beta)}(x_n)} - \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} \frac{(x_n - x^*) \beta}{\Gamma(\beta + 1)} f^{(\beta)}(x_n)
+ \frac{(x_n - x^*)^{3\beta}}{\Gamma(3\beta + 1)} f^{(3\beta)}(x_n).
\]

As a result, the formulation of first-order FNR gives the following:

\[
(x_n + x^*)^{2\beta} = \frac{e_\beta (x_n - x^*)^{3\beta}}{f^{(3\beta)}(x_n) / \Gamma(3\beta + 1)}
\]

After rearrangement, we have

\[
|x_{n+1} - x^*| = e^{\beta / 2} \left| \frac{f^{(3\beta)}(\bar{x}_n)}{f^{(3\beta)}(x_n)} \Gamma(2\beta + 1) \right|^{1/2\beta}.
\]

(28)

Finally,

\[
|x_{n+1} - x^*| = \left( \frac{f^{(3\beta)}(\bar{x}_n)}{f^{(3\beta)}(x_n)} \Gamma(2\beta + 1) \right)^{1/2\beta} |x_n - x^*|^{3/2}.
\]

(29)

By continuity, \( f^{(2\beta)}(x_n) \) converges to \( f^{(2\beta)}(x^*). \) Convergence of \( \bar{x}_n \in (x_n, x^*) \) to \( x^* \) leads to convergence of \( f^{(3\beta)}(\bar{x}_n) \) to \( f^{(3\beta)}(x^*). \) As a result, we have

\[
|x_{n+1} - x^*| \leq M |x_n - x^*|^{3/2},
\]

(30)

if

\[
M > \left( \frac{f^{(3\beta)}(x^*)}{f^{(3\beta)}(x_n)} \Gamma(2\beta + 1) \right)^{1/2\beta} \Gamma(2\beta + 1) \Gamma(3\beta + 1)
\]

or

\[
M^{3\beta} (3\beta + 1) > \frac{f^{(3\beta)}(x^*)}{f^{(3\beta)}(x_n)} \Gamma(2\beta + 1) \Gamma(3\beta + 1).
\]

(31)

for sufficiently large \( n. \)

In general, for the convergence of higher-order FNR, we obtain the following condition:
for sufficiently large $n$. 

\[ |x_{n+1} - x^*| \leq M|f(x_n) - x^*|^{(k+\beta)}/M \]

\[ M > \left( \left| \frac{f^{(k+\beta)}(x^*)}{f^{(k+\beta)}(x_n)} \right| \Gamma((k+1)/(k+\beta)) \right)^{1/(k+\beta)}, \quad k = 1, 2, 3, \ldots, \]

(32)

\[ |x_{n+1} - x_n| \leq 10^{-8} \] and a maximum of 500 iterations are

5. Numerical Examples

In this section, some illustrative examples are presented to show the implementation of first- and second-order FNR which allows us to confirm the obtained results given in the previous section. Matlab R2016b with stopping criterion
utilized. In the tables of corresponding examples, the reached root $|x_{n+1} - x_n|$, $\|f(x_{n+1})\|$, and the number of iterations are shown.

*Example 1.* Let us consider the function $f_1(x) = x^3 - 10x^2 + 34x - 40$ with roots $x_0 = 4$, $x_1 = 3 - i$ and $x_2 = 3 + i$. 

**Figure 2:** Convergence planes of second-order FNR on $f_1(x)$ for Caputo derivative.

**Figure 3:** Convergence planes of first-order FNR on $f_1(x)$ for Riemann–Liouville derivative.

**Figure 4:** Convergence planes of second-order FNR on $f_1(x)$ for Riemann–Liouville derivative.
### Table 3: First-order NR and second-order NR results for $f_2(x)$ with Caputo derivative and initial estimation $x_0 = 0.2$. 

