Review Article

Study of Metric Space and Its Variants

Surjeet Singh Chauhan (Gonder) 1, Prachi Garg 1, and Kamleshwar Thakur 2

1 Department of Mathematics, UIS, Chandigarh University, Punjab, India
2 Department of Electronics and Communication, Eastern College of Engineering, Purbanchal University, Biratnagar, Nepal

Correspondence should be addressed to Kamleshwar Thakur; kamleshwar.thakur@gmail.com

Received 12 April 2022; Accepted 30 July 2022; Published 27 October 2022

The objective of this paper is to present a comparative study of metric space and its variants. This study will provide the structure, gap analysis, and application of metric space and its variants from 1906 to 2021.

1. Introduction

Metric space plays an important role to solve various mathematical problems. A wide range of metric spaces offers a powerful method to learn the optimization and approximation theory, variational inequalities, and several other problems. The word metric originated from the word meter (measure—a class of functions, which consider as generalizing the notion of distance between each pair of elements) [1–4]. The French mathematician Frechet first considered these functions, to generalize the notion of distances and apply them to arbitrary sets. He introduced the definition of metric on a set in his doctoral dissertation, “Less Espaces Abstrait” [5].

A function \( d: X^2 \rightarrow \mathbb{R} \) (where \( X \neq \{\emptyset\} \)) is s.t.b. a distance function (or metric) on \( X \), if it meets the subsequent requirements:

1. \( d(N''_1, N''_2) \geq 0 \) if \( N''_1 \neq N''_2 \) and \( N''_1 = N''_2 \Leftrightarrow d(N''_1, N''_2) = 0 \),
2. \( d(N''_1, N''_2) = d(N''_2, N''_1) \),
3. \( d(N''_1, N''_2) + d(N''_2, N''_3) \geq d(N''_1, N''_3) \), for all \( N''_1, N''_2, N''_3 \in X \).

So the function “d” composed with a set “\( X \)” is titled as metric space, which is denoted by \( (X, d) \).

Example 1. Let \( X = \mathbb{R} \) with \( d(N''_1, N''_2) = |N''_1 - N''_2| \), for all \( N''_1, N''_2 \in X \).

Now it is easy to get that \( (X, d) \) is a metric space.

Example 2. Let \( X = \mathbb{R}^2 \) with \( d((N_1', N_2'), (N_1'', N_2'')) = \sqrt{(N_1' - N_1'')^2 + (N_2' - N_2'')^2} \), for all \( (N_1', N_2'), (N_1'', N_2'') \in X \).

Now it is easy to get that \( (X, d) \) is a metric space.

1.1. Picard’s Convergence Theorem [2]. For nonlinear equations, Picard proved the theorem for showing the existence of the solutions.

Theorem 1. Let \( \psi: [u, v] \rightarrow \mathbb{R} \) and \( \psi: (u, v) \rightarrow \frac{d}{dt} f(R) \). If \( \exists \mu < 1 \), such that

\[
|\psi'(x)| \leq \mu, \forall x \in (u, v).
\]

Then, the sequence \( \{\gamma_n''\} \) in \( (u, v) \) is defined by

\[
\gamma_{n+1}'' = \psi(\gamma_n''),
\]

\( \forall n \geq 0 \) converges to a solution of the equation \( \psi(\gamma) = \gamma'' \).

The iterative sequence \( \{\gamma_n''\} \) is termed as Picard’s iterative sequence.

1.2. Banach’s Fixed-Point Theorem [2]. The existence of a solution for the integral equation was established by Banach in 1922, which is known as Banach’s fixed-point theorem.
Theorem 2. Let \((X, d)\) be a complete metric space, and \(T: X \rightarrow X\) be a contraction mapping (i.e., \(\exists \theta \in [0, 1)\) such that \(d(Tx, Ty) \leq \theta d(x, y), \forall x, y \in X\)). Then, the following axioms hold:

(i) \(T\) has a unique fixed point \(z\) in \(X\);

(ii) \(\forall z'' \in X\), the sequence \(\{z^n\}\) is defined by \(z^1 = z, z^{n+1} = T(z^n)\) for \(n \in \mathbb{N}\);

(iii) \(\forall n \geq 0\) converges to the fixed point \(z\) of \(T\) (i.e., \(Tz = z\)).

Metric space is used in various fields such as real-life situations, convergence, and quantum mechanics. With the help of metric space, those functions that satisfy metric space properties can be used to determine the distance between two points, like in quantum mechanics, that is, the conservation laws naturally lead to metric spaces, which are related to the set of physical quantities. The onion-shell geometry is used in such all metric spaces where max and min distances between the states can geometrically interpret. Also, convergence is depending on the choice of metric but not an inherent function of real numbers [1–4].

2. Extensions of Metric Space to Other Spaces

Recently, several authors have incorporated some generalization/extension of metric space in various ways and prepared a comparative study of metric space and its variants to explore the gap analysis and their applications, starting from 1906 to 2020.

2.1. Period-I (1906–1950)

2.1.1. Quasi-Metric Spaces [6,7]. Wilson [6] introduced the metric space without symmetric condition, which is termed as quasi-metric if it satisfies the following conditions:

\[
\begin{align*}
(1) & \quad d(\gamma_1, \gamma_2) = 0, \forall \gamma_1 \in X; \\
(2) & \quad d(\gamma_1, \gamma_2) = 0 \Rightarrow \gamma_1 = \gamma_2; \\
(3) & \quad d(\gamma_1, \gamma_2) \leq d(\gamma_1, \gamma_3) + d(\gamma_3, \gamma_2), \quad \forall \gamma_1, \gamma_2, \gamma_3 \in X.
\end{align*}
\]

Then, quasi-metric \(d\) together with set \(X\) is titled as quasi-metric space and represented by \((X, d)\).

Example 3. Let \(\mathbb{R}\) be set of all real numbers and \(p > 0\). Let \(d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} + [0]\) be a mapping, which is defined as follows:

\[
d(\gamma_1, \gamma_2) = \begin{cases} 
\gamma_2 - \gamma_1, & \text{if } \gamma_2 \geq \gamma_1, \\
p(\gamma_2 - \gamma_1), & \text{if } \gamma_2 < \gamma_1.
\end{cases}
\]

It is easy to get that \(d\) is a quasi-metric on \(\mathbb{R}\).

Remark 1.

(1) Nearly all “fixed-point theorems” used for "\(\Psi_f\) contraction” in quasi-metric space (q.m.s) have been proved.

(2) Quasi-Metric Space is applied for proving Hamilton–Jacobi equation’s existence and uniqueness, iterated function systems, fractal theory, shape-memory alloys, in the rate-independent models for plasticity, etc [7].

A mapping \(T: X \rightarrow X\) is called as the following:

(1) “Forward \(\Psi_f\) contraction (resp. backward \(\Psi_f\) contraction)” if quasi-metric space \((X, d)\) and functions \(F \in \mathcal{F}\) and \(\psi \in \Psi\) satisfy the following conditions,

\[
T \gamma_1 \neq T \gamma_2 \Rightarrow F(d(T \gamma_1, T \gamma_2)) \leq \psi d(\gamma_1, \gamma_2),
\]

(resp. \(T \gamma_1 \neq T \gamma_2 \Rightarrow F(d(T \gamma_1, T \gamma_2)) \leq \psi d(\gamma_2, \gamma_1)\)).

(2) “Forward Picard operator (backward Picard operator)” if quasi-metric space \((X, d)\) has a unique fixed point \(\xi \in X\) and \(T \gamma_1 \rightarrow \xi\) for every \(\gamma_1 \in X\).

2.1.2. Probabilistic Metric Space [8–10]. The generalized probabilistic theory was proposed by Menger, which played a significant part in the development of metric space. In particular, he suggested replacing the number \(d(p, q)\) with the real function \(F_{pq}\) whose function is \(F_{pq}''\), for any real number \(\gamma_1\). It is defined as the probability that is the distance between \(p\) and \(q\) is less than \(\gamma_1\).

A function \(F: X \times X \rightarrow [0, 1]\), where \(X \neq \emptyset\), is s.t.b. probabilistic metric space, if met the following requirements:

\[
\begin{align*}
F_{ab}(\gamma_1) & = 1, \forall \gamma_1 > 0 \Rightarrow a = b \quad \forall a, b \in X; \\
F_{ab}(\gamma_1) & = F_{ba}(\gamma_1), \quad \forall a, b \in X; \\
F_{ac}(\gamma_1) & = 1, F_{bc}(\gamma_1) = 1; \\
F_{ab}(\gamma_1) & = 1, \forall a, b, c \in X \text{ and } d(\gamma_1, \gamma_2) \in (0, \infty).
\end{align*}
\]

Then, probabilistic metric space is represented as \((X, F)\).

Remark 2. With the help of probabilistic metric space, the existence of the random fixed-point theorem was solved. Also, constrained file migration and metrical task systems use a randomized algorithm that is based on the probabilistic metric approximation.

A mapping \(F: X \rightarrow X\) is titled as follows:

\[
\begin{align*}
\text{1) } B\text{-contraction if } (X, F) & \text{ be a probabilistic metric space and } \exists \theta \in (0, 1) \text{ such that } \forall \gamma_1, \gamma_2 \in X \text{ and } \theta > 0 \text{ satisfy the following implication:} \\
F_{\gamma_1, \gamma_2}(\theta \gamma_1) & \geq F_{\gamma_1, \gamma_2}(\theta). \\
\text{2) } C\text{-contraction if } (X, F) & \text{ be a probabilistic metric space and } \exists \theta \in (0, 1) \text{ such that } \forall \gamma_1, \gamma_2 \in X \text{ and } \theta > 0 \text{ satisfy the following implication:} \\
F_{\gamma_1, \gamma_2}(\theta) & > 1 - \theta \Rightarrow F_{\gamma_1, \gamma_2}(\theta) > 1 - \theta. 
\end{align*}
\]
A mapping $F: X \rightarrow 2^X$ is called $(\Psi, C)$-contraction (where $\Psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\Psi\left(\frac{1}{n}\right) = 0$, if $(X, F)$ be a probabilistic metric space and $\exists \theta \in (0, 1)$ such that $\forall \alpha, \beta \in X$ and all $\theta > 0$ satisfy the following implication:

$$F_{\alpha, \beta}(\theta) > 1 - \theta \Rightarrow \forall \sigma \in \mathbb{E} F_{\alpha, \beta}(\theta),$$

such that

$$F_{\alpha, \beta}(\Psi(\theta)) > 1 - \Psi(\theta).$$

2.2. Period-II (1951–2000)

2.2.1. Two-Metric Space [11–15]. Gahler [15] first attempted to generalize the ordinary distance function to measure the distance between three points. Two-metric space is somewhere dissimilar to the metric space, which provides a unique nonlinear structure.

A mapping $d: X^2 \rightarrow \mathbb{R}$ is s.t.b. 2-metric on $X (X \neq \emptyset)$, if satisfied the subsequent axioms:

1. $d(\alpha, \beta, \gamma) > 0, (\forall \alpha, \beta, \gamma \in X)$, and $d(\alpha, \beta, \gamma) = 0$, only if two or more points are the same.
2. $d(\alpha, \beta, \gamma) = d(\beta, \gamma, \alpha) = d(\gamma, \alpha, \beta)$, $\forall \alpha, \beta, \gamma \in X$.
3. $d(\alpha, \beta, \gamma) \leq d(\alpha, \beta, \delta) + d(\beta, \delta, \gamma)$, $\forall \alpha, \beta, \gamma, \delta \in X$.
4. $d(\alpha, \beta, \gamma) \leq d(\alpha, \delta, \gamma) + d(\beta, \delta, \gamma), \forall \alpha, \beta, \gamma, \delta \in X$.
5. $d(\alpha, \beta, \gamma) \leq \beta d(\alpha, \beta, \delta) + d(\beta, \gamma, \delta), \forall \alpha, \beta, \gamma, \delta \in X$.

Then, 2-metric $d$ together with set $X$ is titled as 2-metric space and represented by $(X, d)$.

Remark 3

1. The “Baire category theorem” and “Cantor intersection theorem” are used for solving problems of fixed point [12].
2. In 2-metric space, Lebesgue integrable function is used as a summable function for each compact $\mathbb{R}_+$ [13].

A mapping $T: (X, \phi) \rightarrow (X, \phi)$ is called contraction in 2-metric space if satisfies the following conditions,

$$\phi(T_2^n, T_3^n, \rho) < \phi(\alpha, \beta, \gamma^n, \rho), \forall \alpha, \beta, \gamma^n, \rho \in X,$$

where $\beta, \gamma^n \neq \gamma^n$ and $\phi(T_2^n, T_3^n, \rho) = 0$, if any two of $\beta, \gamma^n$ and $\rho$ are equal.

A mapping $T: X \rightarrow X$ is entitled as “$A$-contraction” in 2-metric space if satisfies the succeeding conditions,

$$d(T_2^n, T_3^n, \rho) \leq \max\{d(T_2^n, T_3^n, \rho), \phi(T_2^n, T_3^n, \rho), (T_2^n, T_3^n, \rho)), \forall \alpha, \beta, \gamma^n, \rho \in X, \tau \in [0, 1/2] [14].$$

2.2.2. Fuzzy Metric Space [17–19]. After probabilistic metric space, author replaced randomness with fuzziness and presented an innovative theory, called fuzzy metric space (f.m.s.). The idea of probabilistic metric space (p.m.s.) has been expanded to fuzzy metric space. The purpose of introducing the fuzzy metric space was to allocate the non-negative fuzzy number to the distance among two points, which is the most appropriate way to describe the fuzzy metric space.

