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Research Article

Numerical Solution via Operational Matrix for Solving Prabhakar Fractional Differential Equations

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In this work, we apply the operational matrix based on shifted Legendre polynomials for solving Prabhakar fractional differential equations. The Prabhakar derivative is defined in three-parameter Mittag-Leffler function. We achieve this by first deriving the analytical expression for Prabhakar derivative of x^p where p is positive integer, via integration. Hence, for the first time, the operational matrix method for Prabhakar derivative is derived by using the properties of shifted Legendre polynomials. Hence, we transform the Prabhakar fractional differential equations into a system of algebraic equations. By solving the system of algebraic equations, we were able to obtain the numerical solution of fractional differential equations defined in Prabhakar derivative. Only a few terms of shifted Legendre polynomials are needed for achieving the accurate solution.

1. Introduction

The operational matrix method is one of the powerful tools for solving fractional differential equations. This method uses the concept of replacing a symbol with another symbol, i.e., replacing symbol fractional derivative, D^{α}, with another symbol, which is an operational matrix, P^{α}. In [1], the authors had derived shifted Legendre operational matrix for solving fractional differential equations, defined in Caputo sense. Then, researchers started to apply the various types of polynomials to derive the operational matrix for solving various types of fractional calculus problems, including Genocchi operational matrix for fractional partial differential equations [2], Laguerre polynomials operational matrix for solving fractional differential equations with non-singular kernel [3], and Müntz–Legendre polynomial operational matrix for solving distributed order fractional differential equations [4].

Recently, apart from the fractional differential equation defined in Caputo sense, this kind of operational matrix method had been extended to tackle another type of fractional derivative or operator, which includes the Caputo–Fabrizio operator [5] and Atangana–Baleanu derivative [6, 7]. In this research direction, the operational matrix method is either an operational matrix of derivative or operational matrix of integration based on certain polynomials. The operational matrix method is possible to apply to another type of fractional derivatives if there is an analytical expression for x_p (where p is integer positive) in the sense of certain fractional derivatives or operators. Hence, we extend this operational matrix to tackle operator defined by one parameter Mittag–Leffler function, i.e. Antagana–Baleunu derivative [6] to the operator that defined by using three-parameter Mittag–Leffler function, so-called Prabhakar fractional integrals or derivative. In short, we aim to solve the following fractional differential equation defined in Prabhakar sense:

$$D^{\gamma}_{\alpha,\beta,\omega} y(x) = g(x, y(x)), \tag{1}$$

subject to the initial condition $y(0) = a$.

On top of that, Prabhakar introduced a type of convolution-type integral operator, called Prabhakar integral in

[8]. Recently, Prabhakar fractional integrals or derivative had received more and more attention by the researchers [9–11]. This kind of integral had been applied in anomalous dielectrics [12], viscoelasticity [13], kinetic equation [14], and diffusion equation [15]. Besides that, some new concepts or theories were derived to suit this Prabhakar operator, for example, in [16], Hyers–Ulam stability of fractional differential equations with Prabhakar derivatives was investigated, and stability analysis of fractional differential equations with Prabhakar derivative was studied in [17]. Furthermore, since this Prabhakar operator involves more parameters, existing numerical methods may not be applicable for solving the fractional differential equation defined in the Prabhakar sense. Hence, some works had been done including numerical approximation to Prabhakar fractional Sturm–Liouville problem [18], and the Prabhakar derivative was approximated by using series representations in [19]. However, the work for solving numerically the Prabhakar fractional differential equations is still less. Hence, we intend to derive a new operational matrix based on shifted Legendre polynomials to approximate Prabhakar derivative, hence solving the Prabhakar fractional differential equations by using a collocation scheme.

This paper is organized as follows. We will briefly explain some preliminary concepts including the Prabhakar fractional integral and derivative in Section 2. Section 2.3 presents analytical expression for Prabhakar integral and derivative for x^p. Section 3 discusses the derivation of a new operational matrix based on shifted Legendre polynomials for Prabhakar fractional derivative. In Section 4, we explain the new scheme and the error analysis. Some examples for solving fractional differential equations defined in Prabhakar derivative using our proposed method via the new operational matrix will be presented in Section 5. Conclusion and some recommendations are highlighted in Section 6.

