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Research Article

A Logistic Trigonometric Generalized Class of Distribution Characteristics, Applications, and Simulations

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We propose a trigonometric generalizer/generator of distributions utilizing the quantile function of modified standard Cauchy distribution and construct a logistic-based new G-class disbursing cotangent function. Significant mathematical characteristics and special models are derived. New mathematical transformations and extended models are also proposed. A two-parameter model logistic cotangent Weibull (LCW) is developed and discussed in detail. The beauty and importance of the proposed model are that its hazard rate exhibits all monotone and non-monotone shapes while the density exhibits unimodal and bimodal (symmetrical, right-skewed, and decreasing) shapes. For parametric estimation, the maximum likelihood approach is used, and simulation analysis is performed to ensure that the estimates are asymptotic. The importance of the proposed trigonometric generalizer, G class, and model is proved via two applications focused on survival and failure datasets whose results attested the distinct better fit, wider flexibility, and greater capability than existing and well-known competing models. The authors thought that the suggested class and models would appeal to a broader audience of professionals working in reliability analysis, actuarial and financial sciences, and lifetime data and analysis.

1. Introduction and Motivations

Generalizing a classical distribution is an old practice in distribution theory. The earliest work regarding generalizing a distribution was conducted by Pearson [1] using differential equations. Since 1985, the families (or classes) of distributions have been derived adopting the following famous methodologies: differential equation method, transformation techniques, compounding methodology, skewed distributions generation, parametric induction, quantile based approach, transformed (T)-transformer (X) mechanism, exponentiated T-X system, T-R\{Y\} approach, etc., It is possible to more readily manage data that is highly skewed with each new model that is developed because of its enhanced heavy-tailed, tractable core functions, and simplified simulation technique.

The literature review explores the following four critical points which prove the basis for this study. (i) In the statistical literature, a slew of new families and models for algebraically generalizers/distribution's generators $W[G(x)]$ have been introduced in contrast of neglecting trigonometric equations (trigonometric generalizers). (ii) The interest in modelling directional and proportional data led applied researchers to develop and employ trigonometry function-based models capable of handling these datasets more smoothly and economically. (iii) The use of algebraic and trigonometric functions in mixture generalizers is still to be researched and investigated. (iv) Using the stated transformation, any current G class or model may be simply reversed in a subsequent version. The major motives are considered to be influenced and inspired by the generated results in terms of accuracy, adaptability, and goodness of fit (gof) which are included below:

(i) To introduce a generator of distributions based on cotangent function (that is a combination of algebraic and trigonometric functions and generators concurrently).

(ii) To introduce a new G-class called a new logistic-G class of distributions (LCG for short) in trigonometric scenario.

(iii) There are several advantages to the suggested class, including its simplification, lack of non-identifiability, and lack of over-parametrization.

(iv) Because of the injection of the cotangent function, the new CDF can increase flexibility, giving rise to new efficient and flexible models.

(v) Non-generalized and non-exponentiated models, according to the literature, give insufficient gof.

The famous generalizers and corresponding G-families are presented in Table 1.

Moreover, the new generators and G-classes are introduced regarding several purposes. Among them, the recent and remarkable include the following. (A) To present a new family utilizing the additive model structure (see [6]). (B) New generator $W[G(x)]$ is defined with quantile function (see [7]). (C) To achieve better gof and more flexibility than existing classical models (see [8]). (D) Type I half logistic Burr XG family has been constructed by Algarni et al. [9]. (E) The bivariate Weibull-G family based on copula function using odd classes has been introduced by El-Sherpieny et al. [10]. (F) A significant amount of distribution families was proposed using parameter induction (inserting one or maybe more new parameters (s) to the baseline), for example, a new one was put forward by Cordeiro et al. [11]. (G) Adopting the T-X family methodology only (see [12]). (H) Presenting a flexible family which deals with both monotone and non-monotone hazard rate function (see [13]). (I) Using a flexible family to introduced flexible generalized Pareto distribution (see [14]). The whole real line interval $(-\infty, \infty)$ distributions naturally come up when random variables should vary in the infinite real interval and several distributions such as logistic, normal, Laplace, t, Chen, and Gumbel distributions are supported on this interval. It has been used to describe the distribution of income and wealth in a fairly basic way. The logistic distribution has numerous uses in statistical analysis and has a form that resembles the normal distribution.

A previous study [15] introduced logistic distribution whose main functions (cdf and pdf) in a new format are given below:

$$F(x) = \left(1 + e^{-(x-\mu/\sigma)}\right)^{-1},$$
$$f(x) = \frac{e^{-(x-\mu/\sigma)}}{\sigma\left(1 + e^{-(x-\mu/\sigma)}\right)^2}, \ x \in \mathbb{R}. \quad (1)$$

Alzaatreh et al. [16] contributed transformed (T)-transformer (X) family of distributions (for short, T-X family) which expanded vision about generators of distributions ($W[G(x)]$), and in this article, via the T-X method, a new logistic-G class of statistical distributions is proposed adopting a cotangent-based trigonometric generator $[-\cot(\pi G(x))]$.

This paper is outlined as follows. Section 1 is about introduction and motivations while in Section 2, cotangent generator and LCG class are developed. In Section 3, some special models are presented while in Section 4, the characteristics of LCG class are deduced. In Section 5, a new model LCW along with significant properties is discussed and a simulation work is performed in Section 6. The importance of the new class and model is confirmed by two real-life applications using failure and survival datasets in Section 7. Finally, in Section 8, the conclusions are presented.

2. Development of Cotangent Generator

In this part, let us assume that X is a random variable (r.v.) that follows the famous distribution, the standard Cauchy distribution, and its location parameter has value equal to $\mu = 0$ while the other parameter which is called the scale parameter has value equal to $\sigma = 1$; then, its cdf and qf, respectively, are

$$F(x) = \frac{1}{2} + \frac{\tan^{-1} x}{\pi}, \ x > 0, \quad (2)$$

$$Q(u) = \tan[\pi(u - 0.5)] = \tan\left[\pi u - \frac{\pi}{2}\right] = [-\cot(\pi u)]. \quad (3)$$

Replacing u by $G(x)$, $W[G(x)] = [-\cot(\pi G(x))]$, a new trigonometric generator based on cotangent function is achieved having support $(-\infty, \infty)$.

2.1. Genesis of LCG Class. By assuming that $r(t)$ is considered as the pdf of a r.v. $T \in [a, b]$ for $-\infty \le a < b < \infty$ and $W[G(x)] = [-\cot(\pi G(x))]$, it fulfills the T-X family's following requirements. (i) Since $[\pi G(x)] \in [0, \pi]$, then $[-\cot(\pi G(x))] \in (-\infty, \infty)$. (ii) $[-\cot(\pi G(x))]$ is differentiable and monotonically non-decreasing because cotangent function is differentiable and monotonically non-decreasing on $[0, \pi]$. (iii) Since $G(x) \longrightarrow 0$ such that $x \longrightarrow -\infty$, then $[\pi G(x)] \longrightarrow -\infty$ and $[-\cot(\pi G(x))] \longrightarrow -\infty$. Similarly, $G(x) \longrightarrow 1$ as $x \longrightarrow \infty$; then, $[\pi G(x)] \longrightarrow \infty$ and $[-\cot(\pi G(x))] \longrightarrow \infty$.

Now, the main functions of LCG class in T-X format can be written as

$$F(x) = \int_{-\infty}^{-\cot(\pi G(x))} r(t) dt. = R[-\cot(\pi G(x))], \quad (4)$$

$$f(x) = \pi g(x) \csc^2(\pi G(x)) r[-\cot(\pi G(x))], \quad (5)$$

Table 1: The real line supporting generalizers and corresponding G-families.

Range of T	Generator $W[G(x)]$	Models of the T-X family	Inventor(s)
$(-\infty,\infty)$	$\log(G(x)/\overline{G}(x))$	Log odd logistic family	Torabi and Montazeri [2]
$(-\infty,\infty)$	$\log(-\log\overline{G}(x))$	Logistic-X family	Tahir et al. [3]
$(-\infty,\infty)$	$1-(1+[-\log G(x)]^{-\lambda})^{-1}$	Logistic type 2-G family	Hassan et al. [4]
$(-\infty,\infty)$	$\log(-\log G(x))$	Logistic-G family	Mansoor et al. [5]

$$h(x)=\frac{\pi g(x)\csc^2(\pi G(x))r[-\cot(\pi G(x))]}{1-R[-\cot(\pi G(x))]}. \tag{6}$$

2.1.1. LCG Class in Cotangent Scenario. Let T be a logistic r.v. with cdf $R(t)=(1+e^{-t})^{-1}$ and pdf $r(t)=e^{-t}(1+e^{-t})^{-2}$. Putting $W[G(x)]=-\cot[\pi G(x)]$ in (4), the cdf of new class is obtained and presented as

$$F(x)=\left(1+e^{\cot(\pi G(x))}\right)^{-1}. \tag{7}$$

The pdf corresponding to (7) reduces to

$$f(x)=\pi g(x)\csc^2(\pi G(x))e^{\cot(\pi G(x))}\left(1+e^{\cot(\pi G(x))}\right)^{-2}. \tag{8}$$

2.1.2. LCG Class in Tangent Scenario. Putting $W[G(x)]=\tan[\pi(G(x)-0.5)]$ in (4), the cdf of new class may be expressed as below:

$$F(x)=\left(1+e^{-\tan[\pi(G(x)-0.5)]}\right)^{-1}. \tag{9}$$

The pdf corresponding to (9) reduces to

$$f(x)=\pi g(x)\sec^2(\pi(G(x)-0.5))e^{-\tan(\pi(G(x)-0.5))}\left(1+e^{-\tan(\pi(G(x)-0.5))}\right)^{-2}. \tag{10}$$

3. Special Models

Some special models of LCG class with their main functions and corresponding graphs are presented subsequently.

