Research Article

Certain Families of Analytic Functions Characterized by (p, q) -Difference Operator

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The major motivation behind this work is to utilize (p, q) -calculus and inspected the generalized (p, q) -Ruscheweyh differential operator. At the present time, we utilized (p, q) -Ruscheweyh differential operator and examined some new subfamilies of $\mathcal{S}$ and study some essential properties, for example, inclusion and subordination properties.

1. Introduction

The q -calculus is the speculation of ordinary calculus where the limits has been supplanted by q . The chronicled background of q -calculus may be portrayed by its wide variety of use in quantum mechanics, logical number hypothesis, theta functions, Bernoulli and Euler polynomials, Sobolev spaces, and more in the analytic and univalent function theory. Jackson ought to be praised for the specific start of quantum calculus in his work in [1, 2], by presenting the q -derivative and integral, respectively. In q -calculus, we essentially center around q -analogue that arise normally, rather than in arbitrary q -analogues of known results. While zeroing in on advancements and improvement of q -calculus and its applications in explicit fields of numerical and actual science, we will moreover look at q -analogues of a part of the new results in Geometric Function Theory in unit disk. The early work of q -calculus in the field of Geometric Function Theory was finished by Ismail et al. [3], by summing up the class of star-like functions into the class of q -star-like functions.

Assume that $f \in \mathbb{C}$. Then,

$$f \text{ is normalized analytic function} \iff f'(0) - 1 = 0 = f(0) \quad (1)$$

and characterized as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}. \quad (2)$$

The symbol $\mathcal{A}$ stands for the family of all these type of functions. Also, by $\mathcal{S} \subset \mathcal{A}$, comprising of functions, which are univalent in $\mathcal{E}$. By $\mathcal{S}^*(\alpha)$ ($0 \leq \alpha < 1$), we meant the subfamily of $\mathcal{S}$ consisting of star-like functions of order α . If $\alpha = 0$, then $\mathcal{S}^*(\alpha) = \mathcal{S}^*$.

Let $\wp \in \mathbb{C}$ be analytic in $\mathcal{E}$. Furthermore, we say that

$$\begin{aligned} \wp \in \mathcal{P} &\iff \wp(0) = 1, \\ \operatorname{Re}(\wp(z)) &> 0, z \in \mathcal{E}, \end{aligned} \quad (3)$$

and presented as

$$\wp(z) = 1 + \sum_{k=1}^{\infty} c_k z^k. \quad (4)$$

For f_1 and f_2 given by

$$\begin{aligned}
f_1(z) &= z + \sum_{k=2}^{\infty} a_k z^k, \\
f_2(z) &= z + \sum_{k=2}^{\infty} b_k z^k, \\
z &\in \mathcal{E},
\end{aligned} \tag{5}$$

the convolution is characterized as

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (f_2 * f_1)(z), \quad z \in \mathcal{E}. \tag{6}$$

Assume f and g be analytic in $\mathcal{E}$. Then, $f < g$ if there exists ω analytic in $\mathcal{E}$, along $\omega(0) = 0$ and $|\omega(z)| < 1$, in $\mathcal{E}$ such as

$$f(z) = g(\omega(z)), \quad z \in \mathcal{E}. \tag{7}$$

Also, it is observed by [4] that

$$f < g \iff f(0) = g(0) \text{ and } f(\mathcal{E}) \subset g(\mathcal{E}). \tag{8}$$

Kanas and Wisniowska [5] extended the parabolic region and characterized the region $Y_k, 0 \leq k < \infty$ as

$$Y_k = \left\{ w = s + it \in \mathbb{C} : s > k\sqrt{(s-1)^2 + t^2} \right\}, \tag{9}$$

with $1 \in Y_k$. Assume

$$\partial Y_k = \left\{ w = s + it : s^2 = k^2((s-1)^2 + t^2) \right\}. \tag{10}$$

Therefore,

$$\begin{aligned}
k = 0 &\iff Y_0 = s > 0, \\
k = 1 &\iff Y_1 = t^2 < 2s - 1, \\
k \in (0, 1) &\iff Y_k = s > k\sqrt{(s-1)^2 + t^2}.
\end{aligned} \tag{11}$$

Identified with the region Y_k , the accompanying functions are extremal and $\wp_k(\mathcal{E}) \subset Y_k$

$$\wp_k(z) = \begin{cases} \frac{1+z}{1-z}, & k=0, \\ 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z} \right], & 0 < k < 1, \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k=1. \end{cases} \tag{12}$$

