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Research Article

Approximation by Rational Functions in Variable Exponent Morrey–Smirnov Classes

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In this work, the direct theorem of approximation theory in variable exponent Morrey–Smirnov classes of analytic functions, defined on a doubly connected domain of the complex plane bounded by two sufficiently smooth curves, is investigated.

1. Introduction

The classical Morrey spaces were introduced by Morrey in [1] in order to investigate the local behavior of the solutions of elliptic differential equations. Recently, many researchers have investigated function spaces with variable exponents due to their use in several fields of applied mathematics. In particular, function spaces with variable exponents have many applications in areas involving the modeling of electrorheological fluids [2] and image restoration [3]. The variable exponent Morrey spaces introduced in [4] have been studied intensively by various authors (see, for example, [5–7]). The fundamental problem in approximation theory is to express complicated functions by simple functions such as polynomials, wavelets, or rational functions with more useful structures. The theory of approximation is strongly related with the operators and has a considerable number of applications in areas including general marginal distributions such as sampling and machine learning (see [8–10]). Also, the approximation problems in the variable Morrey–Smirnov classes of analytic functions defined on a simply connected domain with a Dini-smooth boundary are proved in [11]. The direct and converse theorems of approximation theory in the classical Morrey–Smirnov classes defined on a simply connected domain with a Dini-smooth boundary were obtained in [12, 13]. Similar results in the variable exponent Smirnov classes were studied in [14, 15].

On a doubly connected domain, the rate of approximation by p–Faber–Laurent rational function in Smirnov classes was studied in [16]. Also, the rate of approximation by Faber–Laurent rational function in Smirnov–Orlicz classes and Smirnov classes with variable exponent was obtained in [17]. The approximation property of $(p-\epsilon)$–Faber–Laurent rational functions in the weighted generalized grand Smirnov classes on doubly connected domains is proved in [18].

In the current paper, approximation one direct theorem of approximation theory in variable exponent Morrey–Smirnov classes of analytic functions, defined on a doubly connected domain bounded by two Dini-smooth curves, is obtained.

2. Preliminaries

Let J denote the interval $[0,2\pi]$ or a Jordan rectifiable curve Γ, and let $\wp$ denote the class of all Lebesgue measurable functions $p(.)\colon \Gamma \longrightarrow [1,\infty[$ such that

$$1 < p_- = \operatorname*{essinf}_{z\in J}\, p(z) \le p^+ = \operatorname*{esssup}_{z\in J}\, p(z) < \infty. \tag{1}$$

We denote by $|J|$ to the Lebesgue measure of J. We say $p(.)\in \wp^{\log}$ if there is a constant c such that

$$\left|p(z_1)-p(z_2)\right| \ln\left(\frac{|J|}{\left|z_1-z_2\right|}\right) \le c, \tag{2}$$

for all $z_1, z_2 \in J$.

For $p(.) \in \wp^{\log}(\Gamma)$, we define $L^{p(.)}(\Gamma)$ the set of all measurable functions f such that

$$\int_{\Gamma} |f(z)|^{p(z)} |dz| < \infty. \tag{3}$$

$L^{p(.)}(\Gamma)$ is a Banach space with respect to the norm

$$f_{L^{p(.)}(\Gamma)} = \inf\left\{\lambda > 0, \int_{\Gamma} \left|\frac{f(z)}{\lambda}\right|^{p(z)} |dz| \le 1\right\}. \tag{4}$$

Let U be a finite simply connected domain with a rectifiable Jordan curve boundary Γ. Denote $U^- = \operatorname{ext}\Gamma$, $\gamma_0 = \{w \in \mathbb{C}: |w| = 1\}$, $D = \operatorname{int}\gamma_0$, and $D^- = \operatorname{ext}\gamma_0$. Let Γ_r be the image of circle $\{w \in \mathbb{C}: |w| = r, 0 < r < 1\}$ under some conformal mapping of D onto U.

For given $1 \le p < \infty$, we denote by $E^p(U)$ the class of analytic functions f in U which satisfies

$$\int_{\Gamma_r} |f(t)|^p |dt| < \infty, \tag{5}$$

uniformly in r.

It is known that every function of class $E^p(U)$ has nontangential boundary values almost everywhere on Γ and the boundary function belongs to $L^p(\Gamma)$ ([19], pp. 438–453).