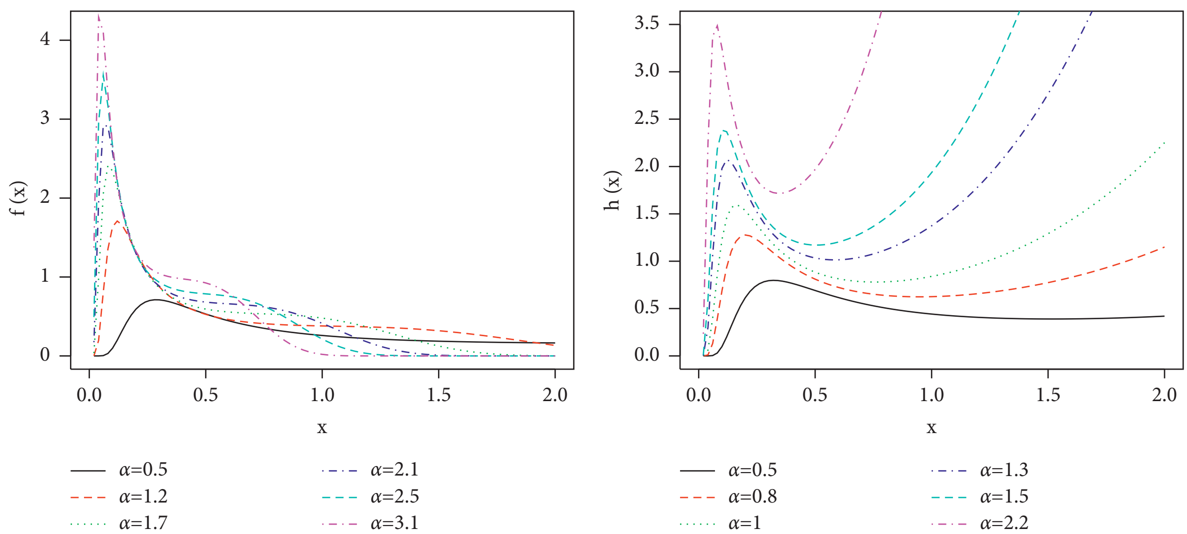
3.1. The Logistic Cot Exponential (LCE) Distribution. By assuming that the variable X follows an exponential distribution, then we may be able to express the central functions for the LCE distribution in the form as follows:

$$\begin{aligned}
F(x)&=\left(1+e^{\cot[\pi(1-e^{-\alpha x})]}\right)^{-1}, x>0,\\
f(x)&=\pi\alpha e^{-\alpha x}\csc^2(\pi(1-e^{-\alpha x}))e^{\cot(\pi(1-e^{-\alpha x}))}\left(1+e^{\cot(\pi(1-e^{-\alpha x}))}\right)^{-2},\\
h(x)&=\pi\alpha e^{-\alpha x}\csc^2(\pi(1-e^{-\alpha x}))\left(1+e^{\cot(\pi(1-e^{-\alpha x}))}\right)^{-1}.
\end{aligned} \tag{11}$$

3.2. The Logistic Cot Lindley (LCLi) Distribution. Assuming X is a Lindley random variable, we can write the new one-parameter LCLi model that has the following cdf, pdf, and hazard function. Figure 1 demonstrates the graph plots of the LCE, and Figure 2 demonstrates the the graph plots of the LCLi.

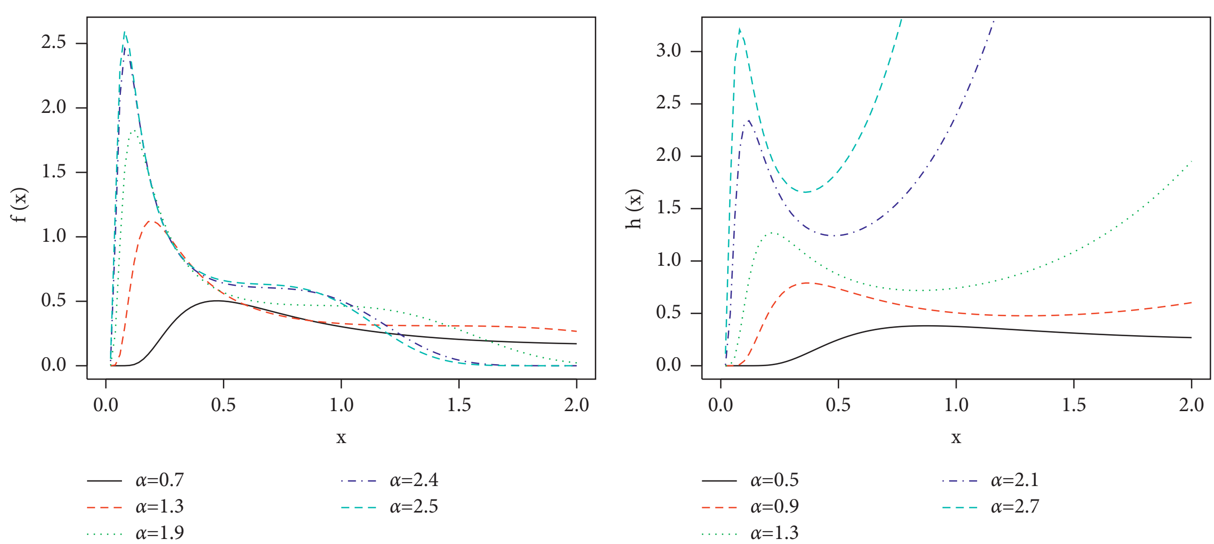
$$\begin{aligned}
F(x)&=\left[1+e^{\cot(\pi(1-e^{-\alpha x}(1+\alpha x/1+\alpha)))}\right]^{-1}, x,\alpha>0,\\
f(x)&=\frac{\pi(1+x)\alpha^2e^{-\alpha x}/1+\alpha\csc^2(\pi(1-e^{-\alpha x}(1+\alpha x/1+\alpha)))\left[e^{\cot(\pi(1-e^{-\alpha x}(1+\alpha x/1+\alpha)))}\right]}{\left[1+e^{\cot(\pi(1-e^{-\alpha x}(1+\alpha x/1+\alpha)))}\right]^2},\\
h(x)&=\pi\frac{(1+x)\alpha^2e^{-\alpha x}}{1+\alpha}\csc^2\left(\pi\left(1-e^{-\alpha x}\left(1+\frac{\alpha x}{1+\alpha}\right)\right)\right)\left[1+e^{\cot(\pi(1-e^{-\alpha x}(1+\alpha x/1+\alpha)))}\right]^{-1}.
\end{aligned} \tag{12}$$

3.3. The Logistic Cot Gamma (LCGa) Distribution. Let X be a gamma random variable; then, the LCGa model has the following main functions:



(a) (b)

Figure 1: Graphical representation for the plots of the LCE: (a) density and (b) hazard rate function using certain values of the parameters.



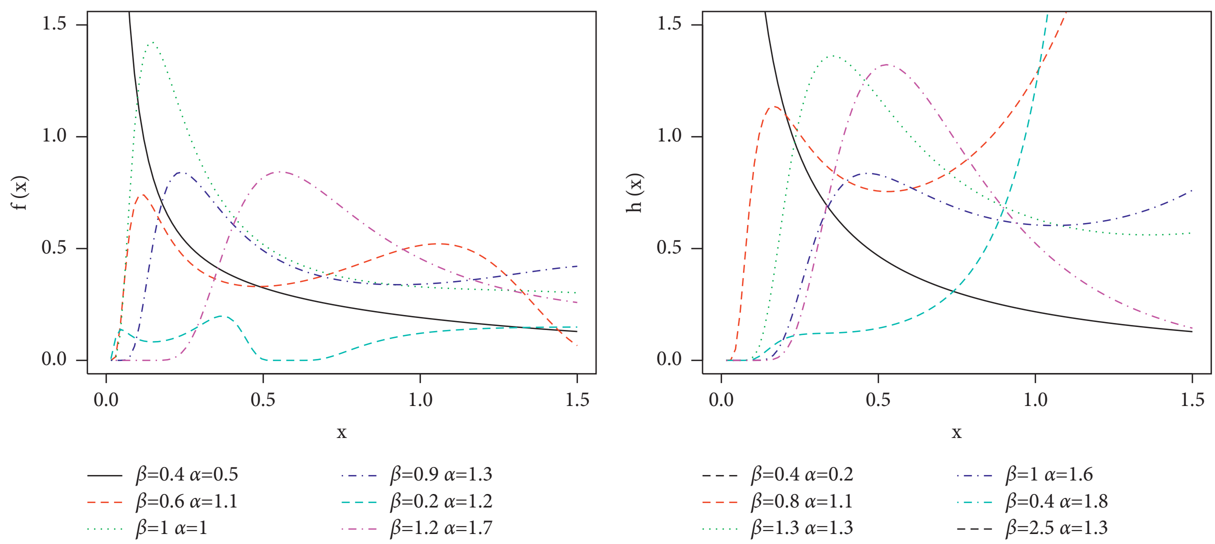
(a) (b)

Figure 2: Graphical representation for the plots of the LCLi: (a) density and (b) hazard rate using certain values of the parameters.

$$F(x) = \left(1 + e^{\cot(\pi\gamma(\alpha,\beta x)/\Gamma\alpha)}\right)^{-1}, x > 0,$$
$$f(x) = \pi \frac{\beta^{\alpha}}{\Gamma\alpha x^{\alpha-1} e^{-\beta x}} \csc^2\left(\pi \frac{\gamma(\alpha,\beta x)}{\Gamma\alpha}\right) e^{\cot(\pi\pi\gamma(\alpha,\beta x)/\Gamma\alpha)}\left(1 + e^{\cot(\pi\pi\gamma(\alpha,\beta x)/\Gamma\alpha)}\right)^{-2}, \tag{13}$$
$$h(x) = \pi \frac{\beta^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x} \csc^2\left(\pi \frac{\gamma(\alpha,\beta x)}{\Gamma\alpha}\right)\left(1 + e^{\cot(\pi\pi\gamma(\alpha,\beta x)/\Gamma\alpha)}\right)^{-1}.$$

The plots of the LCGa are shown as graphs in Figure 3.