Let $M$ is a “fuzzy set” on $X \times X \times [0; \infty)$, that is s.t.b. fuzzy metric (f.m.) on $X (X \neq \emptyset)$, $\ast$ is a continuous t-norm), if met the subsequent axioms: $\forall \alpha, \beta, \gamma^n \in X$

1. $M(\alpha, \beta, 0) = 0$;
2. $M(\alpha, \beta, t) = 1 (\forall t > 0)M(\alpha, \beta, 0) = \beta^n$;
3. $M(\alpha, \beta, 0) = M(\alpha, \beta, 0)$;
4. $M(\alpha, \beta, t) \ast M(\beta^n, \gamma^n, s) \leq M(\gamma^n, \alpha, t + s);
5. M(\alpha, \beta, 0) \ast : [0; \infty) \rightarrow [0; 1]$ is left continuous, $\gamma^n, \alpha, \beta^n \in X$ and $t, s > 0$.

Then, fuzzy metric $M$ together with set $X$ is titled as fuzzy metric space and symbolized as $(X, M; \ast)$.

Example 4. Let $X = \mathbb{R}$ and $M = \exp(|\gamma^n - \beta^n|/|t|), \forall \gamma^n, \beta^n \in X, \tau \in (0, \infty)$.

1. $m \ast n = mn$;
2. $m \ast n = \min\{m, n\}$.

It is easy to get that $M$ is a fuzzy metric on $X = \mathbb{R}$ with a continuous t-norm $\ast$.

Remark 4

1. Gregori and Sapena expanded the “Banach contraction mapping” to the “fuzzy contractive mapping” in the complete fuzzy metric space.
2. Chos et al. proved Banach’s contraction and Edelstein mapping in fuzzy metric space.
3. N. Holland proved Baire’s theorem in fuzzy metric space.

A mapping $T: X \rightarrow X$ is known as the following:

1. “Banach fuzzy contraction” if fuzzy metric space $(X, M, \ast)$ satisfies the subsequent conditions, $M(T_2^n, T_3^n, \rho) \geq M(\beta^n, \gamma^n, \rho), \forall \gamma^n, \beta^n \in X, k \in (0, 1)$, and all $\rho > 0$.
2. “Edelstein fuzzy contraction” if fuzzy metric space $(X, d, \ast)$ satisfies the subsequent conditions, $M(T_2^n, T_3^n, \rho) \geq M(\gamma^n, \beta^n, \rho), \forall \gamma^n, \beta^n \in \mathbb{R}$, and all $\rho > 0$.

2.2.3. b-Metric Space [20–22]. In the b-metric space, $K$ (relaxed triangle inequality) was introduced. If $\beta = 1$, then b-metric and metric space are the same so as compared with the metric space, and b-metric space is the weaker notion. A mapping $d: X \times \mathbb{R} \rightarrow X (X \neq \emptyset)$ is s.t.b. b-metric on $X (X \neq \emptyset)$ if met the subsequent axioms:

1. $d(\alpha, 0) = 0 \Rightarrow \alpha = \beta^n$;
2. $d(\alpha, 0) = d(\beta^n, 0)$ (symmetry);
3. There exists $\beta \geq 1$ satisfying

$$d(\alpha, 0) \leq \beta d(\beta^n, 0) + d(\beta^n, 0), \forall \alpha, \beta^n \in X.$$
Then, \( b \)-metric \( d \) together with set \( X \) is titled as \( b \)-metric space and represented as ordered pair \((X, d)\).

Example 5. Let \( X = \mathbb{N} \) and function \( f(n) = -\lfloor \log_2 n \rfloor \) and \( g(n) = 2n - 2^{f(n)} \) if \( f(n) < 0 \), \( g(n) = (2^n - n)K^{f(n)} \) if \( f(n) = 0 \) and \( g(n) = (2^n - n)K^{f(n)-1} \) and a function \( d \) from \( X \longrightarrow X \) onto \([0, \infty)\) by

\[
d(x, y) = g(|x-y|).
\]

It is easy to show that \((X, d)\) is a \( b \)-metric space.

Remark 5. With the help of \( b \)-metric space, the presence of a unique solution for a nonlinear fractional differential equation is given with an integral boundary condition. Biological, physical science, and economics such real-life problems eventually lead to the linear fractional differential equation and integral equation [20–22]. Further extension of \( b \)-metric space can be seen in the paper of Chauhan and Kaur [23].

2.2.4. \( D \)-Metric Space [24–26]. Since 2-metric space was not continuous and convergent for all three points, so Dhage [24] introduced the new generalized class of the ordinary metric space. The convergence of \( D \)-metric defines a Hausdorff topology and (sequentially) continuity for all three points.

A mapping \( D: X^3 \longrightarrow [0, \infty) \) is s.t.b. \( D \)-metric on \((X, d)\), if it meets the following conditions given below:

1. \( D(2^n_1, 2^n_2, 2^n_3) = 0 \iff 2^n_1 = 2^n_2 = 2^n_3 \) (coincidence)
2. \( D(2^n_1, 2^n_2, 2^n_3) = D(p(2^n_1, 2^n_2, 2^n_3)) \), where \( p \) is a permutation of \( 2^n_1, 2^n_2, 2^n_3 \) (symmetry), and
3. \( D(2^n_1, 2^n_2, 2^n_3) \leq D(2^n_1, 2^n_2, 2^n_3) + D(2^n_3, 2^n_4, 2^n_5) \forall 2^n_1, 2^n_2, 2^n_3, 2^n_4, 2^n_5 \in X \) (tetrahedral inequality).

Then, \( D \)-metric \( d \) together with set \( X \) is titled as \( D \)-metric space and represented as ordered pair \((X, d)\).

Example 6. Let \( X = \mathbb{R}^+ \) and \( D: X^3 \longrightarrow \mathbb{R}^+ \) be defined as

\[
D(2^n_1, 2^n_2, 2^n_3) = \max \{|2^n_1 - 2^n_2|, |2^n_2 - 2^n_3|, |2^n_3 - 2^n_1|\}, \forall 2^n_1, 2^n_2, 2^n_3 \in X.
\]

It is easy to show that \((X, D)\) is a \( D \)-metric space.

Remark 6

1. The analysis is obtained from the results of nonlinear self-mapping, which satisfies the particular form (completeness and boundedness) of contraction condition in \( D \)-metric space, to prove the existence of a unique fixed point [25].
2. Singh et al. introduced semicompatibility in \( D \)-metric space for “fixed-point theorems” with the help of the orbit concept [26].

Every \( D \)-continuous function \( T: (X, D) \longrightarrow (X, D) \) has a fixed point in \( X \) if \( D \)-metric space has a \( D \)-fixed-point property.

Every \( D \)-weakly continuous function \( T: (X, D) \longrightarrow (X, D) \) has a fixed point in \( X \) if and only if \( D \)-metric space has a \( D \)-weakly fixed-point property.

2.2.5. Partial Metric Space [27–29]. The partial metric space is a component of dataflow networks, denotational semantic study. Compared with the standard metric, the main difference is that there is no need for the self-distance of arbitrary variables is to be zero.

A function \( p: \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \) is s.t.b. partial metric on \( (P \neq \{\emptyset\}) \), if it satisfies the subsequent axioms:

\[
\forall 2^n_1, 2^n_2, 2^n_3 \in P:
\]

1. \( 2^n_1 = 2^n_2 \iff p(2^n_1, 2^n_2) = p(2^n_2, 2^n_1) = p(2^n_2, 2^n_2) \);
2. \( p(2^n_2, 2^n_3) \leq p(2^n_2, 2^n_2) \);
3. \( p(2^n_1, 2^n_2) = p(2^n_2, 2^n_1) \);
4. \( p(2^n_2, 2^n_3) \leq p(2^n_2, 2^n_2) + p(2^n_2, 2^n_3) - p(2^n_2, 2^n_3) \).

Then, partial metric \( p \) together with the set \( P \) is titled as partial metric space and represented as ordered pair \((P, p)\).

Example 7. Let \( X = \mathbb{R}^+ = [0, \infty) \) and \( p: X \times X \longrightarrow \mathbb{R}^+ \) be a function defined by \( p(2^n_1, 2^n_2) = \max \{2^n_1, 2^n_2\} \), for all \( 2^n_1, 2^n_2 \in X \).

Then, it is easy to see that \((X, p)\) is a partial metric, but it is not a metric. Indeed, for any \( x > 0 \), we have \( p(x, x) = x \neq 0 \).

Remark 7. Through the experience of computer science, the idea of nonzero self-distance for metric space was inspired. Also, it gave an extension of the “Banach contraction principle” in the complete partial metric space [30].

A mapping \( T: X \longrightarrow X \) is titled as:

1. Contraction for every partial metric space \((X, p)\) if \( \exists k \in [0, 1) \) so that \( \forall 2^n_1, 2^n_2 \in X \) satisfies the following condition:

\[
p(T2^n_1, T2^n_2) \leq kp(2^n_1, 2^n_2).
\]

2. Asymptotic regular for every partial metric space \((X, p)\) if at point \( 2^n_1 \in X \) satisfies the following condition:

\[
\lim_{n \longrightarrow \infty} d(T^n2^n_1, T^{n+1}2^n_1) = 0.
\]

2.2.6. Rectangular Metric Space [31, 32]. The rectangular metric space was introduced with the most general triangle inequality in which four points are implemented instead of three points.

A generalized metric (rectangular metric) is a function \( d: \mathbb{R}^2 \longrightarrow [0, \infty) \) on \( R \neq \{\emptyset\} \), if it satisfied the subsequent axioms:

\[
\forall 2^n_1, 2^n_2, 2^n_3, 2^n_4 \in R \text{ with } 2^n_1 \neq 2^n_2 \text{ and } 2^n_3 \neq 2^n_4 \neq [2^n_1, 2^n_2] : \]

1. \( d(2^n_1, 2^n_2) = 0 \iff 2^n_1 = 2^n_2 \);
(2) \( d(\zeta^n_1,\zeta^n_2) = d(\zeta^n_3,\zeta^n_4); \)
(3) \( d(\zeta^n_1,\zeta^n_2) \leq d(\zeta^n_1,\zeta^n_3) + d(\zeta^n_3,\zeta^n_4) \) (quadrilateral inequality).

Then, rectangular metric \( d \) together with the set \( R \) is titled as partial metric space and represented as ordered pair \((R,d)\).

Example 8. Let \( R_1 = [0,2], R_2 = \{1, 1/2, 1/3, \ldots \}, R = R_1 \cup R_2 \). Define \( d: R \times R \rightarrow [0,\infty) \) as follows:

\[
d(\zeta^n_1,\zeta^n_2) = \begin{cases} 
0, & \zeta^n_1 = \zeta^n_2, \\
1, & \zeta^n_1 \neq \zeta^n_2, \zeta^n_1 \in R_1 \text{ or } \zeta^n_2 \in R_2, \\
\zeta^n_1, & \zeta^n_1 \in R_1, \zeta^n_2 \in R_2, \\
\zeta^n_2, & \zeta^n_1 \in R_1, \zeta^n_2 \in R_2.
\end{cases}
\]

Then, \((R,d)\) is a complete rectangular metric space.

Remark 8. Budhia et al. contributed to some innovative fixed-point results in the "rectangular metric space" using a fractional-order functional differential equation.

A mapping \( T: X \rightarrow X \) is called as the following:

(1) "Caccioppoli-type fixed-point theorem" for complete metric space \((X,d)\) if for every \( n \in \mathbb{N}, \forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X \) where \( \delta_n > 0 \) distinct from \( \zeta^n_1, \zeta^n_2 \) and \( 0 < \delta_n < 1 \) fulfills the subsequent condition:

\[
d(T^{\delta_n} \zeta^n_1, T^{\delta_n} \zeta^n_2) < \delta_n \left[ d(\zeta^n_1, T \zeta^n_1) + d(\zeta^n_2, T \zeta^n_2) \right].
\]

Then, if series \( \sum_{n=0}^{\infty} \delta_n \) is convergent, then \( T \) has a unique fixed point in \( X \).

(2) "Kannan fixed-point theorem" for complete metric space \((X,d)\) if for every \( n \in \mathbb{N}, \forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X \) and \( 0 < \delta_n < 1/2 \) fulfills the subsequent condition:

\[
d(T \zeta^n_1, T \zeta^n_2) < \delta_n \left[ d(\zeta^n_1, T \zeta^n_1) + d(\zeta^n_2, T \zeta^n_2) \right].
\]

If \( X \) is \( T \)-orbitally complete in \((X,d)\), then \( T \) has a unique fixed point in \( X \).


2.3.1. Fuzzy Quasi-Metric Space [33]. The associated impression of fuzzy metric space was expanded with a quasi-metric sense and then made known to the fuzzy quasi metric space.

Let \( M \) is a fuzzy set mapping on \( X \times X \times (0,\infty) \), that is, s.t.b. fuzzy quasi-metric (f.q.m.) on \( X(\neq \emptyset), \ast \) \((-\emptyset \neq \emptyset)) \) continuous \( t \)-norm), if it satisfied the subsequent axioms: \((\forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X, t > 0)\):

(1) \( M(\zeta^n_1, \zeta^n_2, 0) = 0; \)
(2) \( \zeta^n_1 = \zeta^n_2 \) if \( M(\zeta^n_1, \zeta^n_2, t) = M(\zeta^n_2, \zeta^n_1, t) = 1; \)
(3) \( M(\zeta^n_2, \zeta^n_3, t) \ast M(\zeta^n_3, \zeta^n_4, s) \leq M(\zeta^n_1, \zeta^n_4, t + s); \)
(4) \( \zeta^n_1, \zeta^n_2 : (0,\infty) \rightarrow (0, 1) \) is left continuous.

Then, fuzzy quasi-metric \( M \) together with the set \( X \) is titled as partial metric space and represented as ordered pair \((X,M)\).

Example 9. Let \((X,d)\) be a fuzzy quasi-metric space and be a usual multiplication \( \forall \zeta^n_1, \zeta^n_2 \in [0,1], \) and \( M \) is a fuzzy set mapping on \( X \times X \times (0,\infty) \) by

\[
M_d(\zeta^n_1, \zeta^n_2, t) = \frac{t}{t + d(\zeta^n_1, \zeta^n_2, t)}.
\]

Then, \((X,M_d)\) is a fuzzy quasi-metric space.

Remark 9.

(1) Gregori and Romaguera proved a quasi-metrizable topology, which is generated via fuzzy quasi-metric space.