Two fascinating subfamilies of $\mathcal{S}$ are $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ which consist of k -uniformly convex and k -uniformly star-like functions, respectively, and characterized by [5, 6] for $k \geq 0$ as

$$\begin{aligned}
k-\mathcal{UCV} &= \left\{ g \in \mathcal{S} : \operatorname{Re} \left(1 + \frac{zg''(z)}{g'(z)} \right) > k \left| \frac{zg''(z)}{g'(z)} \right|, z \in \mathcal{E} \right\}, \\
k-\mathcal{ST} &= \left\{ g \in \mathcal{S} : \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > k \left| \frac{zg'(z)}{g(z)} - 1 \right|, z \in \mathcal{E} \right\}.
\end{aligned} \tag{13}$$

Note that

$$g \in k-\mathcal{UCV} \iff zg' \in k-\mathcal{ST}. \tag{14}$$

Geometrically, $g \in k-\mathcal{UCV}$ means that the image of $\mathcal{E}$ under $[(zg'(z)) / g'(z)]$ is contained in Y_k , where Y_k is defined by (9).

In recent years, researchers are using these conic domain defined by (9) very efficiently and obtained very excellent results some of them are [7, 8] and reference therein.

Remark 1. All through this study, we mean by $\mathcal{H}$ the family of functions h that are convex univalent in $\mathcal{E}$ along $h(0) = 1$, except if in any case referenced.

2. (p, q)-Calculus

Expanding the concept of q -calculus, (p, q) -number was freely considered by [9]. Let $0 < q < p \leq 1$; then, (p, q) -number is characterized for $j \in \mathbb{N}$ by

$$[j]_{(p,q)} = \frac{p^j - q^j}{p - q} = p^{j-1} + p^{j-2}q + \dots + pq^{j-2} + q^{j-1}, \tag{15}$$

which is the common generalization of q -number. Note that $[j]_{(1,1)} = j$. Also, for $q \in \mathbb{N}$, we obtain

$$\begin{aligned}
[q+1]_{(p,q)} &= q^q + p[q]_{(p,q)}, \\
[q+j]_{(p,q)} &= q^{q+j-1} + p^j [q]_{(p,q)}.
\end{aligned} \tag{16}$$

Furthermore,

$$\begin{aligned}
[q+j]_{(p,q)} - p^j [q]_{(p,q)} &= \frac{p^{q+j} - q^{q+j}}{p - q} - \frac{p^j (p^q - q^q)}{p - q} \\
&= \frac{q^q (p^j - q^j)}{p - q} \\
&= q^q [j]_{(p,q)}.
\end{aligned} \tag{17}$$

In [9], for $f \in \mathbb{C}$, (p, q) -derivative of is characterized as

$$\mathfrak{d}_{(p,q)} f(z) = \begin{cases} \frac{f(qz) - f(pz)}{(q-p)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (18)$$

Moreover,

$$f(z) = z^k \iff \mathfrak{d}_{(p,q)} f(z) = [k]_{(p,q)} z^{k-1}. \quad (19)$$

Also, by utilizing (2), we attain

$$\mathfrak{d}_{(p,q)} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{(p,q)} a_k z^{k-1}. \quad (20)$$

Note, if $p = 1$, at that point

$$\mathfrak{d}_{(1,q)} f(z) = \mathfrak{d}_q f(z). \quad (21)$$

Also, $\mathfrak{d}_{(1,1)} f(z) = f'(z)$. It is notable [10] that

$$\begin{aligned} \mathfrak{d}_{(p,q)} (v(z)u(z)) &= v(pz)\mathfrak{d}_{(p,q)} u(z) \\ &\quad + u(qz)\mathfrak{d}_{(p,q)} v(z), \end{aligned} \quad (22)$$

$$\begin{aligned} \text{or } \mathfrak{d}_{(p,q)} (v(z)u(z)) &= u(pz)\mathfrak{d}_{(p,q)} v(z) \\ &\quad + v(qz)\mathfrak{d}_{(p,q)} u(z). \end{aligned} \quad (23)$$

The (p, q) -integral of f presented in [10] is given by

$$\frac{1}{z(p-q)} \int_0^z \mathfrak{d}_{(p,q)} t = \sum_{j=0}^{\infty} \left(\frac{q^j}{p^{j+1}} \right) f \left(\frac{q^j}{p^{j+1}} z \right), \quad \text{if } \left| \frac{p}{q} \right| > 1. \quad (24)$$

Note that if $p = 1$, definition (24) reduces to the well-known Jackson integral defined in [2]. We refer to a notion of (p, q) -operators, that is, (p, q) -operator, that play imperative job in the hypothesis of hypergeometric series, operator theory, and quantum physics. The utilization of q -calculus was started by Jackson.