Also, suppose that ϕ^* is the conformal mapping of U^- onto D^- normalized by

$$\phi^*(\infty) = \infty, \qquad \lim_{z \longrightarrow \infty} \frac{\phi^*(z)}{z} > 0, \tag{6}$$

and let ψ^* be the inverse of ϕ^*. Let ϕ_1^* be the conformal mapping of U on to D^-, normalized by

$$\phi_1^*(0) = \infty, \quad \lim_{z \longrightarrow 0} z\,\phi_1^*(z) > 0. \tag{7}$$

The inverse mapping of ϕ_1^* will be denoted by ψ_1^*.

The functions ψ^* and ψ_1^* have in some deleted neighborhood of ∞ representations

$$\begin{aligned} \psi^*(w) &= \alpha w + \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \cdots + \frac{\alpha_k}{w^k} + \cdots, \quad \alpha > 0, \\ \psi_1^*(w) &= \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \cdots + \frac{\beta_k}{w^k} + \cdots, \quad \beta_1 > 0. \end{aligned} \tag{8}$$

The functions

$$\begin{aligned} &\frac{\psi^{*\prime}(w)}{\psi^*(w) - z}, \quad z \in U, \\ &\frac{\psi_1^{*\prime}(w)}{\psi_1^*(w) - z}, \quad z \in U^-, \end{aligned} \tag{9}$$

are analytic in the domain D^-, and the following expansions hold [20–23]:

$$\begin{aligned} \frac{\psi^{*\prime}(w)}{\psi^*(w) - z} &= \sum_{k=0}^{\infty} \frac{F_k(z)}{w^{k+1}}, \qquad z \in U, w \in D^-, \\ \frac{\psi_1^{*\prime}(w)}{\psi_1^*(w) - z} &= \sum_{k=1}^{\infty} -\frac{\widetilde{F}_k(1/z)}{w^{k+1}}, \quad z \in U^-, w \in D^-, \end{aligned} \tag{10}$$

where $F_k(z)$ and $\widetilde{F}_k(1/z)$ are the Faber polynomials of degree k with respect to z and $1/z$ for the continuums $\overline{U}$ and $\overline{\mathbb{C}}\backslash U$, respectively.

Let Γ be a rectifiable Jordan curve in the complex plane with length ℓ and let $\Gamma(t, r) = \Gamma \cap B(t, r)$, $t \in \Gamma$, $r > 0$, where $B(t, r) = \{z \in \mathbb{C}: |z - t| < r\}$. *The classical Morrey spaces* $L^{p,\lambda}(\Gamma)$ for given $0 \le \lambda \le 1$ and $1 \le p < \infty$ are defined as the set of functions $f \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} = \sup_{z \in \Gamma\, 0 < r < \ell} r^{-\lambda/p} \|f\|_{L^p(\Gamma(t,r))} < \infty. \tag{11}$$

Let $U = \operatorname{int}\Gamma$, we define the *classical Morrey–Smirnov classes* $E^{p,\lambda}(U)$ for $0 < \lambda \le 1$ and $1 < p < \infty$ as

$$E^{p,\lambda}(U) = \left\{f \in E^1(U),\ f \in L^{p,\lambda}(\Gamma)\right\}. \tag{12}$$

We define $\|f\|_{E^{p,\lambda}(U)} \coloneqq \|f\|_{L^{p,\lambda}(\Gamma)}$.

Definition 1. Let $p(.)\colon \Gamma \longrightarrow [1, \infty[$ be a Lebesgue measurable function satisfying the condition (1), and let $\lambda(.)\colon \Gamma \longrightarrow [0, 1]$ be a measurable function. We define the *variable exponent Morrey spaces* $L^{p(.),\lambda(.)}(\Gamma)$ as the set of Lebesgue measurable functions f defined on Γ, such that

$$\mu_{p(.),\lambda(.)}(f) = \sup_{t \in \Gamma,\, 0 < r \le \ell} r^{-\lambda(t)} \int_{\Gamma(t,r)} |f(s)|^{p(s)}\, ds < \infty. \tag{13}$$

$L^{p(.),\lambda(.)}(\Gamma)$ becomes a Banach space with respect to the norm:

$$\|f\|_{L^{p(.),\lambda(.)}(\Gamma)} = \inf\left\{u > 0\colon \mu_{p(.),\lambda(.)}\left(\frac{f}{u}\right) \le 1\right\}. \tag{14}$$

We define the variable exponent Morrey–Smirnov class $E^{p(.),\lambda(.)}(U)$ as

$$E^{p(.),\lambda(.)}(U) \coloneqq \left\{f \in E^1(U),\ f \in L^{p(.),\lambda(.)}(\Gamma)\right\}. \tag{15}$$

If we define $\|f\|_{E^{p(.),\lambda(.)}(U)} \coloneqq \|f\|_{L^{p(.),\lambda(.)}(\Gamma)}$, the class $E^{p(.),\lambda(.)}(U)$ becomes a Banach space.