3.4. The Logistic Cot Dagum (LCD) Distribution. By assuming that for the possibility that X is a Dagum random variable, then the LCD distribution has the following primary functions:



(a) (b)

FIGURE 3: Graphical representation for the plots of the LCGa: (a) density and (b) hazard rate using certain values of the parameters.

$$F(x) = \left[1 + e^{\cot\left(\pi\left(\left(1+\alpha x^{-\beta}\right)^{-\lambda}\right)\right)}\right]^{-1}, x > 0,$$

$$f(x) = \frac{\pi\alpha\beta\lambda x^{-(\beta+1)}\left(1 + \alpha x^{-\beta}\right)^{-(\lambda+1)}\csc^2\left(\pi\left(\left(1 + \alpha x^{-\beta}\right)^{-\lambda}\right)\right)e^{\cot\left(\pi\left(\left(1+\alpha x^{-\beta}\right)^{-\lambda}\right)\right)}}{\left[1 + e^{\cot\left(\pi\left(\left(1+\alpha x^{-\beta}\right)^{-\lambda}\right)\right)}\right]^2}, \tag{14}$$

$$h(x) = \pi\alpha\beta\lambda x^{-(\beta+1)}\left(1 + \alpha x^{-\beta}\right)^{-(\lambda+1)}\csc^2\left(\pi\left(\left(1 + \alpha x^{-\beta}\right)^{-\lambda}\right)\right)\left[1 + e^{\cot\left(\pi\left(\left(1+\alpha x^{-\beta}\right)^{-\lambda}\right)\right)}\right]^{-1}.$$

The graphs in Figure 4 show the plots of the LCD.

4. Mathematical Properties of LCG Class

Here, important properties of the new class are presented.

4.1. The Inverse Function for Both pdf and cdf (Quantile Function). We present an additional property of X which is qf:

$$Q(u) = Q_G(v). \tag{15}$$

Here, $Q_G(v) = G^{-1}(v)$ is the parental qf, whereas $v = 1/\pi\cot^{-1}(\ln((1-u)/u))$. So, we can write the quantile density function $Q'(u)$ as follows:

$$Q'(u) = \frac{\left[\pi^{-1}\cot^{-1}(\ln(1-u)/u)\right]^{-2}}{\pi u(1-u)\left(1 + (\ln(1 - u/u))^2\right)}. \tag{16}$$

4.2. Useful Reliability Functions. In this part of the paper, we will concentrate our efforts to introduce the most important reliability functions. First we will define the survival function (sf) $S(x)$, after that we write the equation of the hazard rate function (hrf) $h(x)$, and the reversed hazard rate function $r(x)$ is as below, and the cumulative hazard rate function (chrf) $H(x)$ and mills' ratio $m(x)$ are, respectively, given below.

$$S(x) = \frac{e^{\cot(\pi G(x))}}{1 + e^{\cot(\pi G(x))}},$$

$$h(x) = \pi g(x)\csc^2(\pi G(x))\left(1 + e^{\cot(\pi G(x))}\right)^{-1},$$

$$r(x) = \pi g(x)\csc^2(\pi G(x))e^{\cot(\pi G(x))}\left(1 + e^{\cot(\pi G(x))}\right)^{-1},$$

$$H(x) = \ln\left(\frac{e^{\cot(\pi G(x))}}{1 + e^{\cot(\pi G(x))}}\right)^{-1},$$

$$m(x) = \frac{1 + e^{\cot(\pi G(x))}}{\pi g(x)\csc^2(\pi G(x))^2}. \tag{17}$$

4.3. The Density Function Analytical Formulas. By solving (18), we can get the roots of the equation, and we can call it the solutions of the density function of the new class; it is possible to provide an analytical description of the forms of the new density:

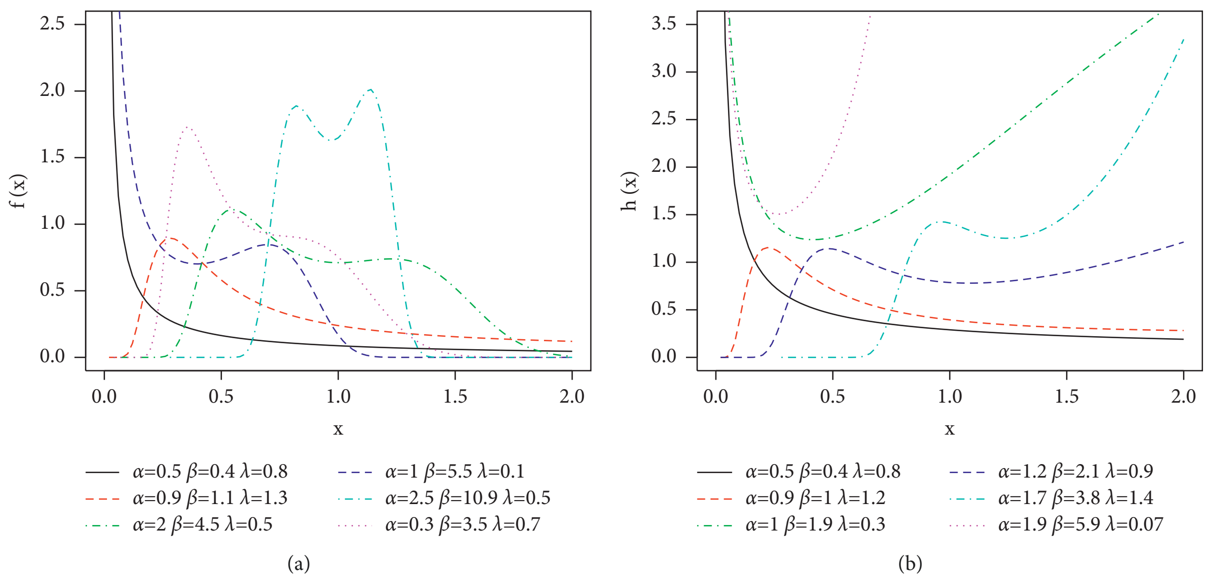


FIGURE 4: Graphical representation for the plots of the LCD: (a) density and (b) hazard rate using certain values of the parameters.

$$\frac{g'(x)}{g(x)} + \pi g(x)\left[\frac{\csc^2(\pi G(x))2e^{\cot(\pi G(x))}}{1+e^{\cot(\pi G(x))}} - 2\cot(\pi G(x)) - \csc^2(\pi G(x))\right] = 0. \tag{18}$$

(18) can be multi-rooted equation. Let $\lambda(x) = d^2 \log[f(x)]/dx^2$; then,

$$\begin{aligned}\lambda(x) = {} & \frac{g''(x)g(x) - \left(g'(x)\right)^2}{g(x)^2} + \pi g'(x)\left[\frac{\left[\csc^2(\pi G(x))\right]2e^{\cot(\pi G(x))}}{1+e^{\cot(\pi G(x))}} - 2\cot(\pi G(x)) - \csc^2(\pi G(x))\right] \\ & + \pi g(x)\left[2\pi g(x)\csc^2(\pi G(x))(1+\cot(\pi G(x)))\right] + \frac{\pi g(x)}{(1+\cot(\pi G(x)))^2} \\ & \left[2\left(1+e^{\cot(\pi G(x))}\right)\left(-2\pi g(x)\csc^2(\pi G(x))\right)e^{\cot(\pi G(x))}\cot(\pi G(x)) + \csc^2(\pi G(x)) \right. \\ & \left. + 2\pi g(x)\left(\csc^2(\pi G(x))\right)^2\left(e^{\cot(\pi G(x))}\right)^2\right].\end{aligned} \tag{19}$$

4.4. The Hazard Function Analytical Formulas. In order to find the roots of (20), we must obtain the crucial points of the hrf $h(x)$.

$$\frac{g'(x)}{g(x)} - 2\pi g(x)\cot(\pi G(x)) - \frac{\pi g(x)\csc^2(\pi G(x))e^{\cot(\pi G(x))}}{1+e^{\cot(\pi G(x))}} = 0. \tag{20}$$

As we can see, this equation has many solutions or we can say many roots are available for this equation. Suppose that $\lambda^{'}(x) = d^2\log[h(x)]/dx^2$. We have

$$\begin{aligned}\lambda^{'}(x) = {} & \frac{g''(x)g(x) - (g'(x))^2}{g(x)^2} - 2\pi\left[g'(x)\cot(\pi G(x)) - \pi g(x)^2\csc^2(\pi G(x))\right] \\ & + \frac{\pi}{\left[1 + e^{\cot(\pi G(x))}\right]^3}\left[\begin{array}{c} g'(x)\csc^2(\pi G(x))e^{\cot(\pi G(x))} - \pi g(x)^2\csc^2(\pi G(x))^2 e^{\cot(\pi G(x))} \\ -2\pi g(x)^2\csc^2(\pi G(x))\cot(\pi G(x))e^{\cot(\pi G(x))} \end{array}\right] \\ & - \frac{\pi}{\left[1 + e^{\cot(\pi G(x))}\right]^4} \times \pi g(x)^2\left(\csc^2(\pi G(x))\right)^2\left[e^{\cot(\pi G(x))}\right]^2. \end{aligned} \tag{21}$$

Using any numerical software and (18) and (20), we can investigate local maximums and minimums, as well as inflexion points.