As of late, Akbar et al. [11] started the possibility of symmetric (p, q) -calculus, presented symmetric (p, q) -derivative, and examined another class of star-like functions along intriguing results.

2.1. (p, q) -Generalized Pochhammer Symbol. The investigation of star-like functions via (p, q) -calculus and various analytic functions classes were reported in [11, 12], respectively. The (p, q) -Gamma function is characterized by the accompanying relation:

$$\begin{aligned} \Gamma_{(p,q)}(r) &= [r]_{(p,q)} \Gamma_{(p,q)}(r-1), \\ \Gamma_{(p,q)}(1) &= 1. \end{aligned} \quad (25)$$

Expanding the concept of q -generalized Pochhammer symbol, the (p, q) -shifted factorial ((p, q) -generalized symbol) is characterized as

$$[r]_{(p,q)} = \begin{cases} 1, & k = 0, \\ [r]_{(p,q)} [r+1]_{(p,q)} \cdots [r+k-1]_{(p,q)}, & k \in \mathbb{N}. \end{cases} \quad (26)$$

Note

$$\left([r]_{(1,q)} \right)_k = \left([r]_q \right)_k \text{ and } (r)_{(1,1)k} = (r)_k. \quad (27)$$

Now, by generalizing the concept of [13], we introduced the (p, q) -Ruscheweyh differential operator as

$$\mathfrak{R}_{(p,q)}^q f(z) = f(z) * \mathfrak{F}_{(p,q)}^{q+1}(z), \quad q > -1, 0 < q < p \leq 1, z \in \mathcal{E}, \quad (28)$$

where

$$\mathfrak{F}_{(p,q)}^q(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{[k-1]_{(p,q)}! \Gamma_{(p,q)}(1+q)} z^k, \quad (29)$$

$$= z + \sum_{k=2}^{\infty} \frac{\left([q+1]_{(p,q)} \right)_{k-1}}{[k-1]_{(p,q)}!} z^k. \quad (30)$$

2.2. Special Cases. If $p = 1, q \iff 1^-$, then

$$\mathfrak{R}_{(1,1)}^q f(z) = f(z) * \mathfrak{F}_{(1,1)}^{q+1}(z) = \mathfrak{R}^q f(z), \quad (31)$$

see [14].

Also, if $p = 1$, then

$$\mathfrak{R}_{(1,q)}^q f(z) = \mathfrak{R}_q^q(z) = f(z) * \mathfrak{F}_q^{q+1}(z), \quad (32)$$

see [13].

Therefore,

$$\mathfrak{R}_{(p,q)}^q f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{\Gamma_{(p,q)}(1+q)[k-1]_{(p,q)}!} a_k z^k, \quad z \in \mathcal{E}. \quad (33)$$

Using (p, q) -derivative defined by (19), we observed that

$$\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) = 1 + \sum_{k=2}^{\infty} \frac{\Gamma_{(p,q)}(k+q)}{\Gamma_{(p,q)}(1+q)[k-1]_{(p,q)}!} [k]_{(p,q)} a_k z^{k-1}. \quad (34)$$

If $p = 1, q \iff 1^-$ and $q = 0$, then

$$\mathfrak{d}_{(1,1)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} = f'(z), \quad z \in \mathcal{E}. \quad (35)$$

Utilizing (35) and $q \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, it can, without much of a stretch, be confirmed that

$$\begin{aligned} [q+1]_{(p,q)} \mathfrak{R}_{(p,q)}^q f(z) &= q^q z \left(\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^q f(z) \right) \right) \\ &\quad + p[q]_{(p,q)} \mathfrak{R}_{(p,q)}^q f(z). \end{aligned} \quad (36)$$

By taking $p = 1$, we obtain

$$[q+1]_q \mathfrak{R}_q^q f(z) = q^q z \left(d_q \left(\mathfrak{R}_q^q f(z) \right) \right) + [q]_q \mathfrak{R}_q^q f(z), \quad (37)$$

see [15].