2. Preliminaries

2.1. Prabhakar Fractional Integral and Derivative. In this section, we will present some basic concepts related to Prabhakar fractional integral and derivative.

Definition 1. The one-parameter Mittag-Leffler function is defined as follows:

$$E_\alpha z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \operatorname{Re}(\alpha) > 0. \tag{2}$$

Definition 2. The three-parameter generalization of Mittag-Leffler function is given by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \operatorname{Re}(\alpha) > 0. \tag{3}$$

Definition 3. For $f \in L^1(a,b)$, Prabhakar fractional integral is defined by

$$I^{\gamma}_{\alpha,\beta,\omega,a+} f(x) = \int_a^x (x-\tau)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\omega(x-\tau)^{\alpha}\right) f(\tau)\mathrm{d}\tau, \tag{4}$$

where $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\gamma, \alpha, \beta, \omega \in \mathbb{C}$.

Normally, the definition in (4) can be written as

$$I^{\gamma}_{\alpha,\beta,\omega,a+} f(x) = \int_a^x e^{\gamma}_{\alpha,\beta}(x-\tau;\omega) f(\tau)\mathrm{d}\tau, \tag{5}$$

where

$$e^{\gamma}_{\alpha,\beta}(x;\omega) = x^{\beta-1} E^{\gamma}_{\alpha,\beta}(\omega x^{\alpha}). \tag{6}$$

Using (6), we have the following Prabhakar fractional derivative and regularized Prabhakar derivative.

Definition 4. For $0 < \beta < 1$, $f(x) \in L^1[a,b]$, and Prabhakar fractional derivative (in Riemann–Liouville sense) is defined by

$${}^{R}D^{\gamma}_{\alpha,\beta,\omega,a+} f(x) = \frac{d^m}{dx^m} \int_a^x e^{-\gamma}_{\alpha,m-\beta}(x-\tau;\omega) f(\tau)\mathrm{d}\tau, \tag{7}$$

where $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\gamma, \alpha, \beta, \omega \in \mathbb{C}$.

Definition 5. For $f(x) \in AC^m(a,b)$, $0 \le a < x < b \le \infty$, the regularized Prabhakar derivative (in Caputo sense) is defined by

$${}^{c}D^{\gamma}_{\alpha,\beta,\omega,a+} f(x) = \int_{a+}^x e^{-\gamma}_{\alpha,m-\beta}(x-\tau;\omega) \frac{\mathrm{d}^m f(\tau)}{\mathrm{d}\tau^m}\mathrm{d}\tau, \tag{8}$$

where $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\gamma, \alpha, \beta, \omega \in \mathbb{C}$.

2.2. Shifted Legendre Polynomials. The analytical form of the shifted Legendre polynomials $L_j(x)$ of degree j is given by

$$L_j(x) = \sum_{h=0}^{j} \frac{(-1)^{j+h}(j+h)!}{(j-h)!(h!)^2} x^h. \tag{9}$$

A function $y(x)$, which is square integrable in $[0,1]$, can be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j L_j(x), \tag{10}$$

where the coefficient c_j is given by

$$c_j = (2j+1)\int_0^1 y(x) L_j(x)\mathrm{d}x, \quad j = 0,1,2,\ldots. \tag{11}$$

(11) can be written like that due to the orthogonality condition of shifted Legendre polynomials

$$\int_0^1 L_i(x) L_j(x)\mathrm{d}x = \begin{cases} \dfrac{1}{2i+1} & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{12}$$

In this work, only the first $N+1$ term of shifted Legendre polynomials is considered. So, we have

$$y(x) = \sum_{j=0}^{N} c_j P_j(x) = \mathbf{C}^T \mathbf{L}(x), \tag{13}$$

where

$$\mathbf{C}^T = [c_0, c_1, c_2, \dots, c_N], \quad \mathbf{L}(x) = [L_0(x), L_1(x), L_2(x), \dots, L_N(x)]^T. \tag{14}$$

2.3. Analytical Expression for Prabhakar Integral and Derivative for x^p. In this section, we find the analytical expression for Prabhakar integration as well as Prabhakar derivative for $f(x) = x^p$ where p is positive integer.