4.5. Linear Representation. The cdf of LCG class presented in (7) can be written as follows:

$$F(x) = \left(1 + e^{\cot(\pi G(x))}\right)^{-1}. \tag{22}$$

Using the relation $(1 + e^t)^{-1} = 1 - (1 + e^t)^{-1}$, $F(x)$ may be easily formulated as

$$F(x) = 1 - \left(1 + e^{-\cot(\pi G(x))}\right)^{-1}. \tag{23}$$

Through WolframAlpha, $(1 + x)^{-1} = \sum_{i=0}^{\infty}(-1)^i x^i$, for $|x| < 1$. We can demonstrate the cdf (2.6) in this form after applying this series and exponent series $e^x = \sum_{j=0}^{\infty} x^j/j!$, respectively.

$$F(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(i)^j(-1)^{i+j+1}}{j!}(\cot(\pi G(x)))^j. \tag{24}$$

Now, for the term $(\cot(\pi G(x)))^j$, we will use the power series to expand it $(\cot(x))^s = \sum_{i=0}^{\infty} a_i(s)(x)^{2i-s}$, such that $a_0(s) = 1$, $a_1(s) = -(s/3)$, $a_2(s) = s((5s - 7)/90)$, etc,. are obtained by the aid of the highly speed MATHEMATICA software see Tahir [17]. Hence, $(\cot(\pi G(x)))^j = \sum_{k=0}^{\infty} a_k(j)(\pi)^{2k-j}(G(x))^{2k-j}$, and

$$F(x) = \sum_{m=1}^{\infty} W_{(i,j,k,l)}H_m(x). \tag{25}$$

We know that $H_m(x)$ is considered as the exponentiated distribution function parameter (m) in the power and

$$W_{i,j,k,l} = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}\frac{(i)^j(-1)^{(i+j+l+m+1)}}{j!}\binom{2k-j}{l}\binom{m}{l}a_k(j)(\pi)^{2k-j}, H_m(x) = G(x)^m. \tag{26}$$

With the expansion of (8), the following formula may be obtained from the previously mentioned idea of exponentiated distributions:

$$f(x) = \sum_{m=1}^{\infty} v_{i,j,k,l}h_{(m)}(x), \tag{27}$$

such that $h_{(m)}(x)$ can be noted as the exponentiated density having a power parameter (m) and

$$\begin{aligned} v_{i,j,k,l} &= \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}\frac{(i)^j(-1)^{(i+j+l+m+1)}}{j!}\binom{2k-j}{l}\binom{m}{l}a_k(j)(\pi)^{2k-j}, \\ a_0(s) &= 1, a_1(s) = -s/3, a_2(s) = s(5s-7)/90, \end{aligned} \tag{28}$$

and

$$h_{(m)}(x) = mg(x)G(x)^{(m-1)}. \tag{29}$$

4.6. Moments, Incomplete Moments, Mean Deviation, Skewness, and Kurtosis. Suppose that we have a random variable called $Y_{(m)}$ that follows the the exp-G distribution that has a power parameter m, i.e., having density $h_m(x)$. Now we gave two formulas for the nth moment of X which follows (27), and the first one can expressed as follows:

$$\mathbb{E}(X^n) == \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty} v_{i,j,k,l}\mathbb{E}(Y_m^n). \tag{30}$$

The nth moment of X can be written in a second form as it may be deduced from (30) in terms of the G qf like the following equations:

$$\begin{aligned}\mathbb{E}(X^n) &= \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty} v_{i,j,k,l}(m)\int_{-\infty}^{\infty} x^n G(x)^{(m-1)} g(x)dx,\\ \mathbb{E}(X^n) &= \sum_{m=0}^{\infty} d_{(m-1)}\tau(n, m-1),\end{aligned} \tag{31}$$

where $d_{(m-1)} = \sum_{i,l=1}^{\infty}\sum_{j,k=0}^{\infty}(m)$ and $\tau_{n,(m-1)} = \int_{-\infty}^{\infty} x^n G(x)^{(m-1)} g(x)dx = \int_0^1 Q_G(u)^n u^{(m-1)}du$. These integrals can be calculated numerically.

4.7. Weighted Moments. In this section, we introduce the equation of the weighted moments. So, we can write the $(r, s)_{th}$ probability weighted moment (PWM) of X as

$$\rho_{r,s} = E\left[X^r F(X)\right]^s = \int_0^{\infty} x^r F(x)^s f(x)dx. \tag{32}$$

Then,

$$F(x)^s = \left[1 + e^{\cot(\pi G(x))}\right]^{-s}. \tag{33}$$

Putting the pdf of LCG class (given below)

$$f(x) = \pi g(x)\left(\csc^2(\pi G(x))\right)e^{\cot(\pi G(x))}\left(1 + e^{\cot(\pi G(x))}\right)^{-2}, \tag{34}$$

in (32), after applying the binomial expansion and exponential series, we get

$$\rho_{r,q} = \sum_{i,j=0}^{\infty}\pi\binom{-(s+2)}{i}\frac{(i+1)^j}{j!}\int_{-\infty}^{\infty} x^r g(x)\left(\csc^2(\pi G(x))\right)[\cot(\pi G(x))]^j dx. \tag{35}$$

Regarding $[\cot(\pi G(x))]^j$, we can use power series expansion $[\cot(x)]^s = \sum_{k=0}^{\infty} a_k(s)(x)^{2k-s}$, such that $a_0(s) = 1$, $a_1(s) = -s/3$, $a_2(s) = s(5s-7)/90$, etc. Similarly, for $[\csc^2(\pi G(x))] = \sum_{l=0}^{\infty} c_l(2)(x)^{2l-2}$, we get

$$\begin{aligned}\rho_{r,q} &= \sum_{i,j,k,l}^{\infty}\binom{-(s+2)}{i}\frac{(i+1)^j}{j!}\pi^{(2(k+l)-j-1)}a_k(j)c_l(2)\int_{-\infty}^{\infty} x^r g(x)G(x)^{2(k+l)-j-2}dx,\\ \rho_{r,q} &= \sum_{i,j,k,l}^{\infty} U_{i,j,k,l}\int_{-\infty}^{\infty} x^r(2(k+l)-j-1)g(x)G(x)^{2(k+l)-j-2}dx,\end{aligned} \tag{36}$$

where $U_{i,j,k,l} = \binom{-(s+2)}{i}((i+1)^j/j!)\pi^{(2(k+l)-j-1)} a_k(j)c_l(2)/2(k+l)-j-1$.

4.8. Generating Function. This section is devoted to introduce the moment generating function (mgf), and it can be written as follows:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty}\left(e^{tx}\right)f(x)dx = \int_{-\infty}^{\infty}\left(e^{tx}\right)\sum_{i,l=1}^{\infty}\sum_{J,k=0}^{\infty} v_{i,J,k,l} mG(x)^{(m-1)} g(x)dx. \tag{37}$$

4.9. Order Statistics. In this section, we will focus our attention on one of the most important properties which is the order statistics. We can easily write the form of I_{th} order statistic density function as follows:

$$f_{I:N}(x) = D\sum_{J=0}^{N-I}(-1)^J\binom{N-I}{J} f(x)F(x)^{J+I-1}, \tag{38}$$

such that $D = N!/[(I-1)!(N-I)!]$.Considering the mathematical advancements described in PWM, $f_{I:N}(x)$ becomes

$$f_{I:N}(x) = \sum_{m,N,p=0}^{\infty} W*_{J,l,m,N,p} h_{(2(p+N)-m-1)}(x), \tag{39}$$

such that $h_{(2(p+N)-m-1)}(x)$ is exp-G density possessing power parameter $(2(p+N)-m-1)$ for $J, l, m, N, p \geq 0$.

$$W*_{J,l,m,N,p} = \sum_{J,l=0}^{\infty} \frac{D(-1)^J \binom{N-I}{J}\binom{-(J+I+1)}{l}(l+1)^m \pi^{2(p+N)-(m+1)} a_p(2) d_n(m)}{m!(2(p+N)-(m+1))}, \tag{40}$$

and

$$h_{(2(p+N)-m-1)}(x) = (2(p+N)-m-1) g(x) G(x)^{(2(p+N)-m-1)-1}. \tag{41}$$

It should come as no surprise that the density of the LCG order statistics is a linear combination of exp-G densities; this is extremely evident, as revealed by (39), which is the main result to be demonstrated.

4.10. Entropy Measures. The Shannon entropy is defined as $\eta_X = \mathbb{E}\{-\log[f(X)]\}$, and for the LCG class, it may be formulated as the following equation:

$$\eta_X = E[-\log f(x)] = -\log \pi - E(\log g(x)) - E\left[\log\left(\csc^2(\pi G(x))\right)\right] - E[\cot(\pi G(x))] + 2E\left[\log\left(1 + e^{\cot(\pi G(x))}\right)\right]. \tag{42}$$

Proof. Alzaatreh et al. [16] deduced Shannon entropy of T-X family. Since here $W[G(x)] = [-\cot(\pi G(x))]$, adopting the same methodology, we get η_X as

$$\eta_X = \mathbb{E}\left[\log\left\{g\left[G^{-1}\left(e^{-e^T}\right)\right]\right\}\right] - \mathbb{E}\left(e^T\right) + \mu_T + \eta_T, \tag{43}$$

where μ_T is the mean of r.v. T. Using (43), we can easily prove the Shannon entropy of the LCG class given in (42) where T follows logistic distribution.