Also, if $p = 1, q \iff 1^-$, we observe

$$(\varrho + 1)\mathfrak{R}^{\varrho} f(z) = z(\mathfrak{R}^{\varrho} f(z))' + \varrho \mathfrak{R}^{\varrho} f(z), \quad (38)$$

see [14].

Using $(\mathfrak{p}, \mathfrak{q})$ -derivative defined by (19), we define the class $k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho)$ by the following.

Definition 1. Assume that $f \in \mathcal{A}$ is defined by (2) and $h \in \mathcal{H}$ is given as in Remark 1. Then, for $0 < \varrho < \mathfrak{p} \leq 1$,

$$f \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff \left(\frac{z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f(z))}{\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f(z)} \right) \prec \wp_k(z), \quad (39)$$

$$k \geq 0, z \in \mathcal{E},$$

where $\wp_k$ are defined by (12).

We can define the corresponding class $k - \mathcal{UCV}_{(\mathfrak{p}, \mathfrak{q})}(\varrho)$ as

$$f \in k - \mathcal{UCV}_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff z(\mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})} f) \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho). \quad (40)$$

Equivalently,

$$(\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f) \in k - \mathcal{ST} \iff f \in k - \mathcal{ST}_{(\mathfrak{p}, \mathfrak{q})}(\varrho). \quad (41)$$

Also,

$$f \in \mathcal{S}^*_{(\mathfrak{p}, \mathfrak{q})}(\varrho) \iff (\mathfrak{R}^{\varrho}_{(\mathfrak{p}, \mathfrak{q})} f) \in \mathcal{S}^*. \quad (42)$$

3. Main Results

To get our main results, the accompanying lemmas are helpful.

Lemma 1. Assume that $s = s_1 + is_2, t = t_1 + it_2$ and $\chi: D \subset \mathbb{C} \times \mathbb{C} \iff \mathbb{C}$ along

- (i) $\chi(s, t)$ is continuous in D
- (ii) $\chi(1, 0) > 0$ and $(1, 0) \in D$
- (iii) $\text{Re}(\chi(is_2, t_1)) \leq 0$, for $(is_2, t_1) \in D$ and $t_1 \leq -(1/2)(1 + s_2^2)$

Suppose that $\wp(z) = 1 + c_1 z + c_2 z^2 + \dots$ belongs to $\mathcal{P}$ along $(\wp, z\wp') \in D \forall z \in \mathcal{E}$. If $\text{Re}(\chi(\wp(z), z\wp'(z))) > 0$, then $\text{Re}(\wp(z)) > 0$; for details, see [16].

The attendant lemma is $(\mathfrak{p}, \mathfrak{q})$ -analogue of the lemma presented by [17].

Lemma 2. Assume that $h \in \mathcal{H}$ be defined as in Remark 1. Furthermore, let $\beta_0, \gamma_0 \in \mathbb{C}$ along $\beta_0 \neq 0$. If $\wp \in \mathcal{P}$, fulfill $\text{Re}(\beta_0 \wp(z) + \gamma_0) > 0$ and

$$\wp(z) + \frac{z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})} f(z)}{\beta_0 \wp(z) + \gamma_0} \prec h(z) \iff \wp(z) \prec h(z), \quad z \in \mathcal{E}. \quad (43)$$

The accompanying lemma is $(\mathfrak{p}, \mathfrak{q})$ -analogue of the lemma presented by [15] and will be required for later use.

Lemma 3. Assume that $h \in \mathcal{H}$ and $s(z) = 1 + s_1 z + s_2 z^2 + \dots$ be analytic in $\mathcal{E}$. Furthermore, if

$$s(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(s(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \prec h(z), \quad z \in \mathcal{E}. \quad (44)$$

Then, for $[\gamma]_{(\mathfrak{p}, \mathfrak{q})} \neq 0$, we have

$$s(qz) \prec t(qz) \prec h(z), \quad (45)$$

where

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \int_0^z t^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h(t) d_{(\mathfrak{p}, \mathfrak{q})} t \quad (46)$$

$$= (\mathfrak{p} - \mathfrak{q}) [\gamma]_{(\mathfrak{p}, \mathfrak{q})} \sum_{j=0}^{\infty} \frac{\mathfrak{q}^{j[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}}{\mathfrak{p}^{(j+1)[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} h\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z\right),$$

is the best super-ordinate of (46).