(2) In the fuzzy quasi-metric space (f.q.m.s.), the "contractive principle" was demonstrated with an appliance to the word domain.

A mapping \( T: X \rightarrow X \) is titled as "\( B \)-contraction" on fuzzy quasi-metric space \((X,M,\ast)\) that fulfills the following circumstances,

\[
M(T \zeta^n_1, T \zeta^n_2, kp) \geq M(\zeta^n_1, \zeta^n_2, \rho), \forall \zeta^n_1, \zeta^n_2 \in X, k \in (0,1), \) and all \( \rho > 0, \) where \( k \) is called the contraction constant of \( T \) [34].

2.3.2. G-Metric Space [35, 36]. In the G-metric space, there is a modification that the distance of three variables is equal to zero if all the variables are equal, and if one of them is different, then the distance is always positive.

A mapping \( G: X^3 \rightarrow R^+ \) is s.t.b. G-metric (more specifically generalized metric) on \( X(\neq \emptyset), \ast \) if it satisfied the subsequent axioms:

(1) \( G(\zeta^n_1, \zeta^n_2, \zeta^n_3) = 0 \) if and only if \( \zeta^n_1 = \zeta^n_2 = \zeta^n_3; \)
(2) \( G(\zeta^n_1, \zeta^n_2, \zeta^n_3) > 0 \forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X, \) with \( \zeta^n_1 \neq \zeta^n_2; \)
(3) \( G(\zeta^n_1, \zeta^n_2, \zeta^n_3) \leq G(\zeta^n_1, \zeta^n_2, \zeta^n_3), \forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X \) with \( \zeta^n_1 \neq \zeta^n_2; \)
(4) \( G(\zeta^n_1, \zeta^n_2, \zeta^n_3) = G(\zeta^n_3, \zeta^n_2, \zeta^n_1) = G(\zeta^n_2, \zeta^n_1, \zeta^n_3), \) (symmetry);
(5) \( G(\zeta^n_1, \zeta^n_2, \zeta^n_3) \leq G(\zeta^n_1, \zeta^n_2, \zeta^n_2) + G(\zeta^n_2, \zeta^n_2, \zeta^n_3), \forall \zeta^n_1, \zeta^n_2, \zeta^n_3 \in X \) (rectangle inequality).

Then, G-metric \( G \) together with the set \( X \) is titled as G-metric space and represented as ordered pair \((X,G)\).

Example 10. Let \((R,d)\) be the usual metric space. We define \( G \) be

\[
G(\zeta^n_1, \zeta^n_2, \zeta^n_3) = d(\zeta^n_1, \zeta^n_2) + d(\zeta^n_2, \zeta^n_3) + d(\zeta^n_1, \zeta^n_3),
\]

for all \( \zeta^n_1, \zeta^n_2, \zeta^n_3 \in R \). Then, it is easy to see that \((R,G)\) is a G-metric space.

Remark 10

(1) In G-metric space, a theorem was introduced for the solution of integral equations.

(2) Existence and uniqueness of the "fixed-point theorem" in generalized metric space (or G-metric space) are proved.
2.3.3. \(D^*\) -Metric Space [37]. When most of the \(D\)-metric space theorems were invalid, then the description of \(D\)-metric space had been possibly updated by Sedghi et al. [37] and introduced \(D^*\)-metric space. With some modifications, the basic properties of \(D^*\)-metric space (\(D^*\)-m.s.) were introduced.

A mapping \(D^*: X^3 \rightarrow 0\), is s.t.b. generalized metric (or \(D^*\)-metric) on \((X \neq \emptyset)\), if fulfilled the subsequent conditions:

\[
\begin{align*}
& (1) \ D^* (x, y, z) \geq 0; \\
& (2) \ D^* (x, y, z) = 0 \Rightarrow x = y = z; \\
& (3) \ D^* (x, y, z) = D^* (p[x, y, z]), \quad \text{(symmetry)} \\
& (4) \ D^* (x, y, z) \leq D^* (x, y, z) + D^* (y, z, x). 
\end{align*}
\]

Then, \(D^*\)-metric \(D^*\) together with the set \(X\) is titled as \(D^*\)-metric space and represented as ordered pair \((X, D^*)\).

\[
D^* (M_1, M_2, M_3) = \phi [D^* (S_1, T_1, T_2), D^* (S_2, M_2, M_3), D^* (T_2, M_1, M_3), D^* (T_2, M_2, M_2)]. 
\]

That, \(M, S\) and \(T\) have a common unique fixed point in \(X\).

2.3.4. Cone Metric Space [38–42]. The cone metric space was introduced when the real numbers were replaced by ordering Banach’s space.

Let \(E\) stand for “real Banach space” and \(C, E\).

A function \(d: X^2 \rightarrow E\) (where \(X \neq \emptyset\)) is s.t.b. a cone metric on \(X\), if it meets the subsequent requirements:

\[
\begin{align*}
& (1) \ d(N_1, N_2) > 0, \forall N_1, N_2 \in X \quad \text{and} \quad N_1 = N_2 \Leftrightarrow d(N_1, N_2) = 0; \\
& (2) \ d(N_1, N_2) = d(N_2, N_1), \forall N_1, N_2 \in X; \\
& (3) \ d(N_1, N_2) + d(N_2, N_3) \geq d(N_1, N_3) \quad \text{for all} \quad N_1, N_2, N_3 \in X.
\end{align*}
\]

Then, function “\(d\)” composed of set “\(X\)” is titled as a cone metric space and represented as \((X, d)\).

Example 11. Let \(X = \mathbb{R}\), then we define

\[
D^* (2^n, \alpha 2^n, \beta 2^n) = \begin{cases} 
0, & \text{if } 2^n = \alpha 2^n = \beta 2^n \\
\max \{2^n, \alpha 2^n, \beta 2^n\}, & \text{o.w.}
\end{cases}
\]

Then, it is easy to check that \((X, D^*)\) is \(D^*\)-metric space.

Remark 11. A “fixed-point theorem” was developed for a class of mapping in complete \(D^*\)-metric space together with the condition of weakly commuting mappings.

A mapping \(S, T: X \rightarrow Y\) is s.t.b \((X, M) \subset S(X) \cap T(X)\);

\(2) \exists \phi \in \Phi\) such that for all \(2^n, 2^n \in X\),

\[
\text{Example 11. Let } X = \mathbb{R}, \text{ then we define }
\]

\[
D^* (2^n, \alpha 2^n, \beta 2^n) = \begin{cases} 
0, & \text{if } 2^n = \alpha 2^n = \beta 2^n \\
\max \{2^n, \alpha 2^n, \beta 2^n\}, & \text{o.w.}
\end{cases}
\]

Then, it is easy to check that \((X, D^*)\) is \(D^*\)-metric space.

Then, \(T\) has a unique fixed point in \(X\). For any \(N^n \in X\), iterative sequence \(\{T^n N^n\}\) converges to the fixed point.

Theorem 4. Let \((X, d)\) be a sequentially compact cone metric space, and \(P\) be a regular cone.

It is supposed that the mapping \(T: X \rightarrow X\) fulfills the following contractive condition:

\[
d(TN^n, TN^n) < k d(N^n, N^n), \forall N^n, N^n \in X, N^n \neq N^n. \tag{23}
\]

Then, \(T\) has a unique fixed point in \(X\).

Theorem 5. Let \((X, d)\) be a complete cone metric space, and \(P\) a normal cone with normal constant \(K\). It is supposed that the mapping \(T: X \rightarrow X\) has a unique fixed point in \(X\) if fulfills the contractive condition:

\[
d(TN^n, TN^n) \leq k d(TN^n, N^n), \forall N^n, N^n \in X, \tag{24}
\]

where \(k \in [0, 1/2]\) is a constant. For any \(N^n \in X\), iterative sequence \(\{T^n N^n\}\) converges to the fixed point.

2.3.5. Multiplicative Metric Space [43, 44]. Initially, the usual metric with \(\mathbb{R}^+\) was not complete. To overcome this problem, Bashirov et al. proposed the concept of multiplicative metric space with a multiplicative value of \(x \in \mathbb{R}^+\) and multiplicative distance function \(d\).
A mapping \( d: X^2 \rightarrow \mathbb{R}^+ \) is s.t.b. multiplicative metric on \( X (X \neq \emptyset) \), when it meets the following conditions:

1. \( d(2_1^n, 2_2^n) \geq 1, \forall 2_1^n, 2_2^n \in X; \)
2. \( d(2_1^n, 2_2^n) = 1 \Leftrightarrow 2_1^n = 2_2^n; \)
3. \( (2_1^n, 2_2^n) = d(2_3^n, 2_4^n), \forall 2_3^n, 2_4^n \in X; \)
4. \( d(2_1^n, 2_2^n) \leq d(2_3^n, 2_4^n)d(2_5^n, 2_6^n) \) for all \( 2_1^n, 2_2^n, 2_3^n, 2_4^n \in X \) (multiplicative triangle inequality).

Then, multiplicative metric \( d \) together with the set \( X \) is titled as multiplicative metric space and represented as ordered pair \((X, d)\).

**Example 13.** Let \( R^n \) be the set of all \( n \)-tuples of non-negative real numbers. Let \( d: R^n \times R^n \rightarrow R \) be a mapping defined as follows:

\[
d(2_1^n, 2_2^n) = \left\{ \begin{array}{ll}
\sum_{i=1}^{n} |2_1^i - 2_2^i| & \text{if } 2_1^n \geq 1, \\
1 & \text{if } 2_1^n < 1.
\end{array} \right.
\]

It is easy to get that \((R^n, d)\) is a multiplicative metric space.

**Remark 13.** We gave the description of multiplicative contraction mapping and “fixed-point theorems” on a complete multiplicative metric space.

A mapping \( T: X \rightarrow X \) is called as follows:

1. “Multiplicative contraction” of multiplicative metric space if \( \exists r \in [0, 1], \forall 2_1^n, 2_2^n \in X \) fulfills the subsequent condition as is follows:

\[
d(T^n 2_1, T^n 2_2) \leq r[d(2^n 1, 2^n 2)].
\]

2. “Multiplicative Kannan’s contraction” of multiplicative metric space if \( \exists r \in [0, 1/2], \forall 2_1^n, 2_2^n \in X \) fulfills the subsequent condition is as follows:

\[
d(T^n 2_1, T^n 2_2) \leq [d(2^n 1, T^n 2_2)d(2^n 2, T^n 2_1)].
\]

3. “Multiplicative Chatterjea’s” contraction of multiplicative metric space if \( \exists r \in [0, 1/2], \forall 2_1^n, 2_2^n \in X \) fulfills the subsequent condition is as follows:

\[
d(T^n 2_1, T^n 2_2) \leq [d(2^n 1, T^n 2_2)d(2^n 2, T^n 2_1)].
\]

**2.3.6. Vector Metric Space [45, 46].** Cevik and Altun were made known to the vector metric space in which metric space takes the value on a Riesz space, i.e., Riesz space replaced from \( \mathbb{R} \).

The function \( d: X \times X \rightarrow E \) is s.t.b. vector metric on \((X \neq \emptyset, E)-Riesz space\) if it satisfied the following:

1. \( d(2_1^n, 2_2^n) = 0 \) if \( 2_1^n = 2_2^n, \)
2. \( d(2_1^n, 2_2^n) \leq d(2_3^n, 2_4^n) + d(2_5^n, 2_6^n) \forall 2_3^n, 2_4^n, 2_5^n, 2_6^n \in X. \)

Then, vector metric \( d \) together with the set \( X \) is titled as vector metric space and termed as triple \((X, d, E)\) (briefly \( X \) with the default parameters omitted).

**Example 14.** Let a Riesz space “E” and a mapping \( d: E \times E \rightarrow E \) be defined by the following:

\[
d(2^n 1, 2^n 2) = |2^n 1 - 2^n 2|.
\]

It is easy to show a vector metric space.

**Remark 14.** Cevik and Altun gave “Banach’s contraction principle” in the vector metric space.

Let \( V \) be an \( E \)-complete vector metric space (v.m.s.), (where “\( E \)” is Archimedean).

The mapping \( T: V \rightarrow V \) satisfies the contractive condition

\[
d(T^n 2_1, T^n 2_2) \leq \theta d(2^n 1, 2^n 2), \forall 2^n 1, 2^n 2 \in V, \quad \text{where } \theta \in [0, 1) \text{ is a constant.}
\]

Then, \( T \) has a unique fixed point in \( V \) and iterative sequence \( 2^n 1, V_0 \subseteq V \), defined through \( 2^n 1 = T^n (2^n 1 - 1) \) for \( n \in N, E \)-converges on the fixed point of \( T \).

2.4. Period-IV (2011–2021)

2.4.1. Complex-Valued Metric Space [47, 48]. The explanation of the complex-valued metric spaces (c.m.s.) occurred when replaced range \( C \) (with a certain order) form range \( \mathbb{R} \) (with a normal order), i.e., the generalization of the contractive condition in metric space made much of the concept available. There was a considerable amount of content available in generalized metric spaces. In order to satisfy a rational inequality, complex-valued metric space was thus introduced and some results were also developed for mapping.

A mapping \( d: X^2 \rightarrow C \) is s.t.b. complex-valued metric space (c.m.s.) on \((X \neq \emptyset, C)\), if fulfills the subsequent condition:

1. \( (1) d(2_1^n, 2_2^n) \geq 0, \forall 2_1^n, 2_2^n \in X \)And \( d(2_1^n, 2_2^n) = 0 \Leftrightarrow 2_1^n = 2_2^n, \)
2. \( (2) d(2_1^n, 2_2^n) = d(2_2^n, 2_1^n), \) for all \( 2^n 1, 2^n 2 \in X; \)
3. \( (3) d(2_1^n, 2_2^n) \leq d(2_3^n, 2_4^n) + d(2_5^n, 2_6^n), \forall 2_3^n, 2_4^n, 2_5^n, 2_6^n \in X. \)

Then, complex-valued metric \( d \) together with the set \( X \) is titled as complex-valued metric space and represented as ordered pair \((X, d)\).