Definition 2. A smooth curve Γ is called *Dini-smooth* if

$$\int_0^{\delta} \frac{\Omega(\sigma, s)}{s}\, ds < \infty, \quad \delta > 0, \tag{16}$$

where $\sigma(s)$ is the angle, between the tangent line of Γ and the positive real axis expressed as a function of arclength s with the modulus of continuity $\Omega(\sigma, s)$, where

$$\Omega(\sigma, s) := \sup_{|s_1 - s_2| \leq s} |\sigma(s_1) - \sigma(s_2)|, \quad s > 0. \tag{17}$$

Definition 3. Let $p(.)\colon \gamma_0 \longrightarrow [1, \infty)$ and $\lambda(.)\colon \gamma_0 \longrightarrow [0, 1]$ be measurable functions such that

$$0 \leq \lambda_- := \operatorname*{essinf}_{t\in\gamma_0} \lambda(t) \leq \lambda^+ := \operatorname*{esssup}_{t\in\gamma_0} p(t) < 1. \tag{18}$$

Also, assume that $p(.) \in \wp^{\log}$. For $f \in L^{p(.,)\lambda(.)}(\gamma_0)$, we define the operator

$$(\nu_h f)(w) = \frac{1}{h}\int_0^h f\left(w e^{it}\right)\mathrm{d}t, \quad w \in \gamma_0, 0 < h < \pi. \tag{19}$$

The operator ν_h is bounded linear operator on $L^{p(.,)\lambda(.)}(\gamma_0)$ [24]. Hence, we can define the modulus of smoothness of $f \in L^{p(.,)\lambda(.)}(\gamma_0)$ as

$$\Omega(f, \delta)_{p(.,)\lambda(.)} = \sup_{0<h\leq\delta} \|f - \nu_h f\|_{L^{p(.,)\lambda(.)}(\gamma_0)}, \quad \delta > 0. \tag{20}$$

The function $\Omega(f, \delta)_{p(.,)\lambda(.)}$ is a continuous, nonnegative, and nondecreasing on $[0, \infty)$ satisfying the following properties for any $f, g \in L^{p(.,)\lambda(.)}(\gamma_0)$:

$$\begin{gathered}\lim_{\delta\longrightarrow 0} \Omega(f, \delta)_{p(.,)\lambda(.)} = 0,\\ \Omega(f + g, \delta)_{p(.,)\lambda(.)} \leq \Omega(f, \delta)_{p(.,)\lambda(.)} + \Omega(g, \delta)_{p(.,)\lambda(.)}, \quad \delta > 0,\\ \Omega(f, n\delta)_{p(.,)\lambda(.)} \leq n\,\Omega(f, \delta)_{p(.,)\lambda(.)}, \quad n = 1, 2, \ldots, \delta > 0.\end{gathered} \tag{21}$$

Suppose that G is an arbitrary doubly connected domain in the complex plane $\mathbb{C}$, bounded by two rectifiable Jordan curves L_1 and L_2. Without loss of generality, we may assume that the closed curve L_2 is inside the closed curve L_1 and $0 \in \operatorname{int} L_2$. Let $G_1^0 := \operatorname{int} L_1, G_1^\infty := \operatorname{ext} L_1, G_2^0 := \operatorname{int} L_2$,and $G_2^\infty := \operatorname{ext} L_2$.

We denote by $w = \phi(t)\,(w = \phi_1(t))$ the conformal mapping of $G_1^\infty\,(G_2^0)$ onto domain D^- normalized by the conditions:

$$\phi(\infty) = \infty,\ \lim_{t\longrightarrow\infty} \frac{\phi(t)}{t} > 0,\ \phi_1(0) = \infty,\ \lim_{t\longrightarrow 0} t\phi_1(t) > 0. \tag{22}$$

Moreover, ψ and ψ_1 will denote the inverse mappings of ϕ and ϕ_1, respectively.