Rényi entropy is

$$I_\delta(f) = \frac{1}{1-\delta}\log[I(\delta)], \tag{44}$$

such that $I_\delta(f) = \int_{\mathfrak{R}} f^\delta(x)dx, \delta > 0$ and $\delta \neq 1$.

$$f^\delta(x) = \left[\pi g(x)\left(\csc^2(\pi G(x))\right)e^{\cot(\pi G(x))}\left(1 + e^{\cot(\pi G(x))}\right)^{-2}\right]^\delta. \tag{45}$$

We get the following result after performing the expansion by the aid of power series:

$$f^\delta(x) = \pi^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{-2\delta}{i}\frac{(\delta+i)^j}{j!}[g(x)]^\delta\left(\csc^2(\pi G(x))\right)^\delta \cot\ (\pi G(x))^j. \tag{46}$$

After incorporating the result, the Rényi entropy will reduce to

$$I_\delta(f) = \frac{1}{1-\delta}\log\ \int_{-\infty}^{\infty}\left[\pi^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{-2\delta}{i}\frac{(\delta+i)^j}{j!}[g(x)]^\delta\left(\csc^2(\pi G(x))\right)^\delta \cot\ (\pi G(x))^j\right]. \tag{47}$$

Now, we can write the final equation of the Rényi entropy as follows:

$$I_\delta(f) = \frac{1}{1-\delta}\log\left[\sum_{i,j=0}^{\infty} D_{i,j,\delta}\int_{-\infty}^{\infty} [g(x)]^\delta \left(\csc^2(\pi G(x))\right)^\delta \cot\,(\pi G(x))^j\right], \tag{48}$$

where $D_{i,j,\delta} = \pi^\delta \binom{-2\delta}{i}(\delta+i)^j/j!$. □

4.11. Some Transformations to Develop New G-Classes

(i) If the r.v $Y\sim$ standard logistic distribution, then $X = \cot^{-1}(Y)/\pi \sim$ logistic cotangent-G class of distributions.

(ii) If the r.v $Y\sim$ standard logistic distribution, then $X = (\cot^{-1}(Y)/\pi)^{(1/\alpha)} \sim$ logistic cotangent exponentiated-G class of distributions.

(iii) If the r.v $Y\sim$ standard logistic distribution, then $X = (\cot^{-1}(\log Y^\alpha)/\pi) \sim$ generalized logistic cotangent-G class of distributions.

(iv) If the r.v $Y = (u^{1/\alpha}-1)\sim$ standard logistic distribution, then $X = (\cot^{-1}(\log Y)/\pi) \sim$ generalized logistic cotangent exponentiated-G class of distributions.

5. LCW Distribution

In this section, a two-parameter special model logistic cot Weibull (LCW) with its properties is presented.

5.1. Methodology. Taking $G(x)$, as the cdf of Weibull distribution while $g(x)$ as the corresponding density, it follows the two-parameter LCWm we can write three of cdf, and its corresponding pdf and associated with its hazard function, as formulated below respectively, are:

$$\begin{aligned}
F(x) &= \left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}, x>0,\\
f(x) &= \pi\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}\csc^2\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-2},\\
h(x) &= \pi\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}\csc^2\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}.
\end{aligned} \tag{49}$$

The graphs represented in Figure 5 demonstrate the plots of the LCW.

5.2. Reliability Functions. We have

$$\begin{aligned}
S(x) &= 1-F(x) = 1-\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}, x>0.\\
h(x) &= \pi\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}\csc^2\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1},\\
r(x) &= \frac{\pi\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}\csc^2\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-2}}{\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}},\\
H(x) &= -\ln[S(x)] = \ln\left(1-\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}\right)^{-1},\\
m(x) &= \frac{1-\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-1}}{\pi\lambda\alpha x^{\alpha-1}e^{-\lambda x^\alpha}\csc^2\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\left[1+e^{\cot\left(\pi\left(1-e^{-\lambda x^\alpha}\right)\right)}\right]^{-2}}.
\end{aligned} \tag{50}$$

5.3. Quantile Function. This section is devoted to demonstrate the quantile function equation of LCW which is

$$Q(u) = Q_G(v), \tag{51}$$

where $Q_G(v) = G^{-1}(v)$ is the parent qf and $v = [(-1/\lambda)\ln(1+(1/\pi)\cot^{-1}(\ln((1-u)/u)))]^{-\alpha}$.

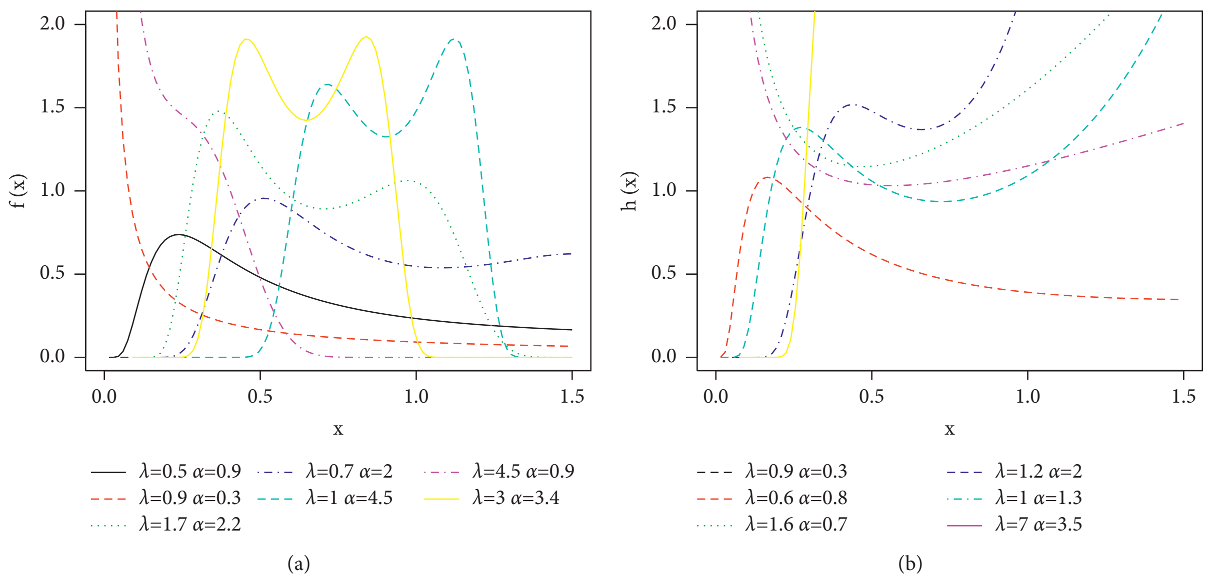


FIGURE 5: Graphical representation for the plots of the LCW: (a) density and (b) hazard rate for some parametric values.

5.4. Residual and Reverse Residual Life. The residual life has several uses in probability and statistics and risk assessment. Suppose that X represents a unit's lifespan and $X \geq 0$ with $P(X = 1)$; then, the r.v. $X_t = (t - X|X \leq t)$, for a fixed $t > 0$, is known as time since failure. The residual lifetime of LCW r.v. X is denoted by $R_t(x)$ and is defined as

$$R_t(x) = \frac{e^{\cot\left(\pi\left(1-e^{-\lambda(x+t)^{\alpha}}\right)\right)}\left(1 + e^{\cot\left(\pi\left(1-e^{-\lambda x^{\alpha}}\right)\right)}\right)}{e^{\cot\left(\pi\left(1-e^{-\lambda x^{\alpha}}\right)\right)}\left(1 + e^{\cot\left(\pi\left(1-e^{-\lambda(x+t)^{\alpha}}\right)\right)}\right)}. \tag{52}$$

Additionally, the reversed hazard rate function $\overline{R}_t(x)$ is written as

$$\overline{R}_t(x) = \frac{e^{\cot\left(\pi\left(1-e^{-\lambda(x-t)^{\alpha}}\right)\right)}\left(1 + e^{\cot\left(\pi\left(1-e^{-\lambda x^{\alpha}}\right)\right)}\right)}{e^{\cot\left(\pi\left(1-e^{-\lambda x^{\alpha}}\right)\right)}\left(1 + e^{\cot\left(\pi\left(1-e^{-\lambda(x-t)^{\alpha}}\right)\right)}\right)}. \tag{53}$$

5.5. Stochastic Ordering. Stochastic ordering is a tool for analysing the constitutional properties of stochastic structures that are intricate. There are several forms of stochastic orderings that can be used to sort random variables according to their distinguished properties. Suppose S and K are independent random variables with cdfs $F(S)$ and $F(K)$, respectively; then, S is said to be smaller than K iff it satisfies the following.

(i) Stochastic order ($S \leq_{st} (K)$) if $F_S(S) \geq F_K(S)$ for all S.
(ii) Hazard rate order ($S \leq_{hr} (K)$) if $h_s(S) \geq h_K(S)$ for all S.
(iii) Mean residual life order ($S \leq_{mrl} (K)$) if $m_s(S) \geq m_k(S)$ for all S.
(iv) Likelihood ratio order ($S \leq_{lr} (K)$) if $f_X(S)/f_y(S)$ decrease in S.