Proof. Assume that $h \in \mathcal{H}$ and

$$s(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(s(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \prec h(z), \quad z \in \mathcal{E}. \quad (47)$$

To find the best dominant of (47), we have to find the solution of corresponding differential equation:

$$t(qz) + \frac{\mathfrak{p}^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} z \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))}{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} = h(z), \quad z \in \mathcal{E}. \quad (48)$$

This means that, making use of product rules (22) or (23), we obtain

$$[\gamma]_{(\mathfrak{p}, \mathfrak{q})} z^{\gamma-1} h(z) = (\mathfrak{p}z)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))$$

$$+ [\gamma]_{(\mathfrak{p}, \mathfrak{q})} z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} t(qz) = (\mathfrak{p}z)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}(t(z))$$

$$+ t(qz) d_{(\mathfrak{p}, \mathfrak{q})}\left(z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}\right) = \mathfrak{d}_{(\mathfrak{p}, \mathfrak{q})}\left(z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}} t(z)\right). \quad (49)$$

Consequently,

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \int_0^z t^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h(t) d_{(\mathfrak{p}, \mathfrak{q})} t, \quad (50)$$

which is best dominant of (46). By using (24), we can right

$$t(z) = \frac{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}{z^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} z (\mathfrak{p} - \mathfrak{q}) \sum_{j=0}^{\infty} \left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} \right) \left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z \right)^{[\gamma]_{(\mathfrak{p}, \mathfrak{q})}-1} h$$

$$\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z \right) [\gamma]_{(\mathfrak{p}, \mathfrak{q})} (\mathfrak{p} - \mathfrak{q}) \sum_{j=0}^{\infty} \left(\frac{\mathfrak{q}^{j[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}}{\mathfrak{p}^{(j+1)[\gamma]_{(\mathfrak{p}, \mathfrak{q})}}} \right) h\left(\frac{\mathfrak{q}^j}{\mathfrak{p}^{j+1}} z\right), \quad (51)$$

which is required.

If we take $\mathfrak{p} = 1$ in Lemma 3, we obtain the well-known result from [15]. $\square$

Theorem 1. Assume that $k \geq 0, 0 < \mathfrak{q} < \mathfrak{p} \leq 1$,

$$k - \mathcal{UCV}_{(p,q)}(\varrho) \subset k - \mathcal{ST}_{(p,q)}(\varrho), \quad z \in \mathcal{E}. \quad (52)$$

$$\frac{z \mathfrak{d}_{(p,q)}(\mathfrak{R}_{(p,q)}^{\varrho} f(z))}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = \hbar(z), \quad \hbar \in \mathcal{P}. \quad (53)$$

Proof. Assume $f \in k - \mathcal{UCV}_{(p,q)}(\varrho)$ and consider

Presently utilizing (36), we obtain

$$\hbar(z) = \frac{\left([\varrho + 1]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \left(\mathfrak{p} [\varrho]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = \frac{\left([\varrho + 1]_{(p,q)} / \mathfrak{q}^{\varrho}\right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{\mathfrak{p} [\varrho]_{(p,q)}}{\mathfrak{q}^{\varrho}}. \quad (54)$$

Differentiating (53), $(\mathfrak{p}, \mathfrak{q})$ -logarithmically, we have

$$\hbar(z) + \frac{z \mathfrak{d}_{(p,q)} \hbar(z)}{\hbar(z)} = \frac{\mathfrak{d}_{(p,q)} \left(z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right) \right)}{\mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)} < \wp_k(z). \quad (55)$$

Therefore,

$$\hbar(z) + \frac{z \mathfrak{d}_{(p,q)} \hbar(z)}{\hbar(z)} < \wp_k(z), \quad z \in \mathcal{E}. \quad (56)$$

Along these lines by utilizing Lemma 2, we have $\hbar(z) < \wp_k(z)$, and thus,

$$\frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} < \wp_k \iff f \in k - \mathcal{ST}_{(p,q)}(\varrho) \quad (57)$$

which is required.

Note that, by taking specific values to the parameters in Theorem 1, we can get some notable results.

For $\varrho = 0$, $\mathfrak{p} = 1$, and $q \iff 1^-$, we obtain accompanying crucial result, see [6]. $\square$

Corollary 1. Assume that $k \geq 0$ and $\forall z \in \mathcal{E}$,

$$k - \mathcal{UCV} \subseteq k - \mathcal{ST}. \quad (58)$$

Corollary 2. Assuming that $k = 0$ and $\forall z \in \mathcal{E}$, we obtain $\mathcal{C} \subseteq \mathcal{S}^*$.