The level lines of the domains G_1^0 and G_2^0 are defined for $r, R > 1$, by

$$C_r := \{t\colon |\phi(t)| = r\},\ C_R := \{t\colon |\phi_1(t)| = R\}. \tag{23}$$

The Faber polynomials $F_k(t)$ and $\widetilde{F}_k(1/z)$ have the following integral representations [22].

If $t \in \operatorname{int} C_r$, then

$$F_k(t) = \frac{1}{2\pi i}\int_{C_r} \frac{[\phi(\xi)]^k}{\xi - t}\mathrm{d}\xi = \frac{1}{2\pi i}\int_{|w|=r} \frac{\psi'(w)w^k}{\psi(w) - t}\,\mathrm{d}w. \tag{24}$$

If $t \in \operatorname{ext} C_r$, then

$$F_k(t) = [\phi(t)]^k + \frac{1}{2\pi i}\int_{C_r} \frac{[\phi(\xi)]^k}{\xi - t}\,\mathrm{d}\xi. \tag{25}$$

If $t \in \operatorname{int} C_R$, then

$$\widetilde{F}_k\left(\frac{1}{t}\right) = [\phi_1(t)]^k - \frac{1}{2\pi i}\int_{C_R} \frac{[\phi_1(\xi)]^k}{\xi - t}\,\mathrm{d}\xi. \tag{26}$$

If $t \in \operatorname{ext} C_R$, then

$$\widetilde{F}_k\left(\frac{1}{t}\right) = -\frac{1}{2\pi i}\int_{C_R} \frac{[\phi_1(\xi)]^k}{\xi - t}\,\mathrm{d}\xi = -\frac{1}{2\pi i}\int_{|w|=R} \frac{\psi_1'(w)w^k}{\psi_1(w) - t}\,\mathrm{d}w. \tag{27}$$

If $f(z)$ is an analytic function in the doubly connected domain bounded by the curves C_r and C_R, then

$$f(t) = \sum_{k=0}^\infty a_k F_k(t) + \sum_{k=1}^\infty b_k \widetilde{F}_k\left(\frac{1}{t}\right), \tag{28}$$

where

$$\begin{aligned}a_k &= \frac{1}{2\pi i}\int_{|w|=r_1} \frac{f(\psi(w))}{w^{k+1}}\,\mathrm{d}w, \quad 1 < r_1 < r, k = 0, 1, 2, \ldots,\\ b_k &= \frac{1}{2\pi i}\int_{|w|=R_1} \frac{f(\psi_1(w))}{w^{k+1}}\,\mathrm{d}w, \quad 1 < R_1 < R, k = 1, 2, \ldots.\end{aligned} \tag{29}$$

Let $L = L_1 \bigcup L_2^-$ and let G be a doubly connected domain bounded by L_1 and L_2, where L_2 is in L_1. We define the variable exponent Morrey–Smirnov classes $E^{p(.,)\lambda(.)}(G)$ as

$$E^{p(.,)\lambda(.)}(G) = \left\{f \in E^1(G),\ f \in L^{p(.,)\lambda(.)}(L)\right\}. \tag{30}$$

For $f \in E^{p(.,)\lambda(.)}(G)$, the norm is defined by

$$\|f\|_{E^{p(.,)\lambda(.)}(G)} = \|f\|_{L^{p(.,)\lambda(.)}(L)}. \tag{31}$$

Let U be a simply connected domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let U^- be the exterior of Γ. Then, for $f \in L^1(\Gamma)$, the functions f^+ and f^- defined by

$$\begin{aligned}f^+(t) &:= \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - t}\,\mathrm{d}\xi, \quad t \in U,\\ f^-(t) &:= \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - t}\,\mathrm{d}\xi, \quad t \in U^-.\end{aligned} \tag{32}$$

are analytic in U and U^-, respectively, $f^-(\infty) = 0$.

For a given $t \in \Gamma$, the operator S_Γ defined by

$$S_\Gamma(f)(t) := \lim_{\varepsilon\longrightarrow 0^+} \frac{1}{2\pi i}\int_{\Gamma\cap\{\xi\colon |\xi - t| > \varepsilon\}} \frac{f(\xi)}{\xi - t}\,\mathrm{d}\xi, \tag{33}$$

is called the Cauchy singular operator.