The LCW distribution (λ, α) is ordered according to the strongest "likelihood ratio" ordering, as demonstrated in the following theorem. The versatility of two-parameter LCW distribution (λ, α) is demonstrated. Let S follow LCW (λ_1, α_1) and K follow LCW (λ_2, α_2). Then, the likelihood ratio is

$$\frac{f_S(S)}{f_K(S)} = \frac{\lambda_1\alpha_1\pi_1\left(-1/\lambda_1 \log\left(1 - w_1/\pi_1\right)\right)^{1-1/\alpha_1}}{\lambda_2\alpha_2\pi_2\left(-1/\lambda_2 \log\left(1 - w_2/\pi_2\right)\right)^{1-1/\alpha_2}}, \tag{54}$$

where $w_1 = \pi_1\left(1 - e^{-\lambda_1 S^{\alpha_1}}\right)$ and $w_2 = \pi_2\left(1 - e^{-\lambda_2 S^{\alpha_2}}\right)$. Again,

$$\frac{d}{dx}\log\frac{f_S(S)}{f_K(S)} = \frac{-(\alpha_1 - \alpha_2)\left[\log\left(\pi_1\left(e^{-\lambda_1 S^{\alpha_1}}\right)/\pi_1\right)\right]^{-1}\left[\log\left(\pi_2\left(e^{-\lambda_2 S^{\alpha_2}}\right)/\pi_2\right)\right]^{-1}}{\alpha_1\alpha_2\pi_1\left(e^{-\lambda_1 S^{\alpha_1}}\right)\pi_2\left(e^{-\lambda_2 S^{\alpha_2}}\right)}. \tag{55}$$

If $\lambda_1 = \lambda_2 = \lambda$ and $\alpha_1 \geq \alpha_2$, then $(d/ds)[(\log f_s(S)/f_K(S))] < 0$, and hence $S \leq_{lr} (K)$, $S \leq_{hr} (S)$, $S \leq_{mrl} (Y)$ and $S \leq_{st} (K)$.

5.6. Stress-Strength Reliability. In this section, we will introduce and define one of the most important properties of any distribution which is the reliability function R; this function may be represented as

$$R = P(X > K) = \int_{-\infty}^{\infty} F_2(x) f_1(x) dx. \tag{56}$$

Suppose both X and K are LCW independent random variables with parameters α_1, λ_1 and α_2, λ_2 and fixed scale parameter σ. Then,

$$R = \int_0^{\infty} \pi_1 \lambda_1 \alpha_1 x^{\alpha_1 - 1} e^{-\lambda_1 x^{\alpha_1}} \csc^2\left(\pi_1 \lambda_1 \alpha_1 x^{\alpha_1 - 1} e^{-\lambda_1 x^{\alpha_1}}\right) e^{\cot\left(\pi_1\left(\lambda_1 \alpha_1 x^{\alpha_1 - 1} e^{-\lambda_1 x^{\alpha_1}}\right)\right)} \left(1 + e^{\cot\left(\pi_1\left(1 - e^{-\lambda_1 x^{\alpha_1}}\right)\right)}\right)^{-2} \left(1 + e^{\cot\left(\pi_2\left(1 - e^{-\lambda_2 x^{\alpha_2}}\right)\right)}\right)^{-1} dx. \tag{57}$$

After applying the binomial and exponent series expansion, then substituting $u_1 = \pi_1 \lambda_1 \alpha_1 x^{\alpha_1 - 1} e^{-\lambda_1 x^{\alpha_1}}$ and $u_2 = \pi_2 \lambda_2 \alpha_2 x^{\alpha_2 - 1} e^{-\lambda_2 x^{\alpha_2}}$, the above equation reduces to

$$R = \sum_{i=0}^{\infty} \sum_{j=0}^{-2} \int_0^{\infty} \pi_1 \lambda_1^{i+1} \alpha_1 (-1)^i i!^{(-1)} \binom{-2}{j} \left(\frac{-1}{\lambda_1} \log\left(1 - \frac{u_1}{\pi_1}\right)\right)^i \csc^2(u_1) e^{(j+1)\cot(u_1)} \left(1 + e^{(j+1)\cot(u_1)}\right)^{-1} du. \tag{58}$$

Solving complicated integration is very hard but with the aid of super computers and advanced mathematical software, we easily find the value of the hard integral introduced above (Table 2).

5.7. Submodels of LCW. *5.8. Estimation.* One of the most famous estimators is the maximum likelihood equation as we can easily obtain and write the log-likelihood function regarding the distribution's parameters of the distribution under consideration by getting the logarithms function for the likelihood function $\Theta = (\alpha, \lambda)^{\top}$. So as a result, we may formulate the log-likelihood function as follows:

$$\ell = n \log(\alpha\lambda\pi) - (\alpha - 1) \sum_{i=1}^{n} \log x - \lambda \sum_{i=1}^{n} x^{\alpha} + \sum_{i=1}^{n} \log\left(\csc^2\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)\right) + \sum_{i=1}^{n} \cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right) - 2 \sum_{i=1}^{n} \log\left(1 + e^{\cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)}\right). \tag{59}$$

Consider the following formulae to be the score vector's components $U(\Theta)$, or in other words the derivative of the vector with respect to the two parameters:

$$U_{\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x - \lambda \sum_{i=1}^{n} x^{\alpha} \log \alpha - 2\pi\lambda\alpha \sum_{i=1}^{n} x^{(\alpha-1)} e^{-\lambda x^{\alpha}} \cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right) + \pi\lambda\alpha \sum_{i=1}^{n} x^{(\alpha-1)} e^{-\lambda x^{\alpha}} \csc^2\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right) - 2\pi\lambda\alpha \sum_{i=1}^{n} x^{(\alpha-1)} e^{-\lambda x^{\alpha}} \csc^2\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)\left[e^{\cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)}\right]\left[1 + e^{\cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)}\right]^{-1}, \tag{60}$$

$$U_{\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x^{\alpha} - 2\pi x^{\alpha} e^{-\lambda x^{\alpha}} + \pi x^{\alpha} e^{-\lambda x^{\alpha}} \csc^2\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right) - 2\pi x^{\alpha} e^{-\lambda x^{\alpha}} \csc^2\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right) e^{\cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)}\left[1 + e^{\cot\left(\pi\left(1 - e^{-\lambda x^{\alpha}}\right)\right)}\right]^{-1}. \tag{61}$$

The MLEs may be derived by putting the equations in the previous sentence to zero and solving them concurrently (see, for instance, [18]).

6. Results Deduced from the Simulation Work

In this phase of the study, we employed Monte Carlo simulation to assess the distribution's effectiveness all across the estimation procedure. The MLEs of the model

Table 2: Submodels of LCW.

Serial No.	α	Submodel of LCW	Remarks about model
1.	1	Logistic cot exponential (LCE) distribution	New in literature
2.	2	Logistic cot Rayleigh (LCR) distribution	New in literature

Table 3: MSEs, biases, lower bounds, and upper bounds.

Parameter	n	Biases	MSEs	Lower bounds	Upper bounds
I					
α	30	0.063	0.004	0.138	0.189
	50	0.059	0.004	0.139	0.179
	100	0.057	0.003	0.143	0.171
	200	0.056	0.003	0.146	0.165
	500	0.055	0.003	0.149	0.161
	1000	0.055	0.003	0.151	0.159
λ	30	1.448	2.294	1.546	2.350
	50	1.392	2.024	1.590	2.195
	100	1.342	1.824	1.633	2.051
	200	1.323	1.759	1.676	1.970
	500	1.318	1.740	1.725	1.910
	1000	1.316	1.733	1.750	1.881
II					
α	30	0.320	0.111	0.690	0.950
	50	0.296	0.092	0.698	0.895
	100	0.282	0.081	0.714	0.850
	200	0.280	0.079	0.732	0.828
	500	0.276	0.076	0.746	0.806
	1000	0.276	0.076	0.754	0.797
λ	30	1.467	2.364	1.559	2.374
	50	1.394	2.021	1.590	2.198
	100	1.335	1.804	1.627	2.043
	200	1.327	1.771	1.680	1.974
	500	1.319	1.743	1.726	1.911
	1000	1.315	1.731	1.750	1.881

Table 4: Monte Carlo simulation results: biases, MSEs, lower bounds, and upper bounds.

Parameter	n	Biases	MSEs	Lower bounds	Upper bounds
III					
α	30	0.032	0.001	0.069	0.095
	50	0.030	0.001	0.070	0.089
	100	0.029	0.001	0.072	0.085
	200	0.028	0.001	0.073	0.083
	500	0.028	0.001	0.075	0.081
	1000	0.027	0.001	0.075	0.080
λ	30	5.323	34.234	4.391	8.256
	50	4.738	23.996	4.456	7.021
	100	4.470	20.366	4.626	6.315
	200	4.370	19.256	4.789	5.952
	500	4.330	18.796	4.965	5.694
	1000	4.316	18.654	5.060	5.573
IV					
α	30	0.006	0.000	0.014	0.019
	50	0.006	0.000	0.014	0.018
	100	0.006	0.000	0.014	0.017
	200	0.005	0.000	0.015	0.016
	500	0.005	0.000	0.015	0.016
	1000	0.005	0.000	0.015	0.016
λ	30	7.283	63.868	5.585	11.380
	50	6.404	44.189	5.708	9.500
	100	6.006	36.791	5.972	8.441
	200	5.913	35.269	6.255	7.971
	500	5.855	34.396	6.518	7.592
	1000	5.847	34.245	6.668	7.426

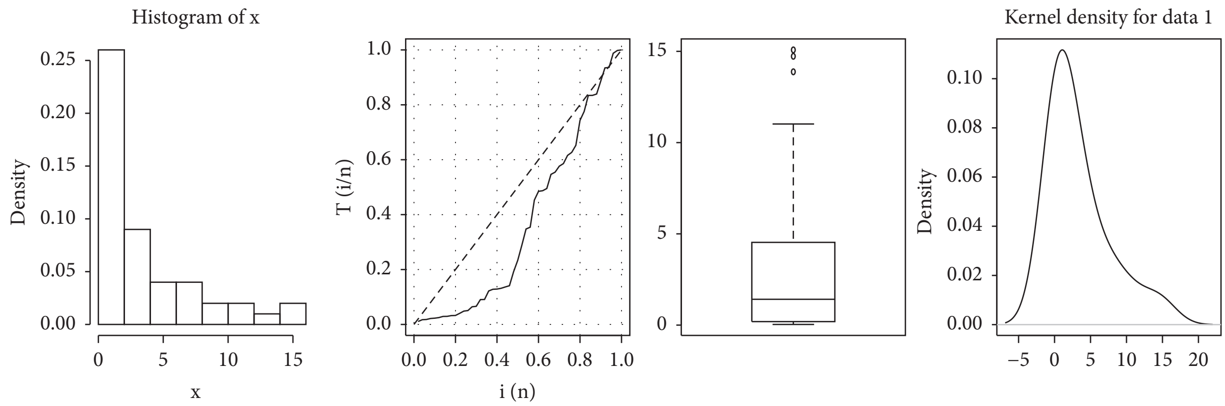


Figure 6: Histogram, TTT plot, box plot, and kernel density for failure time data.