**Example 15.** Let \( X = \mathbb{C} \) be a set of complex number. We define \( d: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \) by

\[
d(2_1^n, 2_2^n) = |2_1^n - 2_2^n| + |2_1^n + 2_2^n|,
\]

where \( 2_1^n = 2_1^n + 2_2^n \) and \( 2_2^n = 2_1^n + 2_2^n. \)

Then, \((C, d)\) is a complex-valued metric space.

**Remark 15.** In the complex-valued metric space (c.m.s.), it provides the extension of “Banach’s fixed-point theorem,”
which helps to give a common unique solution to the
Urysohn integral equations.
(i) \( x(\sigma) = \frac{1}{b} K_1(\sigma, r, x(r))dr + g(\sigma), \)
(ii) \( x(\sigma) = \frac{1}{b} K_2(\sigma, r, x(r))dr + h(\sigma), \)
where \( \sigma \in [a, b], x, g, h \in X. \)

A mapping \( T, f : X \rightarrow X \) be two functions in metric
space \( (X, d) \). A mapping \( T \) is called \( f \)-contraction if
\( \exists \phi \in (0, 1) \) such that
\[
d\left(fT_x^n, fT_y^n\right) \leq \phi d\left(fT_x^n, fT_y^n\right), \forall 2^n, 2^n \in X. \tag{32}\]

2.4.2. Cone \( b \)-Metric Space [49]. The generalization of cone
metric space (C.M.S.) together with b-metric space is termed
as cone b-metric space.

A mapping \( d : X^2 \rightarrow E \) is s.t.b. cone b-metric on
\( X \) (where \( X \neq \emptyset \), \( s \geq 1 \) \( \in \mathbb{R}^+ \)) if for all \( C_1, C_2, C_3 \in X \) met
the following conditions:
1. \( d(C_1, C_2) > 0 \) with \( C_1 \neq C_2 \) and \( \exists C_1, C_2 = 0 \) if and
   only if \( C_1 = C_2; \)
2. \( d(C_1, C_2) = d(C_2, C_1); \)
3. \( d(C_1, C_2) \leq s[d(C_1, C_3) + d(C_3, C_2)]. \)

Then, cone b-metric \( d \) together with the set \( X \) is titled as
cone b-metric space and represented as an ordered pair
\( (X, d) \).

Example 16. Let \( E = I^1, P = \{[u_n] \in E/\{u_n\} \geq 0, \forall n \geq 1, \}\) and
\( d : X \times X \rightarrow E \) such that
\[
d^r(C_1, C_2) = \frac{d(C_1, C_2)}{2^n} \text{, for } n \geq 1. \tag{33}\]

Then, \( (X, d) \) is a cone b-metric space with coefficient
\( r = 2^{1/n} > 1. \)

Remark 16. N. Sharma proved contractive mapping for
“fixed-point theorem” with rational expression and without
normality condition, in cone b-metric space.

Theorem 6. Suppose the mapping \( T : X \rightarrow X \) and
\( (X, d) \) be a complete cone b-metric space with the coefficient
\( s \geq 1 \) that satisfies the following condition, then
\[
d(TC_1, TC_2) \leq \alpha \left[ \frac{d(C_1, TC_1) + d(C_2, TC_2)}{d(C_1, C_2)} \right] + \beta \cdot d(C_1, C_2), \tag{34}\]

for \( C_1, C_2 \in X \) and \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta < 1 \). Then, \( T \) has
a unique fixed point in \( X \).

2.4.3. S-Metric Space [50, 51]. The philosophy of S-metric
spaces is the modification of D-metric and G-metric spaces
in which the 3-dimensional metric space was made known
by Sedghi et al. In S-metric space, the distance between
all three distinct points is always greater than or equal to zero.

A function \( S : X^3 \rightarrow [0, \infty) \) s.t.b. S-metric space
on \( X \neq \emptyset \) if it fulfills the following axioms:\n\( \forall 2^n, 2^n, 2^n, 2^n \in X, \)
\( (1) S(2^n, 2^n, 2^n) \geq 0; \)
\( (2) S(2^n, 2^n, 2^n) = 0 \Leftrightarrow 2^n = 2^n = 2^n; \)
\( (3) S(2^n, 2^n, 2^n) \leq S(2^n, 2^n, 2^n) + S(2^n, 2^n, 2^n) + S(2^n, 2^n, 2^n) \quad \in X. \)

Then, S-metric \( S \) together with the set \( X \) is titled as
S-metric space and represented as an ordered pair \( (X, S) \).

Example 17. Let \( X = \mathbb{R} \) and \( , \) a norm on \( X \), then
\[
S(2^n, 2^n, 2^n) = 2^n + 2^n - 22^n + 2^n + 2^n; \tag{35}\]
is an S-metric on \( X \).

Remark 17. Ozgur and Tas characterized the thought of
a fixed circle and explored the existence and uniqueness of
the “fixed-point theorem” for the self-mapping on S-metric
space. Likewise, a few instances of self-mappings having
fixed circles are given with a mathematical perspective.
Some new contractive mapping and the concept of \( C_s \)
and \( L_s \) mapping on S-metric space were established.

Theorem 7. Let \( (X, S) \) be a complete S-metric space, and
\( A; B : X \rightarrow X \) be the mappings that fulfill the subsequen
t conditions:
\( (1) A(X) \subseteq B(X) \) and either \( A(X) \) or \( B(X) \) is a closed
subset of \( X \),
\( (2) \) The pair \( (A; B) \) is weakly compatible,
\( (3) S(A2^n; A2^n; A2^n) \leq \phi (\max \{S(B2^n; B2^n; B2^n) ; k_1; S(B2^n; B2^n; B2^n) ; k_2 \}) \)
for all \( \forall 2^n, 2^n, 2^n \in X \) and \( 0 < k_1, k_2 < 1 \), where \( \phi \in \emptyset. \)

Then, the maps \( A \) and \( B \) have a unique common fixed
point. If \( B \) is continuous at the fixed point \( p \), then \( A \) is also
continuous at \( p \).

2.4.4. Metric-Like Space [29]. Metric-like space (m.l.s.) is
a generalization of partial metric space, in which all axioms
are the same as the usual metric except first, that is \( a(x, x) \)
may be positive for \( x \in X \).

A function \( \sigma : X^2 \rightarrow R^+ \cup \{0\} \) is a metric-like
space on \( X \neq \emptyset \), if it fulfilled the subsequen axioms:
for any \( 2^n, 2^n, 2^n \in X, \)
\( (1) \sigma(2^n, 2^n) = 0 \Rightarrow 2^n = 2^n; \)
\( (2) \sigma(2^n, 2^n) = \sigma(2^n, 2^n); \)
\( (3) \sigma(2^n, 2^n) \leq \sigma(2^n, 2^n) + \sigma(2^n, 2^n). \)

Then, metric-like \( \sigma \) together with the set \( X \) is titled as
metric-like space and represented as an ordered pair \( (X, \sigma). \)

Example 18. Let \( X = \{0, 1\} \) and let
\[
\sigma(2^n, 2^n) = \begin{cases} 2, \text{if } 2^n = 2^n = 0, \\ 1, \text{otherwise.} \end{cases} \tag{36}\]
It is easy to get \((X, \sigma)\) that is metric-like space.

**Remark 18**

1. As an application, A. Harandi unified and generalized some important results of the “fixed-point theory.”
2. Metric-like space considered new Geraghty-type mappings and proposed “fixed-point theorems” for such space [52].

A topology \(\tau_\sigma\) for each metric-like \(\sigma\) was generated, which is a family of open balls. \(B_\sigma(x, \epsilon) = \{ y \in X : |\sigma(x, y) - \sigma(x, x)| < \epsilon \}, \forall x \in X \) and \(\epsilon > 0\).

Then, a sequence \(\{x_n\}\) converges to a point \(x \in X\) if \(\lim_{n \to \infty} \sigma(x_n, x) = \sigma(x, x)\).

**2.4.5. b-Metric-Like Space** [53, 54]. The \(b\)-metric-like space \((b\text{-m.l.s.})\) is a generalized version of metric-like space in which coefficient of metric space was initiated in the triangle inequality.

A mapping \(\sigma: X^2 \to R^+\) (where \(X \neq \emptyset\)) is \(s\) such \(b\)-metric-like on \(X\) if \(\forall (x_1^n, x_2^n, x_3^n) \in X\) and constant \(s \geq 1\) fulfills the following axioms:

1. \(\sigma(x_1^n, x_2^n) = 0 \implies x_1^n = x_2^n\);
2. \(\sigma(x_1^n, x_2^n) = \sigma(x_2^n, x_1^n)\);
3. \(\sigma(x_1^n, x_3^n) \leq s[\sigma(x_1^n, x_2^n) + \sigma(x_2^n, x_3^n)]\).

Then, \(b\)-metric-like \(\sigma\) together with the set \(X\) is titled as \(b\)-metric-like space and represented as an ordered pair \((X, \sigma)\).

**Example 19.** Let \(X = R^+, \rho > 1\) a constant, and \(\sigma_b: X \times X \to R^+\) be defined by

\[
\sigma_b(x_1^n, x_2^n) = (x_1^n + x_2^n)\rho, \forall x_1^n, x_2^n \in X.
\]

Then, \((X, \sigma_b)\) is a \(b\)-metric-like space with coefficient \(2^{\rho-1}\).

**Remark 19.** Some results of \(b\)-metric space are related to the fixed point and give nontrivial examples to support the concept such as finding the solution of the electric circuit equation with the help of second-order differential equations.

A mapping \(T: X \to X\) is called

1. Banach \(G\)-contraction if \(T\) preserves edges of \(G\), i.e., \(\forall x_1^n, x_2^n \in X\);
   \[
   (x_1^n, x_2^n) \in E(G) \implies (T_{x_1^n}, T_{x_2^n}) \in E(G).
   \]

2. Orbitally continuous if given \(x_1^n \in X\) and any sequence \(\{k_n\}\) of positive integers,

\[
T^{k_n}(x_1^n) \to 2^n_{as} \implies \lim_{n \to \infty} T(T^{k_n}(x_1^n)) \to T(2^n_{as}) \text{ as } n \to \infty.
\]

**2.4.6. Quasi-b-Metric Space** [53–55]. The class of ordinary quasi-metric space was extended to form quasi-b-metric space.

A function \(d: X^2 \to R^+\) is entitled as quasi-b-metric on \(X\) \((\neq \emptyset)\) and constant \(s \geq 1\), if it satisfied the subsequent axioms:

1. \(d(x_1^n, x_2^n) \geq 0, \forall x_1^n, x_2^n \in X\);
2. \(d(x_1^n, x_2^n) = 0 \iff x_1^n = x_2^n\);
3. \(d(x_1^n, x_3^n) \leq s(d(x_1^n, x_2^n) + d(x_2^n, x_3^n)), \forall x_1^n, x_2^n, x_3^n \in X\).

Then, quasi-b-metric \(d\) together with the set \(X\) is titled as quasi-b-metric space and represented as an ordered pair \((X, d)\).

**Example 20.** Let \(X = l_p\), where \(1 \leq p < \infty\), be defined by

\[
l_p = \{ (x_{1n})_{n \geq 1} \in R^\infty : \sum_{n=1}^{\infty} |x_{1n}|^p < \infty \}.
\]

We define \(d: X \times X \to R^+\) by

\[
d(x_1^n, x_2^n) = \begin{cases} 0, \text{if } x_1^n \leq x_2^n; \\
\left( \sum_{n=1}^{\infty} |x_{1n}|^p \right)^{1/p}, \text{if } x_1^n \geq x_2^n. \end{cases}
\]

It is easy to get that \((X, d)\) is a quasi-b-metric space with constant \(p \geq 1\).

**Remark 20**

1. It is proved that some “fixed-point theorems” for a nondecreasing mapping satisfied the nonlinear contraction.
2. Partially well-ordered complete quasi-b-metric space was used to solve the fractional differential equations [55].

A mapping \(T: X \to X\) is called \(a\)-admissible for nonempty set \(X\), if \(\forall x_1^n, x_2^n \in X\), where \(a: X \times X \to [0, \infty)\) be a mapping, then we have

\[
a(x_1^n, x_2^n) \geq 1 \implies a(T_{x_1^n}, T_{x_2^n}) \geq 1.
\]

**2.4.7. Soft Metric Space** [56, 57]. When difficulties occur due to inadequacy of the parameterization tool of the theory, then the new mathematical tool “the soft set" theory was introduced by Molodtsov, for dealing with uncertainties and inherent difficulties of theories. Das and Samanta initiated the analysis of soft metric space (s.m.u.) created on soft-point soft sets, for implementations of soft set theory in real-life problems and other domains for better performance.

Let \(A\) be a nonempty subset of parameters, and \(N\) be the absolute soft set, i.e., \(F(\lambda) = \lambda^n, \forall \lambda \in A\), where \(F(A) = \lambda^n\).
A mapping \( d: SP(2^n) \times SP(2^n) \rightarrow R(\Lambda)^* \) is called soft metric on \( \Lambda \) if \( d \) is known by T. Bag.

Example 21. Let 2\( b \) = A = \{1/n: n \in \mathbb{N} \}, and the mapping \( d: SP(2^n) \times SP(2^n) \rightarrow R(\Lambda)^* \) be defined by

\[
d(\mathbf{y}^a, \mathbf{y}^b) = \| \mathbf{y}^a - \mathbf{y}^b \| + \| \mathbf{y}^b - \mathbf{y}^a \|,
\]

where 0 \( \leq \tau \leq 1/2 \), and then \( f \) has unique fixed point.

Remark 21. The soft metric space is used to prove the soft version of the “contraction principle” in complete soft metric space, which provides applications related to dynamic programming.

Then, it is easy to show a soft metric space on \( 2^b \).

\[
d(f(\mathbf{y}^a), f(\mathbf{y}^b)) \leq \tau d(\mathbf{y}^a, \mathbf{y}^b), \quad \forall \mathbf{y}^a, \mathbf{y}^b \in SP(2^n),
\]

where 0 \( \leq \tau \leq 1/2 \), and then \( f \) has unique fixed point.