| $\beta$  | $x_{n+1}$ | $|x_{n+1} - x_n|$ | $\|f(x_{n+1})\|$ | $n$  |
|----------|----------|----------------|------------------|-----|
| 1st N-R  | 1        | -0.00000000    | 0.000000e+00     | 5   |
| 2nd N-R  | 1        | 6.66798e-10    | 0.000000e+00     | 3   |
| 1st N-R  | 0.98     | -0.00000000    | 2.42649e-09      | 8   |
| 2nd N-R  | 0.98     | 1.07859e-09    | 1.64532e-10      | 8   |
| 1st N-R  | 0.96     | -0.00000000    | 3.91885e-09      | 10  |
| 2nd N-R  | 0.96     | 2.58893e-09    | 3.61075e-10      | 10  |
| 1st N-R  | 0.94     | -0.00000000    | 5.70944e-09      | 12  |
| 2nd N-R  | 0.94     | 4.32641e-09    | 9.30883e-10      | 12  |
| 1st N-R  | 0.92     | -0.00000000    | 2.76674e-09      | 15  |
| 2nd N-R  | 0.92     | 8.17968e-09    | 2.41352e-09      | 14  |
| 1st N-R  | 0.9      | -0.00000000    | 7.94065e-09      | 17  |
| 2nd N-R  | 0.9      | 6.59632e-09    | 2.50229e-09      | 17  |
| 1st N-R  | 0.88     | -0.00000000    | 4.85622e-09      | 21  |
| 2nd N-R  | 0.88     | 9.82148e-09    | 4.59793e-09      | 20  |
| 1st N-R  | 0.86     | -0.00000000    | 7.02051e-09      | 25  |
| 2nd N-R  | 0.86     | 5.90703e-09    | 3.31762e-09      | 25  |
| 1st N-R  | 0.84     | -0.00000000    | 6.69774e-09      | 31  |
| 2nd N-R  | 0.84     | 9.87756e-09    | 6.51845e-09      | 30  |
| 1st N-R  | 0.82     | -0.000000001   | 8.16846e-09      | 39  |
| 2nd N-R  | 0.82     | 6.60392e-09    | 5.03942e-09      | 39  |
| 1st N-R  | 0.8      | -0.00000000    | 7.97894e-09      | 52  |
| 2nd N-R  | 0.8      | 8.52364e-09    | 7.42575e-09      | 51  |
| 1st N-R  | 0.78     | -0.00000000    | 9.79061e-09      | 74  |
| 2nd N-R  | 0.78     | 8.49008e-09    | 8.35597e-09      | 73  |
| 1st N-R  | 0.76     | -0.00000000    | 9.85552e-09      | 125 |
| 2nd N-R  | 0.76     | 9.15524e-09    | 1.00893e-08      | 121 |
| 1st N-R  | 0.74     | -0.00000000    | 9.63986e-09      | 360 |
| 2nd N-R  | 0.74     | 9.78282e-09    | 1.19788e-08      | 317 |
| 1st N-R  | 0.72     | -4.31363578    | 5.71734e-07      | 500 |
| 2nd N-R  | 0.72     | 1.11043e-01    | 1.43663e-01      | 500 |
| 1st N-R  | 0.7      | -4.78460364    | 2.45965e-01      | 500 |
| 2nd N-R  | 0.7      | 7.32260e-01    | 7.50525e-01      | 500 |

### Table 4: First-order NR and second-order NR results for $f_2(x)$ with Riemann–Liouville derivative and initial estimation $x_0 = 0.2$. 

| $\beta$  | $x_{n+1}$ | $|x_{n+1} - x_n|$ | $\|f(x_{n+1})\|$ | $n$  |
|----------|----------|----------------|------------------|-----|
| 1st N-R  | 1        | 0.0000000000   | 0.000000e+00     | 5   |
| 2nd N-R  | 1        | 6.66798e-10    | 0.000000e+00     | 3   |
| 1st N-R  | 0.98     | -0.00000000    | 2.42649e-09      | 8   |
| 2nd N-R  | 0.98     | 9.29969e-10    | 5.95854e-11      | 8   |
| 1st N-R  | 0.96     | -0.00000000    | 3.91885e-09      | 10  |
| 2nd N-R  | 0.96     | 2.40538e-09    | 3.14007e-10      | 10  |
| 1st N-R  | 0.94     | -0.00000000    | 5.70944e-09      | 12  |
| 2nd N-R  | 0.94     | 4.82173e-09    | 9.60702e-10      | 12  |
| 1st N-R  | 0.92     | -0.00000000    | 2.76674e-09      | 15  |
| 2nd N-R  | 0.92     | 3.43115e-09    | 9.26171e-10      | 15  |
| 1st N-R  | 0.9      | -0.00000000    | 7.94065e-09      | 17  |
| 2nd N-R  | 0.9      | 5.89937e-09    | 2.01913e-09      | 18  |
| 1st N-R  | 0.88     | -0.00000000    | 4.85622e-09      | 21  |
| 2nd N-R  | 0.88     | 4.71244e-09    | 1.95934e-09      | 23  |
| 1st N-R  | 0.86     | -0.00000000    | 7.02051e-09      | 25  |
| 2nd N-R  | 0.86     | 5.93748e-09    | 2.90861e-09      | 30  |
| 1st N-R  | 0.84     | -0.00000000    | 6.69774e-09      | 31  |
| 2nd N-R  | 0.84     | 7.64084e-09    | 4.30750e-09      | 43  |
| 1st N-R  | 0.82     | -0.000000001   | 8.16846e-09      | 39  |
| 2nd N-R  | 0.82     | 8.48988e-09    | 5.40319e-09      | 79  |
| 1st N-R  | 0.8      | -0.00000000    | 7.97894e-09      | 52  |
| 2nd N-R  | 0.8      | 1.02936e-02    | 7.19943e-03      | 500 |
| 1st N-R  | 0.78     | -0.00000001    | 9.79061e-09      | 74  |
Table 4: Continued.