Theorem 1. *The Prabhakar integration for $f(x) = x^p$ where p is positive integer can be defined as follows:*

$$I^{\gamma}_{\alpha,\beta,\omega} x^p = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega^k p! x^{\alpha k+\beta+p}}{\Gamma(\gamma)\Gamma(\alpha k + \beta + p + 1)\Gamma(k+1)}. \tag{15}$$

Proof. For sake of simplicity, we let $a = 0$ (that in equation (4)); by using Definition 3 and equations (5) and (6),

$$\begin{aligned} I^{\gamma}_{\alpha,\beta,\omega} x^p &= \int_0^x (x-\tau)^{\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega^k (x-\tau)^{\alpha k}}{\Gamma(\gamma)\Gamma(\alpha k+\beta)k!} \tau^p \mathrm{d}\tau \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega^k \int_0^x (x-\tau)^{\beta-1}(x-\tau)^{\alpha k}\tau^p \mathrm{d}\tau}{\Gamma(\gamma)\Gamma(\alpha k+\beta)k!}. \end{aligned} \tag{16}$$

By using integration by parts, we obtain

$$\int_0^x (x-\tau)^{\alpha k+\beta-1} \tau^p \mathrm{d}\tau = \frac{\Gamma(\alpha k+\beta) p! x^{\alpha k+\beta+p}}{\Gamma(\alpha k+\beta+p+1)}. \tag{17}$$

Substituting (17) into (16) and after some algebra manipulation, we obtain

$$\begin{aligned} I^{\gamma}_{\alpha,\beta,\omega} x^p &= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega^k \Gamma(\alpha k+\beta) p! x^{\alpha k+\beta+p} / \Gamma(\alpha k+\beta+p+1)}{\Gamma(\gamma)\Gamma(\alpha k+\beta)k!} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega^k p! x^{\alpha k+\beta+p}}{\Gamma(\gamma)\Gamma(\alpha k+\beta+p+1)\Gamma(k+1)}. \end{aligned} \tag{18}$$

The expression in Theorem 1 is equivalent to that in Lemma 4 [20]. If $\gamma = \alpha = 1, \beta = 2, \omega = 3, p = 2$, Theorem 1, Definition 3, or Lemma 4 in [20] gives the same result which is $2/81e^{3x} - 1/9x^3 - 1/9x^2 - 2/27x - 2/81$. □

Theorem 2. *The Prabhakar derivative of order $0 < \gamma, \alpha, \beta < 1$ for $f(x) = x^p$ where p is positive integer can be defined as follows:*

$$D^{\gamma}_{\alpha,\beta,\omega} x^p = \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \Gamma(p+1) x^{\alpha k-\beta+p}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+p+1)\Gamma(k+1)}, \tag{19}$$

where γ is not equal to integer positive and $\alpha \neq \beta$.

Proof. Taking $m = 1$ in Definition 4, for sake of simplicity, we let $a = 0$ (that in equation (7)), and using equation (6) and applying the similar approach as in the proof of Theorem 1, we obtain the following results:

$$\begin{aligned} D^{\gamma}_{\alpha,\beta,\omega} x^p &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x e^{-\gamma}_{\alpha,1-\beta}(x-\tau;\omega)\tau^p \mathrm{d}\tau \\ &= \int_0^x -(x-\tau)^{-\beta-1}\beta \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k (x-\tau)^{\alpha k}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+1)k!}\tau^p d\tau \\ &\quad + \int_0^x (x-\tau)^{-\beta} \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k (x-\tau)^{\alpha k} k\alpha}{(x-\tau)\Gamma(-\gamma)\Gamma(\alpha k-\beta+1)k!}\tau^p \mathrm{d}\tau \\ &= -\beta \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \int_0^x (x-\tau)^{\alpha k-\beta-1}\tau^p \mathrm{d}\tau}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+1)k!} \\ &\quad + \alpha \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \int_0^x (x-\tau)^{\alpha k-\beta-1} x^p \mathrm{d}\tau}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+1)\Gamma(k)}. \end{aligned} \tag{20}$$