2.4.8. Partial b-Metric Space [58, 59]. S. Shukla introduced partial b-metric space and the generalization of partial metric space (p.m.s.) together with b-metric space.

A mapping \( b: X^2 \rightarrow R^+ \) is s.t.b. partial b-metric (p.b.-m.) on \( X \neq \emptyset \), if it satisfied the subsequent axioms: (for all \( 2^n_1, 2^n_2, 2^n_3 \in X \),)

\[
(1) \quad b(2^n_1, 2^n_2) = b(2^n_2, 2^n_3) = b(2^n_1, 2^n_3) \Rightarrow 2^n_1 = 2^n_2,
\]

\[
(2) \quad b(2^n_1, 2^n_2) \leq \alpha \cdot b(2^n_2, 2^n_3),
\]

\[
(3) \quad b(2^n_1, 2^n_2) = b(2^n_1, 2^n_3) = b(2^n_2, 2^n_3) \Rightarrow 2^n_1 = 2^n_2,
\]

\[
(4) \quad \exists a \text{ real number } s \geq 1 \text{ such that}
\]

\[
b(2^n_1, 2^n_2) \leq s \cdot [b(2^n_1, 2^n_3) + b(2^n_2, 2^n_3)] - b(2^n_1, 2^n_2).
\]

Then, partial b-metric \( b \) together with the set \( X \) is titled as partial b-metric space and represented as an ordered pair \((X, b)\), and \( s \) is the coefficient of \((X, b)\).

Example 22. Let \( X = \mathbb{R}^+, \alpha > 1 \) a constant, and

\[
b(2^n_1, 2^n_2) = [\max\{2^n_1, 2^n_2\}]^\alpha + [2^n_1 - 2^n_2]^{\alpha}, \quad \forall 2^n_1, 2^n_2 \in X.
\]

Then, \((X, b)\) is a partial b-metric space with coefficient \( s = 2^n \alpha > 1 \).

Remark 22. Analog of “Kannan’s type fixed-point theorem” and “Banach’s contraction theorem” was proved in the partial b-metric space.

Theorem 8. Suppose the mapping \( T: X \rightarrow X \) and let \((X, b)\) be a complete partial b-metric space with the coefficient \( s \geq 1 \) that satisfies the following condition:

\[
b(T^n_1, T^n_2) \leq \beta b(2^n_1, 2^n_2).
\]

For all \( 2^n_1, 2^n_2 \in X \) and \( \beta \in [0, 1) \), then \( T \) has a unique fixed point \( u \in X \) and \( b(u, u) = 0 \).

2.4.9. Fuzzy Cone Metric Space [60]. The class of fuzzy metric space (f.m.s.) was extended and generalized into fuzzy cone metric space.

A mapping \( d: X^2 \rightarrow E^+ (I) \) is s.t.b. fuzzy cone metric (f.c.m.) on \((X \neq \emptyset)\), if it satisfied the subsequent axioms:

\[
(1) \quad d(2^n_1, 2^n_2) = 0 \Rightarrow 2^n_1 = 2^n_2;
\]

\[
(2) \quad d(2^n_1, 2^n_2) = d(2^n_2, 2^n_3), \quad \forall 2^n_1, 2^n_2, 2^n_3 \in X;
\]

\[
(3) \quad d(2^n_1, 2^n_2) \leq d(2^n_1, 2^n_3) + d(2^n_3, 2^n_2), \quad \forall 2^n_1, 2^n_2, 2^n_3 \in X.
\]

Then, fuzzy cone metric \( d \) together with the set \( X \) is titled as fuzzy cone metric space and represented as an ordered pair \((X, d)\).

Example 23. Let \( P \) be any cone and \( X = \mathbb{N} \) and \( 2^n_1 \ast 2^n_2 = 2^n_{1+2^n_2} \) and \( d: X^2 \times int(P) \rightarrow [0, 1] \) be defined by

\[
d(2^n_1, 2^n_2) = \lambda_{2^n_1 \ast 2^n_2}, \quad \forall 2^n_1, 2^n_2 \in X \quad \text{and} \quad t > 0.
\]

(1) In fuzzy cone metric space (f.c.m.s.), some of the basic properties for innumerable types of contraction mappings and for “fixed-point theorems” were established.

(2) Chet et al. provide an application of nonlinear integral equation in fuzzy cone metric space for the existence and uniqueness of the solution.

A mapping \( T: X \rightarrow X \) is titled as fuzzy cone contraction on fuzzy cone metric space \((X, d, *)\) if \( \exists k \in (0, 1) \) satisfies the following:

\[
d(T^n_1, T^n_2) - 1 \leq k \frac{1}{d(2^n_1, 2^n_2)},
\]

\[
\forall 2^n_1, 2^n_2 \in X \text{ and } t > 0. \quad t \text{ is called the contraction constant of } T.
\]

2.4.10. Fuzzy Cone b-Metric Space [61]. The definition of fuzzy cone b-metric space (f.c.b-m.s.), inspired by cone b-metric space (c.b-m.s.) and fuzzy metric space, was made known by T. Bag.
A mapping $d: X^2 \to E^*$ (I) is s.t.b. fuzzy cone $b$-metric (where $X \neq \emptyset$ and $s \geq 1$ be a real number) if it fulfills the following axioms: 

\begin{enumerate}
  \item $d(\tau_n^1, \tau_n^2) \geq 0, \forall \tau_n^1, \tau_n^2 \in X$
  \item $d(\tau_n^1, \tau_n^1) = 0 \Rightarrow \tau_n^1 = \tau_n^2$
  \item $d(\tau_n^1, \tau_n^2) = d(\tau_n^2, \tau_n^1), \forall \tau_n^1, \tau_n^2 \in X$;
  \item $d(\tau_n^1, \tau_n^2) \leq s [d(\tau_n^1, \tau_n^3) + d(\tau_n^3, \tau_n^2)]$
\end{enumerate}

Then, fuzzy cone $b$-metric $d$ together with the set $X$ is titled as fuzzy cone $b$-metric space and represented as an ordered pair $(X, d)$.

**Remark 24.** Some results of “Banach’s contraction mapping” for the “fixed-point theorem” were established in fuzzy cone $b$-metric space.

**Theorem 9.** A mapping $T: X \to X$ on a complete fuzzy cone $b$-metric space $(X, d)$ has a unique fixed point in $X$ if for complete fuzzy cone $b$-metric space $(X, d)$ with the coefficient $s \geq 1$ fulfills the subsequent condition:

\[ d(T\tau_n^1, T\tau_n^2) \leq \beta d(\tau_n^1, \tau_n^2), \]

for all $\tau_n^1, \tau_n^2 \in X$ and where $\beta \in [0, 1)$.

**2.4.11. Fuzzy Soft Metric Space [62–64].** For the fuzzy soft metric space (f.s.m.s.), first fuzzy soft point was introduced in which problems deal with ambiguous data in several fields and that was not effectively demonstrated in mathematics. There are various kinds of mathematical instruments to deal with uncertainties, i.e., the fuzzy set and the soft set that helps to solve the problem difficulty in all areas. In the fuzzy soft metric space (f.s.m.s.), the fuzzy soft open ball and also the closed ball were proposed.

Let $X$ be the absolute fuzzy soft set,

\[ A * E \text{ i.e., } X(x) = \tau, \forall \tau \in E. \]

\[ (A)^* \text{ be the set of all non-negative fuzzy soft real numbers.} \]

A mapping $d: FSC(X) \times FSC(X) \to R(A)^*$ is s.t.b. fuzzy soft metric (f.s.m.s.) on the soft set $X$, if it holds the subsequent axioms:

\begin{enumerate}
  \item $d(\tau_n^1, \tau_n^2) \geq 0, \forall \tau_n^1, \tau_n^2 \in FSC(X)$;
  \item $d(\tau_n^1, \tau_n^1) = 0 \Rightarrow \tau_n^1 = \tau_n^2$;
  \item $d(\tau_n^1, \tau_n^2) = d(\tau_n^2, \tau_n^1), \forall \tau_n^1, \tau_n^2 \in FSC(X)$;
  \item $d(\tau_n^1, \tau_n^2) \leq d(\tau_n^1, \tau_n^3) + d(\tau_n^3, \tau_n^2), \forall \tau_n^1, \tau_n^2, \tau_n^3 \in FSC(X)$.
\end{enumerate}

Then, fuzzy soft metric $d$ together with the set $X$ is titled as fuzzy soft metric space and represented as an ordered pair $(X, d)$.

**Remark 25.** Fuzzy soft metric is applied on soft open balls and soft closed balls.

**Theorem 10.** A mapping $T_p: (E, d) \to (E, d)$ is called fuzzy soft contractive mapping on fuzzy soft metric space $(E, d)$ if there exist two fuzzy soft number $\psi \in [0, 1)$ such that

\[ \psi \left( d(T_p\tau_n^1, T_p\tau_n^2) \right) \leq \psi \left( d(\tau_n^1, \tau_n^2) \right), \forall \tau_n^1, \tau_n^2 \in FSC(X). \]

**Theorem 11.** A mapping $(T, \rho): (E, d) \to (E, d)$ is called fuzzy soft $\psi$-contractive mapping on fuzzy soft metric space $(E, d)$ if there exist two fuzzy soft functions $(a, \phi): FSC(E) \times FSC(E) \to \mathbb{R}^*$ and $\psi \in \Psi$ such that

\[ (a, \phi)(\tau_n^1, \tau_n^2) \leq \psi \left( d(T, \rho)\tau_n^1, T, \rho\tau_n^2 \right), \forall \tau_n^1, \tau_n^2 \in FSC(X). \]

**2.4.12. M-Metric Space [65, 66].** In partial metric space, the self-distance of $x$ is less than the distance between two points so another variable is free to take any value. Thus, M-metric space improved the triangle inequality by replacing it with the minimum of self-distance of $x$ and $y$, also improved the triangle inequality.

A function $m: X^2 \to R^*$ is s.t.b. an $m$-metric (where $X \neq \emptyset$), if it satisfies the following conditions:

\begin{enumerate}
  \item $m(\tau_n^1, \tau_n^1) = m(\tau_n^2, \tau_n^2)$ if $\tau_n^1 = \tau_n^2$;
  \item $m(\tau_n^1, \tau_n^2) \leq m(\tau_n^1, \tau_n^3) + m(\tau_n^3, \tau_n^2)$;
  \item $m(\tau_n^1, \tau_n^2) = m(\tau_n^2, \tau_n^1)$;
  \item $(m(\tau_n^1, \tau_n^2) - m(\tau_n^3, \tau_n^2)) \leq \{m(\tau_n^1, \tau_n^3) - m(\tau_n^3, \tau_n^1)\}$ + $(m(\tau_n^3, \tau_n^1) - m(\tau_n^3, \tau_n^2))$.
\end{enumerate}

Then, $M$-metric $m$ together with the set $X$ is titled as $M$-metric space and represented as an ordered pair $(X, m)$ where

\[ m_{\tau_n^1, \tau_n^2} = \min \{m(\tau_n^1, \tau_n^3), m(\tau_n^2, \tau_n^3)\}, \]

\[ M_{\tau_n^1, \tau_n^2} = \max \{m(\tau_n^1, \tau_n^3), m(\tau_n^2, \tau_n^3)\}. \]

**Example 24.** Let $X = [0, \infty)$, then

\[ m(\tau_n^1, \tau_n^2) = \frac{\tau_n^1 + \tau_n^2}{2}. \]

It is easy to get that $m$ is an $m$-metric space on $X$.

**Remark 26.** M. Asadi et al. generalized the contraction mapping on M-metric space for getting the “fixed-point theorem.”

A mapping $T: X \to X$ has a fixed a unique fixed point on a complete $M$-metric space $(X, m)$ if satisfies the following condition:

\[ \exists k \in [0, 1) \text{such that } m(T\tau_n^1, T\tau_n^2) \leq km(\tau_n^1, \tau_n^2), \forall \tau_n^1, \tau_n^2 \in X. \]
or
\[ \exists k \in \left[ 0, \frac{1}{2} \right] \text{such that} (T_2')_n = \left( T_2'' \right)_n \]
\[ \leq k \left( m \left( T_1', T_2' \right)_n, m \left( T_1'', T_2'' \right)_n \right) \forall \left( T_1', T_2' \right)_n, \left( T_1'', T_2'' \right)_n \in X. \]  

2.4.13. 2-Partial Rectangular Metric Space [67]. Partial rectangular metric space (p.m.s) is the generalized version of rectangular metric space (r.m.s) and the extension of partial metric space (p.m.s).

A function \( \rho: X^2 \rightarrow R \) is s.t. partial rectangular metric (p.r.m) on \( X(\neq \emptyset) \) if satisfied the subsequent axioms:

1. \( \rho \left( T_1', T_2' \right)_n \geq 0, \forall T_1', T_2' \in X; \)
2. \( T_1' = T_2' \Rightarrow \rho \left( T_1', T_2' \right)_n = \rho \left( T_2', T_1' \right)_n = \rho \left( T_2', T_2' \right)_n, \forall T_1', T_2' \in X; \)
3. \( \rho \left( T_1', T_2' \right)_n \leq \rho \left( T_1'', T_2'' \right)_n, \forall T_1'', T_2'' \in X; \)
4. \( \left( \rho \left( T_1', T_2' \right)_n \right) \leq \rho \left( T_1', T_2' \right)_n, \forall T_1', T_2' \in X; \)
5. \( \rho \left( T_1', T_2' \right)_n \leq \rho \left( T_1', T_2' \right)_n + \rho \left( T_1', T_2' \right)_n + \rho \left( T_1', T_2' \right)_n - \rho \left( T_1', T_2' \right)_n, \forall T_1', T_2' \in X \) and for all distinct points \( T_1', T_2' \in X \).

Then, partial rectangular metric \( \rho \) together with the set \( X \) is titled as partial rectangular metric space and represented as an ordered pair \( (X, \rho) \).