| $\beta$ | $x_{n+1}$ | $|x_{n+1} - x_n|$ | $\|f(x_{n+1})\|$ | $n$ |
|---------|-----------|-------------------|-----------------|-----|
| 2nd N-R | 0.78      | -0.07024016       | 1.47568e-01     | 9.83535e-02 | 500 |
| 1st N-R | 0.76      | -0.00000001       | 9.85552e-09     | 1.08611e-08 | 125 |
| 2nd N-R | 0.76      | -0.14147226       | 2.83238e-01     | 1.86505e-01 | 500 |
| 1st N-R | 0.74      | -0.00000001       | 9.63986e-09     | 1.18037e-08 | 360 |
| 2nd N-R | 0.74      | -1.57171790       | 2.40309e+00     | 1.13046e+00 | 500 |
| 1st N-R | 0.72      | -4.31365378       | 5.71734e-07     | 5.23966e-05 | 500 |
| 2nd N-R | 0.72      | -0.00000000       | 9.36668e-09     | 5.10297e-09 | 157 |
| 1st N-R | 0.7      | -4.78460364       | 2.45965e-01     | 1.47957e+00 | 500 |
| 2nd N-R | 0.7      | 0.00000000        | 9.82670e-09     | 5.30280e-09 | 101 |
| 1st N-R | 0.68      | -5.71070798       | 5.01626e-01     | 1.03562e+01 | 500 |
| 2nd N-R | 0.68      | 0.00000000        | 8.29674e-09     | 4.54705e-09 | 85  |
| 1st N-R | 0.66      | -754.27892554     | 7.80349e+01     | 2.58569e+18 | 500 |
| 2nd N-R | 0.66      | -0.00000000       | 8.34916e-09     | 4.76503e09  | 82  |
| 1st N-R | 0.64      | -1411860.53657482 | 1.42625e+05     | 3.96270e+44 | 500 |
| 2nd N-R | 0.64      | 0.00000000        | 9.54192e-09     | 5.80404e-09 | 89  |
| 1st N-R | 0.62      | -1063648380.57639720 | 1.04707e+08 | 4.10943e+67 | 500 |
| 2nd N-R | 0.62      | -0.00000000       | 8.55254e-09     | 5.65282e-09 | 113 |
| 1st N-R | 0.6      | -875382037053.40283000 | 8.36961e+10 | 8.59364e+90 | 500 |
| 2nd N-R | 0.6      | -3.22839026       | 2.60655e-07     | 1.73936e-04 | 500 |

Figure 5: Convergence planes of first-order NR on $f_2(x)$ for Caputo derivative.

Figure 6: Convergence planes of second-order NR on $f_2(x)$ for Caputo derivative.
It can be observed from Tables 1 and 2 that the estimation of second-order FNR is better than the one of the first-order FNR when the order of the derivative is close to one in both Caputo and Riemann–Liouville derivatives. In Figures 1–4, the convergence plane of the polynomial function \( f_1(x) \) is given when \( x_0 \in [-3, 3] \) for various values of \( \beta \).

**Example 2.** Let us consider the function \( f_2(x) = \exp(x) - 1 \), whose only root is \( x_1 = 0 \).

It can be observed from Tables 3 and 4 that the estimation of second-order FNR is better than the one of the first-order FNR when the order of the derivative is close to one in both Caputo and Riemann–Liouville derivatives. In Figures 5–8, the convergence plane of the polynomial function \( f_2(x) \) is given when \( x_0 \in [-10, 10] \) for various values of \( \beta \).

**6. Conclusion**

First- and second-order FNR are developed, and analyzed and applied in this study. Moreover, the convergence of both methods is established. It is shown that second-order FNR gives better results compare to first-order FNR when the order of fractional derivative is close to one in both Caputo and Riemann–Liouville derivatives. It is also shown that the order of convergence for first-order FNR is quadratic while one of the second-order FNR is \( 3/2 \). It is clear from tables that as the fractional parameter increases to one, the number of iterations decreases for both developed methods. Moreover, figures depict that the convergence of approximate solutions is better for \( \beta \in (0.7, 1] \) which can be seen also from the tables. Generally, it is obvious from the obtained formulation that the order of convergence for \( k \)th order FNR is \( (k + 1/k) \). The obtained results are verified by presented examples, too.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
References