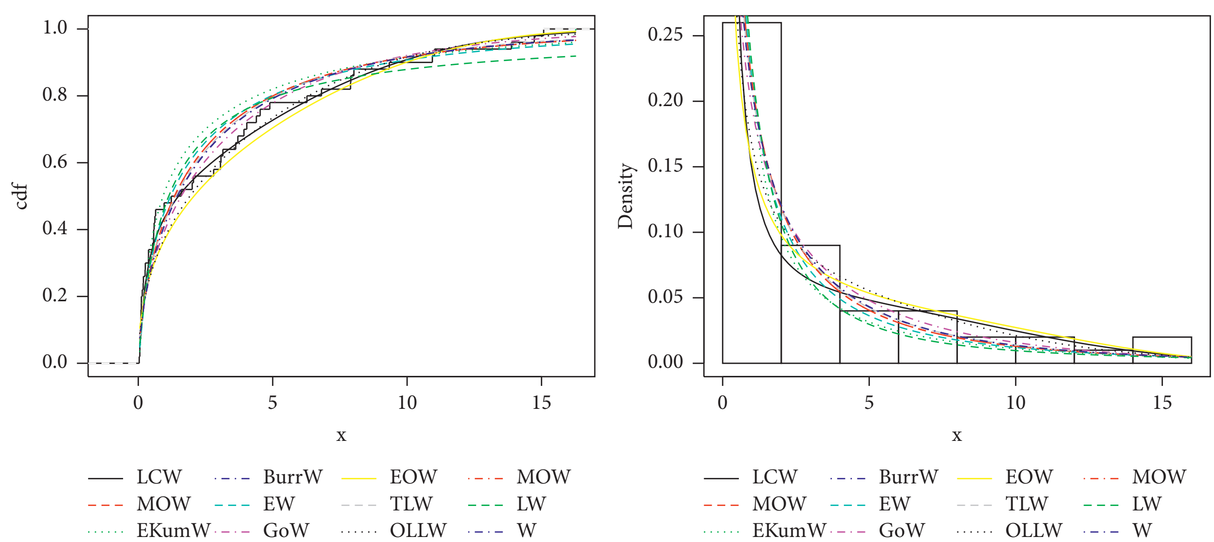


Figure 7: The comparative cdf and pdf of LCW and other models using failure time data.

parameters of the LCW distribution are evaluated. For each sample size $n = 30$, $50, 100, 200, 500, 1000$, this simulation study is repeated 1,000 times. The parametric values are (I) $\alpha = 0.1$, $\lambda = 0.5$, (II) $\alpha = 0.5$, $\lambda = 0.5$, (III): $\alpha = 0.05$, $\lambda = 1.0$, and (IV) $\alpha = 0.01$, $\lambda = 1.2$. The outputs related to the biases of the MLEs, mean square errors (MSEs), lower bounds, and upper bounds are displayed in Tables 3 and 4. As the sample size n increases, in general, the biases, MSEs, L.bounds, and U.bounds of X decrease while the CPs of the confidence intervals are quite close to the 95% nominal levels which indicate that the MLEs have good performance for estimating the parameters of the LCW distribution; also, we will use the conducted results to find the upper and lower bounds for the estimates for the parameters of the model. Tables 3 and 4 contain a summary of all simulation outcomes.

7. Applications and Data Analysis

Two different applications to actual datasets were shown by us in order to demonstrate the utility of the distribution that was suggested (LCW). The criteria of goodness of fit proved that it can be used in place of famous two, three, and four-parameter models and many others. All calculations are performed using the *R* script Adequacy Model. Moreover, the proposed LCW is compared with McDonald Weibull (McW), Exponentiated Kumaraswamy Weibull (EKumW), Burr Weibull (BurrW), Beta Weibull (BW), Gompertz Weibull (GoW), Exponentiated Odd Weibull (EOW), Topp-Leone Weibull (TLW), Odd Log Logistic Weibull (OLLW), Marshall Olkin Weibull (MOW), and many other Weibull based models.

7.1. Application 1: Failure Time Data. The first data correspond to the failure time of 50 components (per 1000 h) which were collected from Murthy et al. [19]. The dataset can be found easily in [19]. We avoid adding the data in the paper as they can be easily accessed. We have provided some statistics on the data used to make the reader comfortable in reading the paper. The summary statistics for this dataset are as follows: $n = 50$, $median = 1.4140$, $\overline{x} = 3.3422$, $s = 4.18$, $Q_1 = 0.2075$, $Q_3 = 4.4988$, skewness = 1.38, and kurtosis = 0.92.

Table 5: MLEs and their standard errors (in parentheses) for failure time data.

Distribution	δ	α	β	λ	θ
LCW	0.5515 (0.0510)	— —	0.5669 (0.0342)	— —	— —
McW	1.7077 (0.0269)	0.1098 (0.0304)	1.1321 (0.0029)	0.0829 (0.0144)	5.2260 (1.6100)
EKumW	4.8184 (0.2392)	7.9489 (8.3781)	0.7365 (0.0001)	0.0624 (0.0279)	0.3578 (0.2505)
BurrW	0.2564 (3.5884)	56.4086 (161.976)	0.1954 (2.0084)	3.4080 (35.0251)	— —
BW	7.8468 (0.0298)	0.0996 (0.0158)	0.5128 (0.0249)	3.6685 (1.6635)	— —
GoW	0.5845 (0.1166)	— —	0.4439 (17.7316)	1.0618 (42.4056)	0.3756 (15.0100)
EOW	1.3143 (1.1942)	0.8260 (0.6978)	0.0920 (0.2778)	— —	0.4360 (0.2273)
TLW	0.8427 (1.2441)	0.0707 (0.0979)	0.4825 (0.4253)	— —	22.5648 (35.2538)
OLLW	0.3300 (0.1721)	0.5287 (0.1784)	1.1378 (0.3284)	— —	— —
MOW	0.3805 (0.2614)	0.6038 (0.5527)	0.7274 (0.1395)	— —	— —
LW	0.8904 (1.5009)	— —	0.9336 (13.3937)	0.9755 (13.9952)	— —
W	0.5412 (0.0994)	— —	0.6612 (0.0747)	— —	— —

Table 6: $\hat{\ell}$, AIC, CAIC, BIC, HQIC, A^*, W^*, K-S, and P values for failure time data.

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P value
LCW	94.8185	193.6372	193.8925	197.4612	195.0934	0.2433	0.0329	0.0851	0.8316
McW	99.1402	208.2805	209.6441	217.8406	211.9210	0.5940	0.0821	0.1174	0.4604
EKumW	100.6444	211.2888	212.6525	220.8489	214.9294	0.9032	0.1458	0.1616	0.1312
BurrW	102.4136	212.8272	213.7161	220.4753	215.7396	0.9604	0.1533	0.1263	0.3706
BW	100.9767	209.9534	210.8423	217.6015	212.8659	0.8871	0.1433	0.1266	0.3684
GoW	101.9336	211.8671	212.7560	219.5152	214.7796	0.9046	0.1415	0.1374	0.2754
EOW	101.6597	211.3194	212.2083	218.9675	214.2318	0.8983	0.1278	0.1494	0.1934
TLW	101.7022	211.4044	212.2933	219.0525	214.3168	0.8963	0.1429	0.1187	0.4473
OLLW	101.0825	208.1650	208.6867	213.9010	210.3493	0.8184	0.1204	0.1491	0.1953
MOW	102.2050	210.4099	210.9317	216.1460	212.5942	0.9519	0.1526	0.1146	0.4917
LW	105.4720	216.9440	217.4658	222.6801	219.1284	1.3274	0.2137	0.1354	0.2907
W	102.3533	208.7066	208.9619	212.5306	210.1628	0.9537	0.1521	0.1269	0.3649

We can easily recognize that Figure 6 represents the graphical representation of the histogram, TTT plot, box plot, and kernel density for failure time data. Figure 7 represents the comparative cdf and pdf of LCW and other models using failure time data.

Table 5 provides the MLEs of the parameters while Table 6 provides the values of AIC, CAIC, BIC, HQIC, A^*, W^*, K-S, and P values for each model. On the basis of the statistics given in these tables, the best fit model is LCW and has the potential to fit right-skewed data with the increasing failure rate.

7.2. Application 2: Survival Time Data. The upcoming data are for the mortality periods, measured in weeks, of 33 individuals with acute myelogenous leukemia which were used by Feigl and Zelen [20]. The dataset is given below. 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 8, 4, 3, 30, 4, 43.