According to the Privalov theorem [19] if one of the functions f^+ or f^- has the nontangential limits a.e. on Γ, then $S_\Gamma(f)(t)$ exists a.e. on Γ and also the other one has the nontangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(t)$ exists a.e. on Γ, then the functions f^+ and f^- have nontangential limits a.e. on Γ. In both cases, the following formulae:

$$f^+(t) = S_\Gamma f(t) + \frac{1}{2} f(t),\ f^-(t) = S_\Gamma f(t) - \frac{1}{2} f(t),$$
$$f(t) = f^+(t) - f^-(t), \tag{34}$$

hold a.e. on Γ.

In Kokilashvili and Meskhi [25], it is proved that, if Γ is a Dini-smooth curve, then the operator S_Γ is bounded on $L^{p(.,)\lambda(.)}(\Gamma)$, i.e., there exists a positive constant c_1 such the following inequality holds for any $f \in L^{p(.,)\lambda(.)}(\Gamma)$:

$$\left\| S_\Gamma(f) \right\|_{L^{p(.,)\lambda(.)}(\Gamma)} \le c_1 \| f \|_{L^{p(.,)\lambda(.)}(\Gamma)}. \tag{35}$$

If L_1 and L_2 are Dini-smooth curves, then by [26] there are positive constants c_2, c_3, c_4, and c_5 such

$$c_2 < \left|\psi'(w)\right| < c_3,\ c_4 < \left|\psi_1'(w)\right| < c_5. \tag{36}$$

Let $L_i\,(i = 1, 2)$ be a Dini-smooth curve, we define the following functions $f_0 := f \circ \psi$ for $f \in L^{p(.,)\lambda(.)}(L_1)$, $p_0 = p \circ \psi$ and $f_1 := f \circ \psi_1$ for $f \in L^{p(.,)\lambda(.)}(L_2)$, $p_1 = p \circ \psi_1$.

If $f \in L^{p(.,)\lambda(.)}(\mathrm{L}_1)$ and $f \in L^{p(.,)\lambda(.)}(\mathrm{L}_2)$, then by (36), $f_0 \in L^{p_0(.,)\lambda(.)}(\gamma_0)$ and $f_1 \in L^{p_1(.,)\lambda(.)}(\gamma_0)$. From (34), we get $f_0^-(\infty) = 0$, $f_1^-(\infty) = 0$ and the following relations hold a.e. on γ_0:

$$f_0(w) = f_0^+(w) - f_0^-(w),\ f_1(w) = f_1^+(w) - f_1^-(w). \tag{37}$$

Using Theorem 6.1 from [24] and taking into account the proof of a similar result in [20], we deduce the following lemma.

Lemma 1. *Let $p(.)\colon \gamma_0 \longrightarrow [1, \infty[$ and $\lambda(.)\colon \gamma_0 \longrightarrow [0, 1]$ be measurable functions. Let $g \in E^{p(.,)\lambda(.)}(D)$ with $p(.) \in \wp^{\log}(\gamma_0)$ and $0 \le \lambda_- \le \lambda^+ < 1$. If $\sum_{k=0}^{n} a_k(g) w^k$ is the $n-$th partial sum of the Taylor series of g at the origin, then for any $n = 1, 2, \ldots$, there is a constant c_6 such the following estimate:*

$$\left\| g(w) - \sum_{k=0}^{n} a_k(g) w^k \right\|_{L^{p(.,)\lambda(.)}(\gamma_0)} \le c_6 \Omega\left(g, \frac{1}{n}\right)_{p(.,)\lambda(.)}, \tag{38}$$

holds.

3. Main Result

From now on, we will assume that the set of rational functions is dense in the space $E^{p(.,)\lambda(.)}(L)$. Our main result is the following.