Example 25. Let \( X = [0, b], \varphi \geq \varphi \geq 3 \) define a mapping \( \rho: X^2 \rightarrow R \) by
\[ \rho \left( T_1', T_2' \right)_n = \begin{cases} \frac{3b + \rho \left( T_1', T_2' \right)_n + \rho \left( T_1', T_2' \right)_n}{2}, & \text{if } T_1', T_2' \in \{1, 2\}, T_1' \neq T_2', \\ \frac{b + \rho \left( T_1', T_2' \right)_n + \rho \left( T_1', T_2' \right)_n}{2}, & \text{otherwise}. \end{cases} \]  

Then, it is easy to see that \( (X, d) \) is the partial rectangular metric space.

Remark 27. Y. Cheng and Y. C. Shiab presented quasi-type contraction for providing the "fixed-point theorem."

A mapping \( T: X \rightarrow X \)

1. is called quasi-contraction for complete partial rectangular metric space \( (X, \rho) \) with constant \( k \in [0, 1] \) so that \( \forall T_1', T_2' \in X \) satisfies the following condition:

\[ \rho \left( T_2', T_2' \right)_n \leq k \max \{ \rho \left( T_1', T_2' \right)_n, \rho \left( T_1', T_2' \right)_n, \rho \left( T_1', T_2' \right)_n, \rho \left( T_1', T_2' \right)_n \}. \]  

Example 26. Let \( (X, d, k) \) be b-metric space. Let \( M_d: X \times X \times [0, \infty) \rightarrow [0, 1] \) be defined by
\[ M_d \left( T_1', T_2', t \right) = \begin{cases} \hinspp \frac{t}{t + d \left( T_1', T_2' \right)_n}, & t > 0, \\ 0, & \text{if } t = 0. \end{cases} \]  

Then, \( (X, M_d, \wedge, k) \) be fuzzy b-metric space.

Remark 28.

1. Abbas et al. proposed the definition of "\( \psi \)-contraction" and monotone "\( \psi \)-contraction" in fuzzy b-metric space.
2. "Banach’s contraction principle" is extended as an application to the linear equation.

Let a mapping \( \mathcal{F}: X \rightarrow X \) be correspondence in fuzzy b-metric space such that \( \forall T_1', T_2' \in X \) and \( \mathcal{F}_1 \in \mathcal{F} \left( T_1' \right)_n, \mathcal{F}_2 \in \mathcal{F} \left( T_2' \right)_n \) satisfy the following condition:
\[ M \left( \mathcal{F}_1, \mathcal{F}_2, t \right) \leq M \left( T_1', T_2', \frac{t}{k} \right), \]  

where \( k \in [0, 1] \) and \( t \in R \). Then, \( \mathcal{F} \) has unique fixed point.
Chauhan and Utreja [69] proved one important result on fuzzy 2-metric space on the same line.

### 2.4.15. Fuzzy Quasi-b-Metric Space [68].
Nadaban extended the perception of fuzzy quasi-metric space (f.q.m.s.) and b-metric space (b-m.s.).

Let $M$ be a fuzzy set in $X \times X \times [0, \infty)$ (where $X \neq \emptyset$), $*$ is a continuous t-norm, and $k \geq 1$ is a real number) is called fuzzy quasi-b-metric space if it satisfies the subsequent condition: $(\forall 2^\alpha, 2^\beta, 2^\gamma) \in X$, we have the following:

1. $M(2^\alpha, 2^\beta, 0) = 0$;
2. $M(2^\alpha, 2^\beta, 2^\gamma, t) = M(2^\alpha, 2^\beta, t) = 1, \forall t > 0] \Rightarrow 2^\gamma = 2^\alpha$;
3. $M(2^\alpha, 2^\beta, k(t + s)) \geq M(2^\alpha, 2^\beta, t) \cdot M(2^\beta, 2^\gamma, s), \forall t, s \geq 0$;
4. $M(2^\alpha, 2^\beta, \cdot)$: $[0, \infty) \rightarrow [0, 1]$ is left continuous, and $\lim_{\alpha \rightarrow \infty} M(2^\alpha, 2^\beta, 0) = 1$.

Then, fuzzy quasi-b-metric $M$ together with the set $X$ is titled fuzzy quasi-b-metric space and represented as quadruple $(X, M, \ast, k)$.

**Example 27.** Let $(X, d, k)$ be quasi-b-metric space. Let $M_d: X \times X \times [0, \infty) \rightarrow [0, 1]$ be defined by

$$M_d(2^\alpha, 2^\beta, t) = \begin{cases} t & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then, $(X, M_d, \land, k)$ be fuzzy quasi-b-metric space.

**Remark 29.** Fuzzy quasi-b-metric space (f.q.b-m.s.) is used to apply in the control theory and convex optimization.

Let $(X, M_d, \land, k)$ be fuzzy quasi-b-metric space and

$$M(2^{\alpha}, 2^{\beta}, t) = \min \{M_d(2^{\alpha}, 2^{\beta}, t), M_d(2^{\beta}, 2^{\gamma}, t)\}. \quad (63)$$

Then, $(X, M, \land, k)$ is a fuzzy b-metric space.

### 2.4.16. M-b-Metric Space [70–72].

The $M_b$-metric space is an enlargement of M-metric space by introducing the coefficient of the $M_b$-metric space (real number $s \geq 1$) in triangle equality. The concept of an extension was based on the simulation functions made known by Khojasteh et al.

A function $m_b: X^2 \rightarrow \mathbb{R}^+$ is s.t.b. $M_b$-metric space on $X \neq \emptyset$, if it satisfied the subsequent conditions: $(\forall 2^\alpha, 2^\beta, 2^\gamma) \in X$, we have

1. $m_b(2^\alpha, 2^\beta, 2^\gamma) = m_b(2^\beta, 2^\gamma, 2^\alpha) = m_b(2^\alpha, 2^\beta, 2^\gamma)$ if $2^\gamma = 2^\alpha$;
2. $m_b(2^\alpha, 2^\beta, 2^\gamma, t) \leq m_b(2^\alpha, 2^\beta, 2^\gamma)$;
3. $m_b(2^\alpha, 2^\beta, 2^\gamma) = m_b(2^\beta, 2^\gamma, 2^\alpha)$;
4. $\exists a$ real number $s \geq 1$ s.t.b. $(\forall 2^\alpha, 2^\beta, 2^\gamma) \in X$, we have $\leq s\left[m_b(2^\alpha, 2^\beta, 2^\gamma) + m_b(|2^\alpha|, |2^\beta|, |2^\gamma|)\right] - m_b(2^\alpha, 2^\beta, 2^\gamma)$.

(64)

Then, $M_b$-metric $M_b$ together with the set $X$ is titled $M_b$-metric space and represented as quadruple $(X, M_b)$. The number $s$ is termed as the coefficient of the $M_b$-metric space $(X, M_b)$.

$$m_b(2^\alpha, 2^\beta, 2^\gamma) = \min \{m_b(2^\alpha, 2^\beta, 2^\gamma), m_b(2^\beta, 2^\gamma, 2^\alpha), m_b(2^\alpha, 2^\gamma, 2^\beta)\}. \quad (65)$$

**Example 28.** Let $X = R^+, \rho > 1$ be a constant, and $M_b: X \times X \rightarrow \mathbb{R}^+$ be defined by

$$M_b(2^\alpha, 2^\beta, 2^\gamma) = \max \{2^\alpha, 2^\beta\} + |2^\gamma - 2^\alpha|, \forall 2^\alpha, 2^\beta, 2^\gamma \in X. \quad (66)$$

Then, $(X, M_b)$ is an $M_b$-metric space with coefficient $s = 2^\rho > 1$.

**Remark 30.** We demonstrate the existence of the “fixed-point theorem” with the assistance of $\eta Z_{mb}$ contraction in complete $M_b$-metric space.

**Theorem 12.** Suppose the mapping $T: X \rightarrow X$ and let $(X, M_b)$ be a complete $M_b$-metric space with the coefficient $s \geq 1$ that satisfies the following condition:

$$M_b(T2^\alpha, 2^\beta) \leq \beta b(2^\alpha, 2^\beta), \quad \beta \in [0, 1]. \quad (67)$$

Then, $T$ has a unique fixed point $u \in X$ and $M_b(u, u) = 0$.

### 2.4.17. Orthogonal Metric Space [73, 74].

Gordji et al. defined a binary relation on the set.

Let $d$ be a usual metric on $(X \neq \emptyset)$ with an orthogonal set $(X, \perp)$. Then, metric with an orthogonal set is termed as orthogonal metric space and represented as $(X, \perp, d)$ (O-metric space).

**Example 29.** Let $X = [0, \infty)$, and we define $2^\alpha \perp 2^\beta$, if $2^\alpha 2^\beta \leq 2^\alpha$ or $2^\alpha 2^\beta \leq 2^\beta$. Then, $\perp = 0 \cup \perp$ is an orthogonal set, and if $(X, d)$ be an usual metric, then $(X, \perp, d)$ is orthogonal metric space.

**Remark 31.** Gordji et al. gave a generalized version of “Banach’s contraction principle” with the orthogonal set, which stands for finding the existence and uniqueness of first-order ordinary differential equations and nonlinear fractional differential equations that are allied with Caputo fractional derivative.

A mapping $T: X \rightarrow X$ is called the following:
Remark 32. Let $X = \mathbb{N}$ define $d : X^2 \rightarrow X$ by

$$d(\Sigma_1^n, \Sigma_2^n) = \begin{cases} 0, & \text{if } \Sigma_2^n = \Sigma_1^n, \\ 4\rho, & \text{if } \Sigma_2^n, \Sigma_1^n \in \{1, 2\} \text{and } \Sigma_2^n \neq \Sigma_1^n, \\ \rho, & \text{if } \Sigma_2^n \text{ or } \Sigma_1^n \notin \{1, 2\} \text{and } \Sigma_2^n \neq \Sigma_1^n, \end{cases}$$

where $\rho > 0$ is a constant. Then, $(X, d)$ is rectangular $b$-metric space with coefficient $s = 4/3 > 1$.

Remark 33. George et al. gave an analog of “Kannan’s fixed-point theorem” and “Banach’s contraction principle” for generalizing several outcomes in fixed-point theory.

A mapping $T : X \rightarrow X$ has a unique fixed point on a complete rectangular $b$-metric space $(X, d)$ with coefficient $s > 1$, if satisfies the following condition:

$$d(T\Sigma_1^n, T\Sigma_2^n) \leq k d(\Sigma_1^n, \Sigma_2^n),$$

where $k \in [0, 1/s] \forall \Sigma_1^n, \Sigma_2^n \in X$ or $d(T\Sigma_1^n, T\Sigma_2^n) \leq k d(\Sigma_1^n, \Sigma_2^n), (\Sigma_1^n, T\Sigma_2^n), (\Sigma_2^n, T\Sigma_1^n)).$ (73)

where $k \in [0, 1/(s + 1)], \forall \Sigma_1^n, \Sigma_2^n \in X.$
Then, extended fuzzy \( b \)-metric \( M_a \) together with the set \( X \) is titled as extended fuzzy \( b \)-metric space and represented as \((X, M_a, \ast, \alpha(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}))\).

**Example 32.** Let \( X = \{1, 2, 3\} \) and let \( d_{ij}: X \times X \to \mathbb{R} \) by \( d(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}) = (\Psi_{\gamma_{1}} - \Psi_{\gamma_{2}})^2 \), then it is easy to get that \((X, d)\) is a \( b \)-metric space. We define the mapping \( \alpha: X \times X \to [1, \infty), \)
\[
\alpha(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}) = 1 + 2 \Psi_{\gamma_{1}} + 2 \Psi_{\gamma_{2}}.
\]  
(75)

Let \( M_{a}: X \times X \times [0, \infty) \to [0, \infty) \) be given by
\[
M_{a}(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}, t) = \begin{cases} 
\frac{t}{t + d(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}})} & \text{if } t > 0, \\
0 & \text{if } t = 0,
\end{cases}
\]  
(76)

and continuous \( t \)-norm \( \ast = \wedge \); then, \( \Psi_{\gamma_{1}} \ast \Psi_{\gamma_{2}} = \Psi_{\gamma_{1}} \wedge \Psi_{\gamma_{2}} \).

Then, \((X, M_{a}, \wedge, k)\) be fuzzy quasi-\( b \)-metric space.

**Remark 34.** Mehmoood et al. established a new kind of “Banach-type fixed-point theorem” with the help of a fuzzy set in the most universal class of fuzzy \( b \)-metric space.

**Theorem 13.** (Banach’s contraction theorem in fuzzy extended \( b \)-metric spaces).

Let \((X, M_{a}, \ast)\) be a \( G \)-complete extended fuzzy \( b \)-metric space with the mapping \( \alpha: X \times X \to [1, \infty) \) such that
\[
\lim_{t \to \infty} M_{a}(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}, t) = 1.
\]  
(77)

Let \( g: X \to X \) be a mapping, which satisfies the condition
\[
M_{a}(g\Psi_{\gamma_{1}}, g\Psi_{\gamma_{2}}, kt) \geq M_{a}(\Psi_{\gamma_{1}}, \Psi_{\gamma_{2}}, t), \forall \Psi_{\gamma_{1}}, \Psi_{\gamma_{2}} \in X,
\]  
(78)

where \( k \in (0, 1) \). Furthermore, \( g \) has a unique fixed point if for an arbitrary \( b_0 \in X \), and \( n, q \in \mathbb{N} \), we have
\[
\alpha\left(b_n, b_{n+q}\right) < \frac{1}{k}.
\]  
(79)

where \( b_n = g^{n}b_0 \).

2.4.21. **Extended \( M_b \)-Metric Space** [70]. The coefficient of \( M_b \)-metric space was replaced with the function \( \theta \) and formed a new generalized version of \( M_b \)-metric space, titled as extended \( M_b \)-metric space.