The summary statistics for this data set are as follows: $n = 33$, $me\ di\ an = 22.00$, $\overline{x} = 42.06$, $s = 46.94$, $Q_1 = 4.0$, $Q_3 = 65.00$, skewness = 1.07, and kurtosis = -0.16. Figure 8 represents the histogram, TTT plot, box plot, and kernel density for survival time data. Figure 9 represents the comparative cdf and pdf of LCW and other models using survival time data.

Table 7 provides the MLEs of the parameters, and Table 8 contains the values of AIC, CAIC, BIC, HQIC, A^*, W^*, K-S,

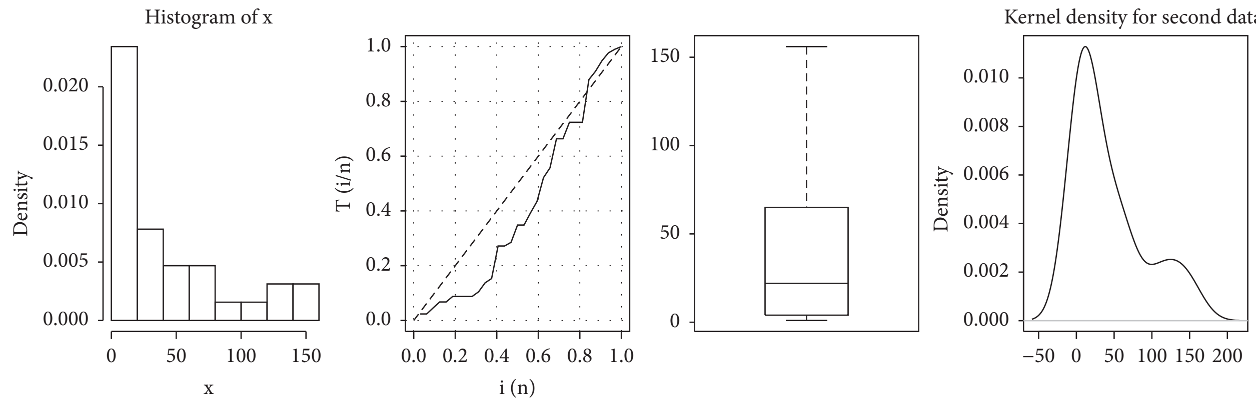


Figure 8: Histogram, TTT plot, box plot, and kernel density for survival time data.

and P values for each model. On the basis of the statistics presented in these tables, the best fitted model is LCW and

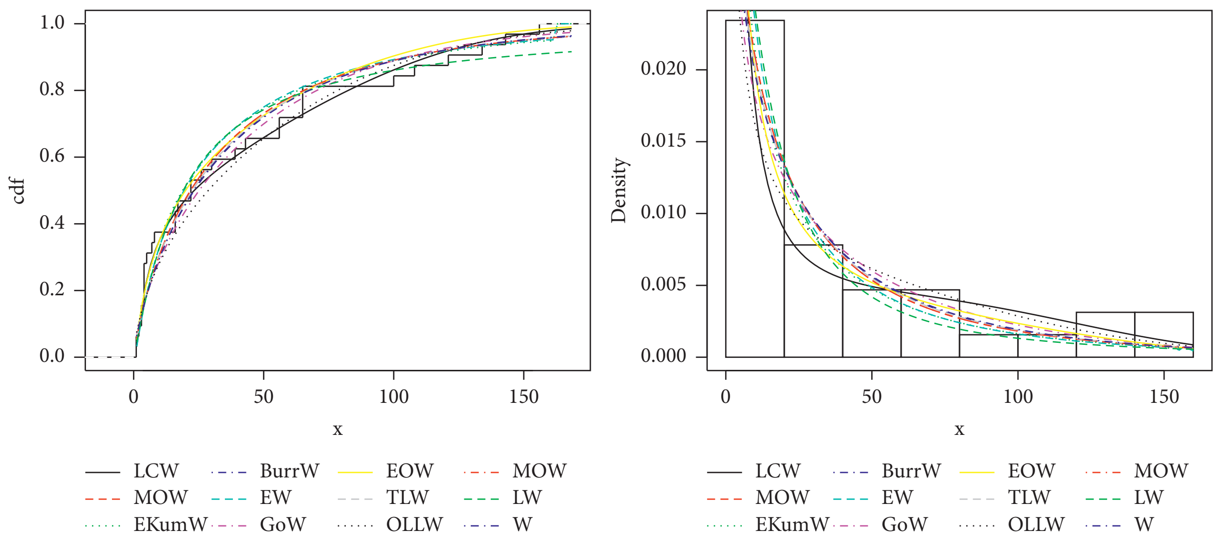


Figure 9: The comparative cdf and pdf of LCW and other models using survival time data.

Table 7: MLEs and their standard errors (in parentheses) for survival time data.

Distribution	δ	α	β	λ	θ
LCW	0.0845 (0.0201)	— —	0.6684 (0.0513)	— —	— —
McW	0.8656 (0.0025)	0.3334 (0.1019)	0.7401 (0.0024)	0.0840 (0.0160)	10.8047 (0.1870)
EKumW	1.3657 (0.0057)	2.5405 (0.1880)	0.6500 (0.0027)	0.0827 (0.0188)	0.9868 (0.2236)
BurrW	0.2303 (5.4000)	26.5926 (66.0689)	0.1917 (3.0635)	4.1946 (67.0145)	— —
BW	1.5107 (0.0058)	0.0866 (0.5643)	0.6303 (0.0022)	4.0026 (2.2471)	— —
GoW	0.7010 (0.1759)	— —	0.1105 (3.2008)	0.5851 (16.9612)	0.2048 (5.9728)
EOW	0.4654 (0.01367)	0.0916 (0.0164)	6.6955 (0.0037)	— —	5.2082 (0.0037)
TLW	0.0072 (0.0083)	3.1717 (7.1664)	0.9891 (0.2090)	— —	0.3050 (0.4779)

Table 7: Continued.

Distribution	δ	α	β	λ	θ
OLLW	0.0090	0.5640	1.2916	—	—
	(0.0113)	(0.1806)	(0.3135)	—	—
MOW	0.0319	0.6025	0.8728	—	—
	(0.0439)	(0.6717)	(0.2050)	—	—
LW	0.7210	—	0.1109	9.8869	—
	(1.7845)	—	(0.8397)	(74.8148)	—
W	0.0845	—	0.6684	—	—
	(0.0201)	—	(0.0513)	—	—

Table 8: $\widehat{\ell}$, AIC, CAIC, BIC, HQIC, A^*, W^*, K-S, and P values for survival time data.

Distribution	$\widehat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K-S	P value
LCW	146.0607	296.1215	296.5353	299.5300	297.0932	0.2228	0.0304	0.0910	0.9534
McW	148.3388	306.6776	308.9853	314.0063	309.1069	0.4105	0.0600	0.1165	0.7776
EKumW	149.4336	308.8673	311.1750	316.1959	311.2965	0.4994	0.0736	0.1163	0.7790
BurrW	150.2305	308.4611	309.9425	314.3240	310.4045	0.5570	0.0796	0.1275	0.6752
BW	149.0827	306.1654	307.6469	312.0284	308.1088	0.4860	0.0736	0.1283	0.6676
GoW	149.8920	307.7839	309.2654	313.6468	309.7273	0.5504	0.0775	0.1255	0.6943
EOW	147.0092	302.0183	303.4998	307.8813	303.9617	0.3088	0.0436	0.0983	0.9163
TLW	149.9365	307.8730	309.3545	313.7360	309.8164	0.5342	0.0765	0.1242	0.7064
OLLW	149.4551	304.9103	305.7674	309.3075	306.3678	0.5393	0.0753	0.1251	0.6983
MOW	150.0520	306.1041	306.9612	310.5013	307.5616	0.5446	0.0785	0.1252	0.6974
LW	152.1233	310.2465	311.1037	314.6437	311.7041	0.7593	0.1183	0.1284	0.6667
W	150.1510	304.3020	304.7158	307.2335	305.2737	0.5522	0.0787	0.1272	0.6784

has the potential to fit right-skewed data with increasing failure rate.

8. Concluding Remarks

A novel logistic-G family of distributions is developed, which employs trigonometric and algebraic generalizers based on cotangent functions. This class has been shown to be more adaptable and useful in a variety of practical applications, particularly survival, dependability, and failure modelling. Furthermore, a two-parameter model (LCW) with various density shapes, as well as hazard rate different shapes, is developed. This work also derives and presents many statistical and mathematical properties of the proposed family.

In parametric estimating, the maximum likelihood method is used, and a Monte Carlo simulation analysis is used to determine whether or not the estimates are suitable. To ascertain which distribution is most suitable for modelling the real datasets, we employ a number of goodness-of-fit measures that decide which one is the superior one among all its competitors. We show that, even with a higher number of parameters, this suggested distribution consistently delivers superior fits than other existing and competing Weibull models. We believe that the suggested class and related models would find wider applicability in sectors such as dependability and survival studies, hydrology, geology, and others.

9. Future Work

In the upcoming work, we will apply the proposed distribution and the new family of distribution to censored sample scheme. We will try different kinds of censoring schemes like type-I and type-II censored sample and we will generate random censored samples from the new distribution. We can extend our work to apply the proposed model to accelerated life test with different types such as constant and partially constant and maybe progressive stress accelerated life tests. At last, we will use different optimality criteria to the censored samples generated from the proposed model.

Data Availability

The data that were utilised to support the conclusions of this research may be found inside the paper itself.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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