Theorem 1. *Let G be a finite doubly connected domain with the Dini-smooth boundary, $L = L_1 \bigcup L_2^-$, and let $p(.)\colon L \longrightarrow [1, \infty[$, and $\lambda(.)\colon L \longrightarrow [0, 1]$. $E^{p(.,)\lambda(.)}(L)$ be the variable Morrey–Smirnov space with $p(.) \in \wp^{\log}(L)$ and $0 \le \lambda_- < \lambda^+ < 1$. If $f \in E^{p(.,)\lambda(.)}(G)$, then for every $n \in \mathbb{N}$, there are a rational function $R_n(f, .)$ and a constant c_7 such that*

$$\| f - R_n(f, .) \|_{E^{p(.,)\lambda(.)}(G)} \le c_7 \left[\Omega\left(f_0, \frac{1}{n}\right)_{p_0(.,)\lambda(.)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(.,)\lambda(.)} \right]. \tag{39}$$

Proof. Let $f \in E^{p(.,)\lambda(.)}(L)$, then $f_0 \in E^{p_0(.,)\lambda(.)}(\gamma_0)$ and $f_1 \in E^{p_1(.,)\lambda(.)}(\gamma_0)$.

Putting $\phi(\xi)$ and $\phi_1(\xi)$ in place of w in (37), we obtain

$$f(\xi) = f_0^+(\phi(\xi)) - f_0^-(\phi(\xi)), \quad \xi \in L_1, \tag{40}$$

$$f(\xi) = f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi)), \quad \xi \in L_2. \tag{41}$$

Let $t \in \operatorname{ext} L_1$, then from (24), we have

$$\sum_{k=0}^{n} a_k F_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i} \int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi)}{\xi - t} \mathrm{d}\xi, \tag{42}$$

and using (40), we get

$$\sum_{k=0}^{n} a_k F_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i} \int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t} \mathrm{d}\xi + \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)}{\xi - t} \mathrm{d}\xi + \frac{1}{2\pi i} \int_{L_1} \frac{f_0^-(\phi(\xi))}{\xi - t} \mathrm{d}\xi. \tag{43}$$

Since $f_0^-(\phi(\xi)) \in E^{p_0(.,)\lambda(.)}(G_1^\infty)$,

$$\frac{1}{2\pi i} \int_{L_1} \frac{f_0^-(\phi(\xi))}{\xi - t} \mathrm{d}\xi = -f_0^-(\phi(t)). \tag{44}$$

Thus,

$$\sum_{k=0}^{n} a_k F_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i} \int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t} \mathrm{d}\xi + \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)}{\xi - t} \mathrm{d}\xi - f_0^-(\phi(t)). \tag{45}$$

Now, for $t \in \operatorname{ext} L_2$, from (27) and (41), we have

$$\begin{aligned} \sum_{k=1}^{n} b_k \widetilde{F}_k\left(\frac{1}{t}\right) &= -\frac{1}{2\pi i} \int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t} \mathrm{d}\xi \\ &\quad - \frac{1}{2\pi i} \int_{L_2} \frac{f(\xi)}{\xi - t} \mathrm{d}\xi. \end{aligned} \tag{46}$$

For any $t \in \operatorname{ext} L_1$, we have

$$\frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)}{\xi - t} \mathrm{d}\xi = \frac{1}{2\pi i} \int_{L_2} \frac{f(\xi)}{\xi - t} \mathrm{d}\xi. \tag{47}$$

Because $\operatorname{ext} L_1 \subset \operatorname{ext} L_2$, the relations (45), (46), and (47) are valid for any $t \in \operatorname{ext} L_1$, and this gives

$$\sum_{k=0}^{n} a_k F_k(t) + \sum_{k=1}^{n} b_k \widetilde{F}_k(1/t) == \sum_{k=0}^{n} a_k \phi^k(t) - f_0^-(\phi(t)) - \frac{1}{2\pi i}\int_{L_1} \frac{f_0^+(\phi(\xi)) - \sum_{k=0}^{n} a_k \phi^k(\xi)}{\xi - t}\, d\xi + \frac{1}{2\pi i}\int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t}\, d\xi. \tag{48}$$

Taking the limit as $t \longrightarrow z \in L_1$ along nontangential path outside L_1 for almost every $z \in L_1$, we get

$$\begin{aligned} f(z) &- \left(\sum_{k=0}^{n} a_k F_k(z) + \sum_{k=1}^{n} b_k \widetilde{F}_k\left(\frac{1}{z}\right)\right) \\ &= f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z) + \frac{1}{2}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z)\right) \\ &+ S_{L_1}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k \phi^k(z)\right) - \frac{1}{2\pi i}\int_{L_2} \frac{f_1^+(\phi_1(\xi)) - \sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t}\, d\xi. \end{aligned} \tag{49}$$