By using integration by parts, we obtain

$$\int_0^x (x-\tau)^{\alpha k-\beta-1}\tau^p \mathrm{d}\tau = \frac{\Gamma(\alpha k-\beta) p! x^{\alpha k-\beta+p}}{\Gamma(\alpha k-\beta+p+1)}. \tag{21}$$

Substituting (21) into (20) and after some algebra manipulation, we obtain

$$\begin{aligned} D^{\gamma}_{\alpha,\beta,\omega} x^p &= -\beta \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \Gamma(p+1) x^{\alpha k-\beta+p}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+p+1)(\alpha k-\beta)\Gamma(k)k} \\ &\quad + \alpha \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \Gamma(p+1) x^{\alpha k-\beta+p}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+p+1)\Gamma(k)(\alpha k-\beta)}, \end{aligned} \tag{22}$$

where $\alpha \neq \beta$. (22) can be further reduced as follows:

$$D^{\gamma}_{\alpha,\beta,\omega} x^p = \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k \Gamma(p+1) x^{\alpha k-\beta+p}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+p+1)\Gamma(k+1)}. \tag{23}$$

In (22), since the denominator consists of $\alpha k - \beta$, we must have $\alpha \neq \beta$. For $m = 1$, Theorem 2 for x^p is applicable for both Prabhakar derivative and regularized Prabhakar derivative. For regularized Prabhakar derivative [10], we have,

$${}^{C}D^{\gamma}_{\alpha,\beta,\omega} x^p = 0, \quad p = 0, 1, \dots, m-1,\ m = \lceil \beta \rceil. \tag{24}$$

The result obtained here can be verified via the series representation of fractional calculus operators involving generalized Mittag-Leffler functions (see Theorem 2.1 in 19). □

3. Operational Matrix for Prabhakar Fractional Derivative

In this section, we will derive the new operational matrix based on shifted Legendre Polynomials for Prabhakar fractional derivative.

Theorem 3. *Suppose $\mathbf{L}(x)$ is the shifted Legendre Polynomials vector*

$$\mathbf{L}(x) = [L_0(x), L_1(x), L_2(x), \ldots, L_N(x)]^T. \quad (25)$$

Let $0<\alpha<1$, $m=1$. Then,

$$D^{\gamma}_{\alpha,\beta,\omega} y(x) = \mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x), \quad (26)$$

where P^{α} is $(N+1)\times(N+1)$ operational matrix of fractional derivative of order α in Prabhakar sense and is defined as follows:

$$\mathbf{P}^{\gamma}_{\alpha,\beta,\omega} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{h=\lceil\beta\rceil}^{\lceil\beta\rceil} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,0} & \sum_{h=\lceil\beta\rceil}^{\lceil\beta\rceil} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,1} & \cdots & \sum_{h=\lceil\beta\rceil}^{\lceil\beta\rceil} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,0} & \sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,1} & \cdots & \sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{h=\lceil\beta\rceil}^{N} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,0} & \sum_{h=\lceil\beta\rceil}^{N} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,1} & \cdots & \sum_{h=\lceil\beta\rceil}^{N} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,N} \end{bmatrix}, \quad (27)$$

where $\theta^{\gamma}_{j,h,\alpha,\beta,\omega,l}$ is given by

$$\theta^{\gamma}_{j,h,\alpha,\beta,\omega,l} = \sum_{h=\lceil\alpha\rceil}^{j} \frac{(-1)^{j+h}(j+h)!}{(j-h)!(h!)^2} \sum_{k=0}^{\infty} \cdot \frac{\Gamma(-\gamma+k)\omega^k\Gamma(h+1)}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+h+1)\Gamma(k+1)} c_l, \quad (28)$$

and c_l can be obtained from inner product via (11) and $\lceil\,\rceil$ is the ceiling function.