A function \( m_{b}: X^2 \to [0, \infty) \) is titled as extended \( M_b \)-metric (\( eM_b \)-m) on \( X \) (where \( \theta: X^2 \to [1, \infty) \) be a function) if the subsequent conditions:

1. \( m_b(\Psi_{\gamma_1}, \Psi_{\gamma_2}) = m_b(\Psi_{\gamma_2}, \Psi_{\gamma_1}) \) if \( \Psi_{\gamma_2} = \Psi_{\gamma_1} \);
2. \( m_{\theta \Psi_{\gamma_1}, \Psi_{\gamma_2}} \leq m_b(\Psi_{\gamma_1}, \Psi_{\gamma_2}) \);
3. \( m_b(\Psi_{\gamma_1}, \Psi_{\gamma_2}) = m_b(\Psi_{\gamma_2}, \Psi_{\gamma_1}) \);
4. \( (m_b(\Psi_{\gamma_1}, \Psi_{\gamma_2}) - m_{\theta \Psi_{\gamma_1}, \Psi_{\gamma_2}}) \leq \theta(\Psi_{\gamma_1}, \Psi_{\gamma_2}) [m_b(\Psi_{\gamma_1}, \Psi_{\gamma_2}) - m_{\theta \Psi_{\gamma_1}, \Psi_{\gamma_2}}] \) for all \( \Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3} \in X \).

Then, extended \( M_b \)-metric \( m_{b} \) together with the set \( X \) is titled as extended \( M_b \)-metric space and represented as the ordered pair \((X, m_{b})\).

**Example 33.** Let \( X = \mathbb{C} ([\alpha, \beta], \mathbb{R}) \), and define \( m_{b}; X^2 \to [0, \infty) \) and \( \theta: X^2 \to [1, \infty) \) by
\[
m_{b}(\Psi_{\gamma_1}(t), \Psi_{\gamma_2}(t)) = \sup_{t \in [\alpha, \beta]} |\Psi_{\gamma_1}(t) - \Psi_{\gamma_2}(t)|^2,
\]  
\(
\theta(\Psi_{\gamma_1}(t), \Psi_{\gamma_2}(t)) = |\Psi_{\gamma_1}(t)| + |\Psi_{\gamma_2}(t)| + 2.
\)

Then, \((X, m_{b})\) is an extended \( M_b \)-metric space.

**Remark 35.** N. Mlaiki et al. provide several fixed-point results on expanded \( M_b \)-metric space (\( M_b \)-m.s.) using the techniques of the classical fixed-point theorems such as “Kannan’s fixed-point theory,” the “Banach fixed-point theory,” and many more.

**Theorem 14.** Let \((X, m_{b})\) be an extended \( M_b \)-metric space and \( m_{b}^\theta; X^2 \to [0, \infty) \) be a function defined as
\[
m_{b}^\theta(\Psi_{\gamma_1}(t), \Psi_{\gamma_2}(t)) = m_{b}(\Psi_{\gamma_1}, \Psi_{\gamma_2}) - 2m_{b}(\Psi_{\gamma_1}, \Psi_{\gamma_3}) + M_{\theta}(\Psi_{\gamma_1}, \Psi_{\gamma_2}).
\]  
(81)

for all \( \Psi_{\gamma_1}, \Psi_{\gamma_2} \in X \). Then, \( m_{b}^\theta \) is extended \( b \)-metric and the pair \((X, m_{b}^\theta)\) is an extended \( b \)-metric space.

2.4.22. **Soft \( S \)-Metric Space** [81]. Aras et al. gave the theory of soft \( S \)-metric space in which \( S \)-metric was further defined on soft points of soft sets and introduced some of their characteristics as well.

Let \( S \) be the absolute soft set, \( E \) be the \( (\neq \{\emptyset\}) \) set of parameters; \( SP(S) \) be a collection of all soft points of \( S \), and \( R(E)^{\prime} \) be the set of all non-negative soft real numbers.

A mapping \( \delta; SP(S) \times SP(S) \times SP(S) \to R(E)^{\prime} \) is s.t. soft \( S \)-metric on \( S \) (the soft set), if it fulfills the subsequent axioms, for each soft points \( \Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3}, \Psi_{\gamma_4} \in SP(S) \),
\[
\begin{align*}
(1) & \quad S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3}) \geq S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_4}) \\
(2) & \quad S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3}) = 0 \iff \Psi_{\gamma_1} = \Psi_{\gamma_2} = \Psi_{\gamma_3} \\
(3) & \quad S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3}) \leq S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_4}) + S(\Psi_{\gamma_2}, \Psi_{\gamma_3}, \Psi_{\gamma_4})
\end{align*}
\]  
(82)

Then, soft \( S \)-metric \( \delta \) together with the set \( S \) is titled as soft \( S \)-metric space and represented as \((S, \delta, E)\).

**Example 34.** Let \( E = \mathbb{N} \) be a parameter set, and \( X = \mathbb{R} \) be defined as
\[
S(\Psi_{\gamma_1}, \Psi_{\gamma_2}, \Psi_{\gamma_3}) = |a - c| + |b - c| + |\Psi_{\gamma_1} - \Psi_{\gamma_3}| + |\Psi_{\gamma_2} - \Psi_{\gamma_3}|.
\]

It is easy to get a soft \( S \)-metric space.

**Remark 36.** The idea of “soft contractive mapping” on soft \( S \)-metric space (s.S-m.s.) was established, and some significant concepts were given, for instance, completeness, Cauchy sequence, soft continuous mapping on soft \( S \)-metric space (s.S-m.s.), and also with the help of “soft contractive
mapping" on soft $S$-metric space (s.S-m.s.), and some of the "fixed-point theorems" have been demonstrated.

A mapping $(\rho_1, \rho_2): (S, g, E) \rightarrow (S, g, E)$ has a unique soft fixed point on a soft sequential compact soft $S$-metric space $(S, g, E)$ if satisfies the following condition:

$$S((\rho_1, \rho_2)(2^n_1), (\rho_1, \rho_2)(2^n_2), (\rho_1, \rho_2)(2^n_3)) \leq a[S((\rho_1, \rho_2)(2^n_1), (\rho_1, \rho_2)(2^n_2), (\rho_1, \rho_2)(2^n_3), (\rho_1, \rho_2)(2^n_4))],$$

(83)

for every $2^n_1, 2^n_2 \in SP(S)$, where $a \in [0, 1/2]$ is a soft constant real number. Then, $(\rho_1, \rho_2)$ have a unique soft fixed point in $SP(S)$.

2.4.23. $g$-Metric Space [82]. Due to today’s large and complex data collection, we developed the notion of distance function that is called $g$-metric in which we extended the ordinary distance function of metric (two points in between) and $G$-metric (three points in between). The $g$-metric is the distance function of degree $n$, i.e., for $n+1$ points.

A function $g$: $X^n \longrightarrow R^n$ is s.t.b. g-metric with order $n$ on $X \neq \varnothing$ (where $X^n = \prod_{i=1}^{n} X$), if it fulfills the following axioms:

1. Positive definiteness

$$\forall x_0, \ldots, x_n, y_0, \ldots, y_n, w \in X \text{ with } t + 1 = n,$$

$$g(x_0, \ldots, x_n, y_0, \ldots, y_n) \leq g(x_0, \ldots, x_n, w, \ldots, w) + g(y_0, \ldots, y_n, w, \ldots, w).$$

(89)

Then, $g$-metric $g$ together with the set $X$ is titled as $g$-metric space and represented as an ordered pair $(X, g)$.

**Example 35.** Let $X \neq \varnothing$ be defined $d$: $X^n \longrightarrow \mathbb{R}_+$ by

$$d(x_0, x_n) = \begin{cases} 0, & \text{if } x_0 = \ldots = x_n, \forall x_0, x_n \in X. \\ 1, & \text{otherwise} \end{cases}$$

(90)

It is easy to get $(X, d)$ that is a $g$-metric space.

**Remark 37.** Several "fixed-point theorems" in the $g$-metric space (g-m.s.) were generalized, which is considered as an oversimplification of "Banach’s contractive mapping" concerning $g$-metric space.

A mapping $T: X \longrightarrow X$ has a unique fixed point on a complete $g$-metric space $(X, g)$ if satisfies the condition:

$$\exists k \in (0, 1) \text{ such that } g(T^n_1, T^n_2, \ldots, T^n_n) \leq kg(T^n_1, T^n_2, \ldots, T^n_n), \forall T^n_1, T^n_2, \ldots, T^n_n \in X.$$

(91)

for every $T^n_1, T^n_2 \in SP(S)$ and $T^n_1 \neq T^n_2$.

A mapping $(\rho_1, \rho_2): (S, g, E) \rightarrow (S, g, E)$ has a unique soft fixed point on a complete soft $S$-metric space $(S, g, E)$ if satisfies the following contraction condition:

$$g(x_0, x_n) = 0 \iff x_0 = \ldots = x_n.$$

(85)

2. Permutation invariant

$$g(x_0, x_n) = g(x_{\tau(0)}, x_{\tau(n)}).$$

(86)

For any permutation $\tau \in [0, 1, n]$,

3. Monotonicity

$$g(x_0, x_n) \leq g(y_0, y_n), \forall (x_0, x_n), (y_0, y_n) \in X^{n+1},$$

(87)

with

$$[x_i: i = 0, n] \subseteq [y_i: i = 0, n].$$

(88)

4. Triangle inequality

$$g(x_0, x_n) \leq g(x_0, x_i) + g(x_i, x_n).$$

(89)

2.4.24. $F$-Metric Space [83, 84]. $F$-metric is a generalization of metric space in which we defined the natural topology $\tau_F$ with the different topological properties.

The mapping $D$: $X^n \longrightarrow [0, +\infty$ (where $X \neq \varnothing$) is s.t.b. $F$-metric, if fulfills the following axioms:

We suppose that $\exists (f, g) \in F \times [0, +\infty)$ such that

1. $(T^n_1, T^n_2) \in X \times X, D(T^n_1, T^n_2) = 0$ if $T^n_1 = T^n_2$;
2. $(T^n_1, T^n_2) = D(T^n_1, T^n_2); \forall (T^n_1, T^n_2) \in X \times X$;
3. $\forall (T^n_1, T^n_2) \in X \times X, \forall \nu \in N, \nu \geq 2, \forall (u_1, u_2) \subset X$ with $D(u_1, u_2) = (T^n_1, T^n_2)$; we have $D((1)^N_1, (2)^N_2) > 0$

$$\Rightarrow D((1)^N_1, (2)^N_2) \leq f \left( \sum_{i=1}^{N-1} D(u_i, u_{i+1}) + g \right).$$

(91)

Then, $F$-metric $D$ together with the set $X$ is titled as $F$-metric space and represented as an ordered pair $(X, D)$. 


Example 36. Let $X = \mathbb{R}$ be defined $d: X^2 \to \mathbb{R}_+$ by

$$d(\mathbf{z}_1, \mathbf{z}_2) = \begin{cases} (\mathbf{z}_1^1 - \mathbf{z}_2^1)^2, & \text{if } (\mathbf{z}_1^1, \mathbf{z}_2^1) \in [0, 3] \times [0, 3], \forall \mathbf{z}_1, \mathbf{z}_2 \in X, \\ \mathbf{z}_1^1 - \mathbf{z}_2^1, & \text{if } (\mathbf{z}_1^1, \mathbf{z}_2^1) \not\in [0, 3] \times [0, 3] \end{cases}$$

with $f(t) = \ln(t)$ and $\rho = \ln(t)$. It is easy to get $(X, d)$ that is a $F$-metric space.

Remark 38. Jelli and Samet proved the modified version of the “Banach contraction principle.” Afterward, Ansari et al. established some circumstances for the existence of a solution in the “fuzzy fixed-point theorem” used for solving the fuzzy Cauchy problem.

A mapping $T: X \to X$ has a unique fixed point on a complete $F$-metric space $(X, \mathcal{D})$ if satisfies the following condition: $\exists k \in (0, 1)$ such that $D(T^2 x, T^2 y) \leq kD(x, y)$, $\forall x, y \in X$.

2.4.25. Controlled Metric Type Space [85]. The controlled metric is an extension of $b$-metric in which we introduced a control function $\alpha(x, z)$ in the $b$-triangle inequality.

The mapping $d: X^2 \to [0, \infty)$ is s.t.b. controlled metric type (where $\varphi: X \times X \to [1, \infty)$ and $X \neq \emptyset$), if fulfills the following axioms:

1. $d(\mathbf{C}_1, \mathbf{C}_2) = 0$ if $\mathbf{C}_1 = \mathbf{C}_2$;
2. $d(\mathbf{C}_1, \mathbf{C}_2) = d(\mathbf{C}_2, \mathbf{C}_1)$;
3. $d(\mathbf{C}_1, \mathbf{C}_2) \leq \varphi(\mathbf{C}_1, \mathbf{C}_3)d(\mathbf{C}_1, \mathbf{C}_3) + \varphi(\mathbf{C}_3, \mathbf{C}_2)d(\mathbf{C}_3, \mathbf{C}_2)$, for all $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3 \in X$.

Then, controlled metric $d$ together with the set $X$ is titled as controlled metric space and represented as an ordered pair $(X, d)$.

Example 37. Let $X = \{0, 1, 2\}$ and a function $d$ be given as

$$d(0, 0) = d(1, 1) = d(2, 2) = 0, \quad d(0, 1) = d(1, 0) = 1, \quad d(0, 2) = d(2, 0) = \frac{1}{2}$$

We take $\beta: X \times X \to [0, \infty)$ to be symmetric and is defined by the following:

$$\beta(0, 0) = \beta(1, 1) = \beta(2, 2) = \beta(0, 2) = 1, \quad \beta(1, 2) = \frac{5}{4}$$

It is easy to get that $d$ that is a controlled metric space.

Remark 39. Mlaki et al. proved the contraction principle on the controlled metric type space.