The rational function $R_n(f,z)$ is defined as

$$R_n(f,z) = \sum_{k=0}^{n} a_k F_k(z) + \sum_{k=1}^{n} b_k \widetilde{F}_k\left(\frac{1}{z}\right). \tag{50}$$

By (51), Minkowski's inequality, and (35), we get

$$\|f - R_n(f,.)\|_{L^{p(.),\lambda(.)}(L_1)} \le c_8 \left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p_0(.),\lambda(.)}(\gamma_0)} + c_9 \left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p_1(.),\lambda(.)}(\gamma_0)}. \tag{51}$$

And from Lemma 1, we obtain

$$f - R_n(f,.)_{L^{p(.),\lambda(.)}(L_1)} \le c_{10} \left\{\Omega\left(f_0, \frac{1}{n}\right)_{p_0(.),\lambda(.)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(.),\lambda(.)}\right\}. \tag{52}$$

Let $t' \in \operatorname{int} L_2$. From (26) and (41), we get

$$\begin{aligned} \sum_{k=1}^{n} b_k \widetilde{F}_k\left(\frac{1}{t'}\right) &= \sum_{k=1}^{n} b_k \phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi)}{\xi - t'}\, d\xi \\ &= \sum_{k=1}^{n} b_k \phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi) - f_1^+(\phi_1(\xi))}{\xi - t'}\, d\xi \\ &- \frac{1}{2\pi i}\int_{L_2} \frac{f(\xi)}{\xi - t'}\, d\xi - f_1^-(\phi_1(t')). \end{aligned} \tag{53}$$

and for any $t' \in \operatorname{int} L_1$, from (24) and (40), we have

$$\sum_{k=1}^{n} a_k F_k(t') = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi)}{\xi - t'}\, d\xi = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t'}\, d\xi + \frac{1}{2\pi i}\int_{L_1} \frac{f(\xi)}{\xi - t'}\, d\xi. \quad (54)$$

Since $\operatorname{int} L_2 \subset \operatorname{int} L_1$, relations (19) and (20) are valid for $t' \in \operatorname{int} L_2$, and this gives

$$\sum_{k=0}^{n} a_k F_k(t') + \sum_{k=1}^{n} b_k \widetilde{F}_k(1/t') = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - t'}\, d\xi - \frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} b_k \phi_1^k(\xi) - f_1^+(\phi_1(\xi))}{\xi - t'}\, d\xi - f_1^-(\phi_1(t')) + \sum_{k=1}^{n} b_k \phi_1^k(t'). \quad (55)$$

Taking the limit as $t' \longrightarrow z \in L_2$ along nontangential path inside L_2 for almost every $z \in L_2$, we get

$$f(z) - \left(\sum_{k=0}^{n} a_k F_k(z) + \sum_{k=1}^{n} b_k \widetilde{F}_k\left(\frac{1}{z}\right)\right) = f_1^+(\phi_1(z)) - \frac{1}{2}\left(\sum_{k=1}^{n} b_k \phi_1^k(z) - f_1^+(\phi_1(z))\right) - S_{L_2}\left(\sum_{k=1}^{n} b_k \phi_1^k(z) - f_1^+(\phi_1(z))\right) - \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\xi) - f_0^+(\phi(\xi))}{\xi - z}\, d\xi. \quad (56)$$

Using Minkowski's inequality and (35), we get

$$\|f - R_n(f,.)\|_{L^{p(.),\lambda(.)}(L_2)} \le c_{11}\left\| f_1^+(w) - \sum_{k=1}^{n} b_k w^k \right\|_{L^{p_0(.),\lambda(.)}(\gamma_0)} + c_{12}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p_1(.),\lambda(.)}(\gamma_0)}. \quad (57)$$

By Lemma 1, we obtain

$$\|f - R_n(f,.)\|_{L^{p(.),\lambda(.)}} \le c_{13} \cdot \left\{ \Omega\left(f_0, \frac{1}{n}\right)_{p_0(.),\lambda(.)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(.),\lambda(.)} \right\}. \quad (58)$$

From (52) and (58), we obtain

$$\|f - R_n(f,.)\|_{E^{p(.),\lambda(.)}(G)} \le c_7 \left\{ \Omega\left(f_0, \frac{1}{n}\right)_{p_0(.),\lambda(.)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(.),\lambda(.)} \right\}. \quad (59)$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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