Theorem 2. Suppose that, for $k \geq 0$, and $\varrho \in \mathbb{N}_0$,

$$\begin{aligned} k - \mathcal{ST}_{(p,q)}(\varrho + 1) &\subseteq k - \mathcal{ST}_{(p,q)}(\varrho) \subseteq \dots \subseteq k \\ &- \mathcal{ST} \subseteq \mathcal{S}^*(\eta_0) \subseteq \mathcal{S}^* \subseteq \mathcal{S}. \end{aligned} \quad (59)$$

Equivalently, for $\eta_0 = k/k + 1$,

$$\begin{aligned} \mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0) &\subseteq \mathcal{S}_{(p,q)}^*(\varrho, \eta_1) \subseteq \dots \subseteq \mathcal{S}_{(p,q)}^* \\ (\varrho - (m - 1), \eta_m) &\subseteq \mathcal{S}_{(p,q)}^*(\varrho - m, \eta_{m+1}), \end{aligned} \quad (60)$$

where $\varrho > m \in \mathbb{N}_0$ and

$$\eta_{m+1} = \frac{\left[2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) - \mathfrak{q}^{\varrho} \right] - \sqrt{\left[2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) - \mathfrak{q}^{\varrho} \right]^2 + 8 \left[2\eta_0 \mathfrak{p} \left([\varrho]_{(p,q)} - [m]_{(p,q)} \right) + \mathfrak{q}^{\varrho} \right]}}{4}. \quad (61)$$

Proof. Firstly, consider

$$k - \mathcal{ST}_{(p,q)}(\varrho + 1) \subseteq k - \mathcal{ST}_{(p,q)}(\varrho), \quad (62)$$

where

or

$$\eta_1 = \frac{\left(2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} [\varrho]_{(p,q)} - \mathfrak{q}^{\varrho} \right) - \sqrt{\left(2\mathfrak{q}^{\varrho} \eta_0 - 2\mathfrak{p} [\varrho]_{(p,q)} - \mathfrak{q}^{\varrho} \right)^2 + 8 \left(2\eta_0 \mathfrak{p} [\varrho]_{(p,q)} + \mathfrak{q}^{\varrho} \right)}}{4}. \quad (64)$$

$$\mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0) \subseteq \mathcal{S}_{(p,q)}^*(\varrho, \eta_1), \quad (63)$$

Let $f \in \mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0)$. Furthermore, let

$$\frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = (1 - \eta_1) p_1(z) + \eta_1, \quad p_1 \in \mathcal{P}. \quad (65)$$

Using (37), we obtain

$$\begin{aligned} & \frac{\left(\left([\varrho + 1]_{(p,q)} \right) / \mathfrak{q}^{\varrho} \right) \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{\mathfrak{p}[\varrho]_{(p,q)}}{\mathfrak{q}^{\varrho}} \\ &= (1 - \eta_1) p_1(z) + \eta_1. \end{aligned} \quad (66)$$

This implies that

$$\frac{[\varrho + 1]_{(p,q)} \mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} = \mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}. \quad (67)$$

Differentiating (p, q) -logarithmically, we attain

$$\begin{aligned} & \frac{z \mathfrak{d}_{(p,q)} \left(\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) \right)}{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)} = (1 - \eta_1) p_1(z) + \eta_1 \\ &+ \frac{(1 - \eta_1) z \mathfrak{d}_{(p,q)} (p_1(z))}{\mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}}. \end{aligned} \quad (68)$$

Since

$$f \in k - \mathcal{ST}_{(p,q)}(\varrho + 1) \subseteq \mathcal{S}_{(p,q)}^*(\varrho + 1, \eta_0), \quad \eta_0 = \frac{k}{k+1}. \quad (69)$$

This means that

$$\begin{aligned} & \operatorname{Re} \left((1 - \eta_1) p_1(z) + \eta_1 + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1) z \mathfrak{d}_{(p,q)} (p_1(z))}{\mathfrak{q}^{\varrho} (1 - \eta_1) p_1(z) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}} \right) \\ &> \eta_0, \quad z \in \mathcal{E}. \end{aligned} \quad (70)$$

Now, we construct a functional $\Psi(s, \nu)$ with $s = s_1 + is_2 = p_1(z)$ and $\nu = \nu_1 + i\nu_2 = z \mathfrak{d}_{(p,q)} (p_1(z))$ as

$$\Psi(s, \nu) = (1 - \eta_1)s + (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu}{\mathfrak{q}^{\varrho} (1 - \eta_1)s + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}}. \quad (71)$$