Proof. From (9), we can write the shifted Legendre polynomials in analytical form and its fractional derivative in Prabhakar sense is given as in the following equation:

$$D^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) = \sum_{h=0}^{j} \frac{(-1)^{j+h}(j+h)!}{(j-h)!(h!)^2} D^{\gamma}_{\alpha,\beta,\omega}(x^h). \quad (29)$$

$D^{\gamma}_{\alpha,\beta,\omega}(x^h)$ can be calculated using Theorem 2.

$$D^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) = \sum_{h=0}^{j} \frac{(-1)^{j+h}(j+h)!}{(j-h)!(h!)^2} \sum_{k=0}^{\infty} \cdot \frac{\Gamma(-\gamma+k)\omega^k\Gamma(h+1)x^{\alpha k-\beta+h}}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+h+1)\Gamma(k+1)}. \quad (30)$$

Let $f(x) = x^{\alpha k-\beta+h}$; by using truncated shifted Legendre polynomials, we have $f(x) = \sum_{l=0}^{N} c_l L_l(x)$. Substituting this in (30), we obtain

$$\begin{aligned} D^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) &= \sum_{l=0}^{N}\left(\sum_{h=\lceil\beta\rceil}^{j} \frac{(-1)^{j+h}(j+h)!}{(j-h)!(h!)^2} \sum_{k=0}^{\infty} \frac{\Gamma(-\gamma+k)\omega^k\Gamma(h+1)}{\Gamma(-\gamma)\Gamma(\alpha k-\beta+h+1)\Gamma(k+1)} c_l\right) L_l(x) \\ &= \sum_{l=0}^{N}\left(\sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,l}\right) L_l(x), \end{aligned} \quad (31)$$

where $\theta^{\gamma}_{j,h,\alpha,\beta,\omega,l}$ is given in (28). Rewriting (31) in vector form, we have

$$D^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) = \left[\sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,0} \sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,1} \cdots \sum_{h=\lceil\beta\rceil}^{j} \theta^{\gamma}_{j,h,\alpha,\beta,\omega,N}\right]\mathbf{L}(x), \quad (32)$$

where $j = \lceil\beta\rceil \ldots N$. For $j = 0, 1, \ldots, \lceil\beta\rceil - 1$, we have

$$D^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) = [0, 0, \ldots 0]\mathbf{L}(x), \quad j = 0, 1, \ldots, \lceil\beta\rceil - 1. \quad (33)$$

Hence, by combining (32) and (33), the Legendre operational matrix for Prabhakar derivative is proved as in (27).

Since this is the first time that operational matrix for Prabhakar derivative is derived, to show that the operational matrix of Prabhakar derivative is correct, we find the exact solution of Prabhakar derivative with $\gamma = 1/2, \alpha = 1/2, \beta = 1/4, \omega = 1/4$ for first few terms of shifted Legendre polynomials. The corresponding operational matrix when $N = 2$ is given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 1.141646881 & 0.8989013133 & -0.1256075578 \\ -0.8989013137 & 0.8036012368 & 1.256894671 \end{bmatrix}. \quad (34)$$

The comparison between the exact solution for Prabhakar derivative for $L_1(x)$ and $L_2(x)$ with the approximation using operational matrix as in (27) is shown in Figures 1 and 2. The accuracy of the approximation can be increased by using bigger N. □

4. Proposed Scheme and Error Analysis

Basically, in order to solve fractional differential equations in Prabhakar sense as in equation (1) using the operational matrix method, the following procedure can be applied.

Step 1. Write each terms of fractional differential equations in terms of shifted Legendre polynomials, say $D^{\gamma}_{\alpha,\beta,\omega} y(x) + y(x) = g(x)$. Using (26), we have $D^{\gamma}_{\alpha,\beta,\omega} y(x) = \mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x)$. Also, we can have $y(x) = \sum_{k=0}^{N} c_k \mathbf{L}(x)$ and $g(x) = \sum_{k=0}^{N} g_k \mathbf{L}(x)$.