A mapping $T: X \to X$ on a complete controlled metric space is $(X, d)$ and satisfies the following condition:

$$d(T^2 x, T^2 y) \leq k d(x, y), \forall x, y \in X,$$

where $\exists k \in (0, 1)$. For $x_0 \in X$, we take $x_{k+1} = T^k x_0$. Suppose that

$$\lim_{n \to \infty} \frac{\beta(2, 2) + \beta(0, 0)}{2} \leq \frac{1}{k}$$

In additional, we assume that $\forall x \in X$ and we have

$$\lim_{n \to \infty} \beta(2, 0) \leq k \beta(0, 0).$$

Then, $T$ has a fixed a unique fixed point.

2.4.26. Rectangular M-Metric Space [86]. Rectangular $M$-metric space is an extension and generalization of the class of $M$-metric space in addition to partial rectangular metric space.

The mapping $m: X \times X \to [0, \infty)$ is s.t.b. rectangular $M$-metric space (where $X \neq \emptyset$) if satisfies the following conditions: $(\forall \mathbf{R}_1, \mathbf{R}_2 \in X)$

1. $m_r(\mathbf{R}_1, \mathbf{R}_2) = m_r(\mathbf{R}_1, \mathbf{R}_2)$.
2. $m_r(\mathbf{R}_1, \mathbf{R}_2) \leq m_r(\mathbf{R}_1, \mathbf{R}_2)$.
3. $m_r(\mathbf{R}_1, \mathbf{R}_2) = m_r(\mathbf{R}_1, \mathbf{R}_2)$.
4. $m_r(\mathbf{R}_1, \mathbf{R}_2) = m_r(\mathbf{R}_1, \mathbf{R}_2) + m_r(\mathbf{R}_1, \mathbf{R}_2) + m_r(\mathbf{R}_1, \mathbf{R}_2)$.

Then, rectangular $M$-metric $m$ together with the set $X$ is titled as rectangular $M$-metric space and represented as an ordered pair $(X, m)$, where

$$m_r(\mathbf{R}_1, \mathbf{R}_2) = \min \{m_r(\mathbf{R}_1, \mathbf{R}_2), m_r(\mathbf{R}_2, \mathbf{R}_1)\}, \forall \mathbf{R}_1, \mathbf{R}_2 \in X.$$
2.4.27. Rectangular $M_b$-Metric Space [71]. Inspired by the theory of rectangular $M$-metric space ($M$-m.s.) and $M_b$-metric space ($M_b$-m.s.), Asim et al. established the rectangular $M_b$-metric space to get well and enlarge the existing results of the literature.

A mapping $m_{rb}: X^2 \to R^*$ is s.t.b. a rectangular $M_b$-metric with coefficient $s \geq 1$, if $m_{rb}$ satisfies the following $(\forall \mathbf{r}, \mathbf{r}_2, \mathbf{r}_3 \in X$ and all distinct $\mathbf{r}_4, \mathbf{r}_5 \in X)$:

$$m_{rb}(\mathbf{r}_1, \mathbf{r}_2) = m_{rb}(\mathbf{r}_1, \mathbf{r}_3) = m_{rb}(\mathbf{r}_2, \mathbf{r}_3)$$

if and only if $\mathbf{r}_1 = \mathbf{r}_2$;

$$m_{rb}(\mathbf{r}_1, \mathbf{r}_2) \leq m_{rb}(\mathbf{r}_2, \mathbf{r}_3)$$

Then, rectangular $M_b$-metric $m_{rb}$ together with the set $X$ is defined as rectangular $M_b$-metric space and represented as an ordered pair $(X, m_{rb})$.

Example 39. Let $X = R^+, \rho > 1$ a constant, and $m_{rb}: X \times X \to R^*$ be defined by

$$m_{rb}(\mathbf{r}_1, \mathbf{r}_2) = \max(\mathbf{r}_1, \mathbf{r}_2) + |\mathbf{r}_1 - \mathbf{r}_2|, \forall \mathbf{r}_1, \mathbf{r}_2 \in X.$$

Then, $(X, m_{rb})$ is a rectangular $M_b$-metric space with coefficient $s = 3\rho^\rho > 1$.

Remark 41. In rectangular $M_b$-metric space, M. Asim et al. proved an equivalence of the “Banach contraction principle.”

A mapping $T: X \to X$ defined on a complete rectangular $M_b$-metric space $X$ with coefficient $s \geq 1$ satisfies the following contraction condition:

$$m_{rb}(T\mathbf{r}_1, T\mathbf{r}_2) \leq km_{rb}(\mathbf{r}_1, \mathbf{r}_2), \forall \mathbf{r}_1, \mathbf{r}_2 \in X,$$

where $k \in [0, 1/s)$. Then, $T$ has a unique fixed point $\mathbf{r}_0$ such that $m_{rb}(\mathbf{r}_0, \mathbf{r}_0) = 0$.

2.4.28. $R$-metric Space [87]. In the $R$-metric space, additionally, an arbitrary relation ($R$) is defined between the points on a nonempty set.

Suppose $(X, d)$ is a metric space (m.s.) and $R$ be a relation on $X (\neq \emptyset)$, then triple $(X, d, R)$ or $X$ is called $R$-metric space.

Example 40. Let $X = [0, \infty)$ equipped with Euclidean metric. We define $xRy$ if $\mathbf{r}_0^\alpha \mathbf{r}_0^\beta \leq (\mathbf{r}_0^\alpha \vee \mathbf{r}_0^\beta)$, where $\mathbf{r}_0^\alpha \vee \mathbf{r}_0^\beta = \mathbf{r}_0^\alpha$ or $\mathbf{r}_0^\beta$.

Then, $(X, d, R)$ is a $R$-metric space.

Remark 42. It was difficult to discover the exact value of the equilibrium point, so Khalegholi et al. tried to provide a structural method for determining the exact value of the fixed point, which is useful to solve a problem related to economics and game theory.

A mapping $T: X \to X$ is called $R$-contraction with Lipschitz constant $0 < k < 1$ if $V_1^\alpha \mathbf{r}_0^\beta \in X$ such that $\mathbf{r}_0^\alpha R_1^\beta$, then we have the following:

$$d(T\mathbf{r}_1, T\mathbf{r}_2) \leq kd(\mathbf{r}_1, \mathbf{r}_2).$$

Theorem 15. A mapping $T: X \to X$ on a complete metric space $(X, d)$ has a unique fixed point in $X$ if for some $\beta \in (0, 1)$ in a complete metric satisfies the following condition $d(T\mathbf{r}_1, T\mathbf{r}_2) \leq \beta d(\mathbf{r}_1, \mathbf{r}_2)$, for all $\mathbf{r}_1, \mathbf{r}_2 \in X$.

2.4.29. Soft $D$-Metric Space [88]. Aras et al. provided the theory of soft $D$-metric space ($D$-m.s.), which was further defined on soft points of soft sets and defined the soft $\bullet$-distance on complete soft $D$-metric by using the designation of soft $D$-metric.

A mapping $D: SP(X) \times SP(X) \times SP(X) \to R(E)$ is s.t.b. soft $D$-metric on $X$ (where $X \neq \emptyset$) if $D$ fulfills the subsequent conditions, for each soft points $\mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma, \mathbf{r}_0^\delta \in SP(X)$,

$$D(\mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma, \mathbf{r}_0^\delta) \geq 0 \Rightarrow \mathbf{r}_0^\alpha = \mathbf{r}_0^\beta = \mathbf{r}_0^\gamma = \mathbf{r}_0^\delta;$$

$$D(\mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma, \mathbf{r}_0^\delta) = D(\mathbf{r}_0^\gamma, \mathbf{r}_0^\delta, \mathbf{r}_0^\alpha, \mathbf{r}_0^\beta);$$

$$D(\mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma, \mathbf{r}_0^\delta) = D(\mathbf{r}_0^\delta, \mathbf{r}_0^\gamma, \mathbf{r}_0^\beta, \mathbf{r}_0^\alpha);$$

Then, soft $D$-metric $D$ together with the set $X$ is titled as soft $D$-metric space and represents as $(X, D, E)$.

Example 41. Let $X$ be a nonempty set, and $E$ be the nonempty set of parameters. If we define a mapping $D: SP(X) \times SP(X) \times SP(X) \to R(E)$ by

$$D(\mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma) = \begin{cases} 0, & \mathbf{r}_0^\alpha \mathbf{r}_0^\beta \mathbf{r}_0^\gamma \in SP(X), \\ 1, & \text{otherwise} \end{cases}, \forall \mathbf{r}_0^\alpha, \mathbf{r}_0^\beta, \mathbf{r}_0^\gamma \in SP(X),$$

then it is easy to show that $D$ is a soft $D$-metric space.

Remark 43.

(1) "Fixed-point theorem" of soft continuous mappings on soft $D$-metric space has been proven.

(2) It is given the "fixed-point theorem" using the principle of soft $\Delta$-distance.

Theorem 16. Let $(X, D, E)$ be a complete $D$-metric space and $\bullet$-distance of $X$. $(f, g): (X, D, E) \to (X, D, E)$ be a soft mapping. Let $X$ be $\bullet$ bounded. Suppose that $\exists$ is a soft real number $r \in R(E), 0 \leq r < 1$, such that
controlled fuzzy metric, if fulfilled the subsequent conditions, \( (X, M_1, \ast) \) such that

1. \( M_1(C_1, C_2, 0) = 0 \);
2. \( M_1(C_1, C_2, t) = 1 \Leftrightarrow C_1 = C_2 \);
3. \( M_1(C_1, C_2, t) = M_1(C_2, C_1, t) \);
4. \( M_1(C_1, C_2, t + s) \geq M_1(C_1, C_3, t) + M_1(C_3, C_2, s) \).
5. \( M_1(C_1, C_2, \ast) \): \([0, \infty) \rightarrow [0, 1]\) is continuous.

Then, controlled fuzzy metric \( M_1 \) is together with the set \( X \) titled as controlled fuzzy metric space and represented as \((X, M_1, \ast)\).

Example 43. Let \( X = A \cup C \), where \( A = \{0, 1\} \) and \( C = \{0, 1\} \). We define \( M_1 : X \times X \times (0, \infty) \rightarrow [0, 1] \) as

\[
M_1(C_1, C_2, t) = \begin{cases} 
1, \text{if } C_1 = C_2, \\
\frac{t}{t + (1/C_1)}, \text{if } C_1 \in A \text{ and } C_2 \in C, \\
\frac{t}{t + 1}, \text{if } C_1 \in C \text{ and } C_2 \in A,
\end{cases}
\]

with a continuous \( t \)-norm \( \ast \) such that \( t_1 \ast t_2 = t_1 \cdot t_2 \). We define \( \lambda \): \( X \rightarrow [1, \infty) \) as

\[
\lambda(C_1, C_2) = \begin{cases} 
1, \text{if } C_1, C_2 \in A, \\
\max \{C_1, C_2\}, \text{if otherwise.}
\end{cases}
\]

It is easy to get \((X, M_1, \ast)\) that is a controlled metric space.

Remark 45. M. S. Sezen et al. proved a “Banach contraction principle” and introduced a new concept of “fixed-point theorem” for several self-mappings, which satisfied fuzzy “\( \ast \)-contraction” condition.

Theorem 17. A mapping \( T : X \rightarrow X \) on a controlled fuzzy metric space \((X, M_1, \ast)\) has a unique fixed point in \( X \) if satisfies the following condition:

\[
M_1(TC_1, TC_2, t) \geq \beta M_1(C_1, C_2, t),
\]

for all \( C_1, C_2 \in X \) and \( \beta > 0 \).

2.4.32. CAT(0) Space [90, 91]. A metric space \( X \) is said to be a CAT(0) space if it is geodesically connected and if every geodesic triangle in \( X \) is at least as “thin” as its comparison triangle in the Euclidean plane. Complete CAT(0) spaces are often called Hadamard spaces.
Definition of (CAT(0) space): let $(X,d)$ be a geodesic space. It is a CAT(0) space if, for any geodesic triangle, $\Delta \subset X$ and $x, y \in \Delta$, $d(x, y) \leq d(\bar{x}, \bar{y})$ where $\bar{x}, \bar{y} \in \Delta$.

Ammayakarn et al. [90] proved some theorems on the existence of best proximity points of multivalued mappings and monotone inclusion problems in CAT(0) spaces.

Ugwunnadi and Okeke [91] studied the approximation of common solutions for a finite family of generalized demimetric mappings and monotone inclusion problems in CAT(0) spaces.

3. Conclusions

In this review paper, it is perfectly understood that the metric space is exceptionally valuable in science and real-life scenarios. Being closely related to the concept of distance, it has many applications in the field of convergence, quantum mechanics, the field of navigation, etc.

In the beginning, metric space was introduced with the concept of ordinary distance function between two points. Later on, its variants were established in the form of generalized, extended, and/or combinational metric spaces with different properties. Some of the properties are metrics, which consider distance function between $n$ points, constant function/control function $a(x, y)$ in triangle inequality, metric without symmetric property, an arbitrary relation, change triangle inequality into quadrilateral inequality, fuzziness between the points on a nonempty set and function between the points on a nonempty set, the concept of nonzero self-distance, an orthogonal property, and many more on the set of $\mathbb{R}^n$, complex space, Riesz space, ordered Banach’s space, etc.

Variants of metric spaces were introduced from an implementation perspective to find the existence and uniqueness of the “fixed-point theorem,” “Banach contraction theorem,” “extension of Banach contraction principle,” and “Kannan fixed-point theorem,” “fuzzy fixed-point theorem” with the support of some of the “contraction principle,” for instance, $A$-contraction, $Z$-contraction mapping, soft set-valued maps, simulation function, $F$-contraction mapping, and $Z_m$-contraction mapping. Consequently, originated “fixed-point theorem” for single-valued mapping and multivalued mapping, which is further connected to find the simple and efficient solution for a Volterra integral equation, and mixed Volterra–Fredholm type integral equation, helpful in developing some computational methods for answering the system of nonlinear matrix equations, in the proof of Picard–Lindelof theorem about finding the existence and uniqueness of certain ordinary differential equations, solution of initial value problems, and so on. Since metric space plays an important role in various real-life scenarios, these are some of the applications; we can still find many more applications based on variants of metric spaces.

Data Availability

The data used to support the findings of this study are included within the article and acknowledged.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