It very well may be seen that, for the functional $\Psi(s, \nu)$,

- (i) $\Psi(s, \nu)$ is continuous in $\mathbb{D} = \mathbb{C} - \left\{ \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} / \mathfrak{q}^{\varrho} (\eta_1 - 1) \right\} \times \mathbb{C}$
- (ii) $(1, 0) \in \mathbb{D} = \mathbb{C} - \left\{ \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} / \mathfrak{q}^{\varrho} (\eta_1 - 1) \right\} \times \mathbb{C}$ and $\operatorname{Re}(\Psi(1, 0)) = 1 - \eta_0 > 0$

Now, consider

$$\begin{aligned} & \operatorname{Re}(\Psi(is, \nu_1)) - \operatorname{Re} \left((1 - \eta)is_2 + (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu_1}{\mathfrak{q}^{\varrho} (1 - \eta_1)is_2 + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)}} \right) \\ &= (\eta_1 - \eta_0) + \frac{\mathfrak{q}^{\varrho} (1 - \eta_1)\nu_1}{\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2} \leq (\eta_1 - \eta_0) - \frac{(\mathfrak{q}^{\varrho} (1 - \eta_1)^2 (1 + s_2^2) + \mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})}{2(\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2)}, \end{aligned} \quad (72)$$

because $\nu_1 \leq -(1 + s_2^2/2)$. Therefore, after some simplifications, we obtain

$$\operatorname{Re}(\Psi(is_2, \nu_1)) = \frac{A + Bs_2^2}{2C}, \quad (73)$$

where

$$\begin{aligned} A &= \left(2(\eta_1 - \eta_0) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right)^2 - \mathfrak{q}^{\varrho} (1 - \eta_1) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right) \right), \\ B &= \left(2\mathfrak{q}^{2\varrho} (\eta_1 - \eta_0) (1 - \eta_1)^2 - \mathfrak{q}^{\varrho} (1 - \eta_1) \left(\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)} \right) \right), \\ C &= 2 \left(\mathfrak{q}^{2\varrho} (1 - \eta_1)^2 s_2^2 + (\mathfrak{q}^{\varrho} \eta_1 + \mathfrak{p}[\varrho]_{(p,q)})^2 \right). \end{aligned} \quad (74)$$

The R.H.S of (76) is negative if $A \leq 0, B \leq 0$. By applying Lemma 1, we finally attain

$$\mathcal{S}_{(p,q)}^*(\varrho, \eta_0) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - 1, \eta_1), \quad (75)$$

and $A \leq 0$ yields

$$\eta_1 = \frac{\left(2q^e \eta_0 - 2p[\varrho]_{(p,q)} - q^e\right) - \sqrt{\left(2q^e \eta_0 - 2p[\varrho]_{(p,q)} - q^e\right)^2 + 8\left(2\eta_0 p[\varrho]_{(p,q)} + q^e\right)}}{4}, \quad (76)$$

where $\eta_0 = k/k + 1$.

Similarly, proceeding as before, one can prove

where

$$\mathcal{S}_{(p,q)}^*(\varrho, \eta_1) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - 1, \eta_2), \quad (77)$$

$$\eta_2 = \frac{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) - q^e\right] - \sqrt{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) - q^e\right]^2 + 8\left[2\eta_0 p\left([\varrho]_{(p,q)} - [1]_{(p,q)}\right) + q^e\right]}}{4}. \quad (78)$$

Continuing this process, we obtain

where

$$\mathcal{S}_{(p,q)}^*(\varrho - (m-1), \eta_m) \subseteq \mathcal{S}_{(p,q)}^*(\varrho - m, \eta_{m+1}), \quad (79)$$

$$\eta_{m+1} = \frac{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) - q^e\right] - \sqrt{\left[2q^e \eta_0 - 2p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) - q^e\right]^2 + 8\left[2\eta_0 p\left([\varrho]_{(p,q)} - [m]_{(p,q)}\right) + q^e\right]}}{4}. \quad (80)$$

By using the principal of mathematical induction for post-quantum calculus, one can prove that formula (83) is true for all $m \in \mathbb{N}_0$ and the proof is complete.