Step 2. From $\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) + \sum_{k=0}^{N} c_k \mathbf{L}(x) = \sum_{k=0}^{N} g_k \mathbf{L}(x)$, we obtain a system of algebraic equation for the unknown variables c_k. Solving the system to obtain the values for c_k, the solution of fractional differential equations in Prabhakar sense is given by $y(x) = \sum_{k=0}^{N} c_k \mathbf{L}(x)$.

For the error estimation for the numerical scheme, let us consider the residual correction procedure which can be used to estimate the absolute error. From equation (1), i.e., $D^{\gamma}_{\alpha,\beta,\omega} y(x) = g(x, y(x))$,

$$D^{\gamma}_{\alpha,\beta,\omega} y(x) - g(x, y(x)) = 0. \quad (35)$$

If $N \longrightarrow \infty$, using the operational matrix via shifted Legendre polynomials and approximate $g(x, y(x))$ via shifted Legendre polynomials, we obtain

$$\left|\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x) - g(x, y_N(x))\right| \approx 0, \quad N \longrightarrow \infty. \quad (36)$$

However, in practical, N is finite, say m term of shifted Legendre polynomials had been used. There will be a small error, e_m.

$$\begin{aligned} &\left|D^{\gamma}_{\alpha,\beta,\omega} y(x) - \mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\mathbf{L}(x)\right|_2 \\ &\quad + \left|g(x, y_\infty(x)) - g(x, y_N(x))\right|_2 = e_m, \quad N = m. \end{aligned} \quad (37)$$

Let e_m^* is the approximation solution of (1) obtained by the operational matrix method, if $\|e_m - e_m^*\|\epsilon$ are sufficiently small, then the absolute errors e_m can be estimated by e_m^*. Hence, the optimal value m (i.e. N) can be obtained.

5. Numerical Examples

In this section, some examples are presented to illustrate the applicability and accuracy of this new operational matrix for the Prabhakar derivative. All the computations are done by using the software Maple.

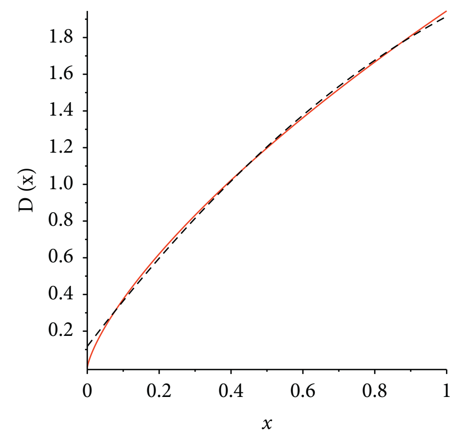

FIGURE 1: Comparison of the exact solution and approximation for $L_1(x)$.

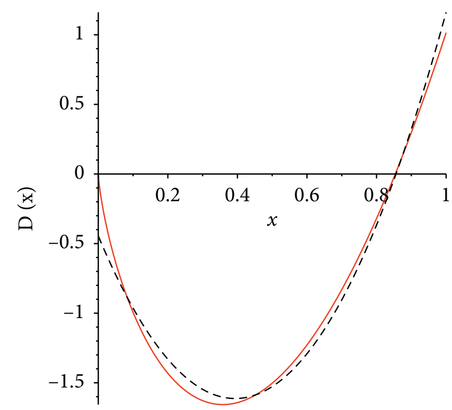

FIGURE 2: Comparison of the exact solution and approximation for $L_2(x)$.

Example 1. Consider a simple fractional differential equation defined in Prabhakar derivative.

$$\begin{aligned} D^{\gamma}_{\alpha,\beta,\omega} y(x) = &-\frac{32x^{7/4}}{315\Gamma(3/4)\pi}\left(7\sqrt{2x}\left(\Gamma\left(\frac{3}{4}\right)\right)^2 {}_2F_2\left(\frac{1}{4},\frac{3}{4};\frac{3}{2},\frac{13}{4};x\right)\right. \\ &\left. - 15\,{}_2F_2\left(\frac{-1}{4},\frac{1}{4};\frac{1}{2},\frac{11}{4};x\right)\pi\right), \end{aligned} \quad (38)$$

where $\gamma = 1/2, \alpha = 1/2, \beta = 1/4, \omega = 1$. The exact solution is given by $y(x) = x^2$.

By using the proposed method, we obtain the following approximation as shown in Table 1. Here, we compare it with the iteration method introduced recently in [17] with $h = 0.02$. From the numerical result, by using very small terms, i.e., $N = 2$, we were able to obtain a good result compared to the iteration method in [17].

Table 1: Comparison of the absolute errors obtained by proposed method with the iteration method used in [17] for Example 1.

x	Exact solution	Abs. Error (iteration method)	Abs. Error (proposed method)
0.1	0.01	2.67306E-03	3.28100E-03
0.2	0.04	4.29289E-03	5.83288E-03
0.3	0.09	5.99314E-03	7.65566E-03
0.4	0.16	7.75504E-03	8.74932E-03
0.5	0.25	9.56803E-03	9.11388E-03
0.6	0.36	1.14254E-02	8.74932E-03
0.7	0.49	1.33229E-02	7.65566E-03
0.8	0.64	1.52571E-02	5.83288E-03
0.9	0.81	1.72261E-02	3.28100E-03

Table 2: Comparison of the absolute errors obtained by proposed method with the iteration method used in [17] for Example 2.

x	Exact solution	Abs. Error (iteration method)	Abs. Error (proposed method)
0.1	0.02	$1.94868E-03$	$2.68860E-03$
0.2	0.08	$3.93195E-03$	$4.51364E-03$
0.3	0.18	$5.92093E-03$	$5.47514E-03$
0.4	0.32	$7.91259E-03$	$5.57308E-03$
0.5	0.50	$9.90584E-03$	$4.80748E-03$
0.6	0.72	$1.19001E-02$	$3.17832E-03$
0.7	0.98	$1.38952E-02$	$6.85620E-04$
0.8	1.28	$1.58908E-02$	$2.67063E-03$
0.9	1.62	$1.78870E-02$	$6.89044E-03$

Example 2. Consider a simple fractional differential equation defined in Prabhakar derivative.

$$D^{\gamma}_{\alpha,\beta,\omega}y(x)+y(x)=2x^2+\frac{32\sqrt{2}\Gamma(3/4)x^{5/4}}{5\pi}{}_3F_3\left(\frac{-1}{6},\frac{1}{6},\frac{1}{2};\frac{1}{3},\frac{2}{3},\frac{9}{4};\frac{x}{64}\right)$$
$$-\frac{72x^{19/12}}{133\Gamma(7/12)}{}_3F_3\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};\frac{2}{3},\frac{4}{3},\frac{31}{12};\frac{x}{64}\right)-\frac{9x^{23/12}}{506\Gamma(11/12)}{}_3F_3\left(\frac{1}{2},\frac{5}{6},\frac{7}{6};\frac{4}{3},\frac{5}{3},\frac{35}{12};\frac{x}{64}\right), \tag{39}$$

where $\gamma=1/2,\alpha=1/3,\beta=3/4,\omega=1/4$. The exact solution is given by $y(x)=2x^2$.

By using the proposed method, we obtain the following approximation as shown in Table 2. Here, we compare it with the iteration method introduced recently in [17] with smaller $h=0.01$. From the numerical result, by using very small terms, i.e., $N=2$, we were able to obtain a good result compared to the iteration method in [17].

As can be seen from both examples, the absolute error for iteration method introduced in [17] will increase when the iteration step is increasing. This will not occur for the operational matrix method.

6. Conclusion

In this paper, we had successfully derived a new operational matrix based on shifted Legendre polynomials for solving fractional differential equations in Prabhakar sense. Only a few terms of shifted Legendre polynomials are needed to obtain a good approximation. Using the same process, the operational matrix can be derived using other types of polynomials. For future work, we hope to extend the existing fractional calculus problem such as in [21, 22] to Prabhakar fractional derivative.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

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