Note by allocating special values for parameters involved in Theorem 2, we gain some notable results. Assuming $p = 1$ and $q \iff 1^-$, in Theorem 2, we acquire notable results from [14]. $\square$

Corollary 3. Assume that $p = 1$ and $q \iff 1^-$, which implies

$$k - \mathcal{ST}(\varrho + 1) \subseteq k - \mathcal{ST}(\varrho) \quad (81)$$

Equivalently,

$$\mathcal{S}^*(\varrho + 1, \eta_0) \subseteq \mathcal{S}^*(\varrho, \eta_1) \quad (82)$$

where

$$\eta_1 = \frac{(2\eta_0 - 2\varrho - 1) - \sqrt{(2\eta_0 - 2\varrho - 1)^2 + 8(2\eta_0\varrho + 1)}}{4},$$

$$\eta_0 = \frac{k}{k+1}. \quad (83)$$

In this portion, we generalize some subordination results presented in [15].

Theorem 3. Let $\alpha > 0, \varrho > 0, 0 < q < p \leq 1, -1 \leq \mathcal{D} < \mathcal{C} \leq 1$. If $f \in \mathcal{A}$ presented by (2) fulfills

$$(1 - \alpha) \frac{\Re_{(p,q)}^e f(z)}{z} + \alpha \frac{\Re_{(p,q)}^{e+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad (84)$$

then, for $\eta \geq 1$, we obtain

$$\operatorname{Re}(t(z))^{1/\eta} > \left(\frac{[\varrho + 1]_{(p,q)}}{\alpha q^e p^{[\gamma]_{(p,q)}}} \int_0^1 s \left(([\varrho + 1]_{(p,q)}) / \left(\alpha_q^{ep^{[\gamma]_{(p,q)}}} \right) \right)^{-1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} d_{(p,q)} s \right)^{1/\eta}. \quad (85)$$

Proof. Let

$$g(z) = \frac{\mathcal{R}_{(p,q)}^e f(z)}{z}, \quad g \in \mathcal{P}. \quad (86)$$

Differentiating this equation (p, q) -logarithmically, we obtain

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{z \mathfrak{d}_{(p,q)} \left(\frac{\mathcal{R}_{(p,q)}^e f(z)}{z} \right)}{\mathcal{R}_{(p,q)}^e f(z)} - 1. \quad (87)$$

Making use of (37), we can obtain

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{[\varrho + 1]_{(p,q)}}{q^{\varrho}} \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} - \frac{p[\varrho]_{(p,q)}}{q^{\varrho}} - 1. \quad (88)$$

This implies from (16) that

$$\frac{z \mathfrak{d}_{(p,q)}(g(z))}{g(z)} = \frac{[\varrho + 1]_{(p,q)}}{q^{\varrho}} \left(\frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{\mathfrak{R}_{(p,q)}^{\varrho} f(z)} \right) \frac{q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} = \left(\frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z) - \mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} \right). \quad (89)$$

Equivalently,

$$g(z) + \frac{\alpha q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} = (1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E}. \quad (90)$$

Therefore,

$$g(z) + \frac{\alpha q^{\varrho} z \mathfrak{d}_{(p,q)}(g(z))}{[\varrho + 1]_{(p,q)}} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E} \quad (91)$$

or

$$g(z) + \frac{p^{[y]_{(p,q)}} z \mathfrak{d}_{(p,q)}(g(z))}{\left([\varrho + 1]_{(p,q)}\right) \left(\alpha q^{\varrho} p^{[y]_{(p,q)}}\right)} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad z \in \mathcal{E}. \quad (92)$$

Using Lemma 3, we obtain

$$g(z) < t(z) = \frac{[\varrho + 1]_{(p,q)}}{\alpha q^{\varrho} p^{[y]_{(p,q)}}} z^{-\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]_{(p,q)}}\right)\right)} \int_0^z s^{\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]_{(p,q)}}\right)\right) - 1} \frac{1 + \mathcal{C}w(z)}{1 + \mathcal{D}w(z)} \mathfrak{d}_{(p,q)} s. \quad (93)$$

Therefore,

$$Re(t(z)) > \frac{[\varrho + 1]_{(p,q)}}{\alpha q^{\varrho} p^{[y]_{(p,q)}}} \int_0^1 s^{\left(\left([\varrho + 1]_{(p,q)}\right) / \left(\alpha q^{\varrho} p^{[y]_{(p,q)}}\right)\right) - 1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} \mathfrak{d}_{(p,q)} s. \quad (94)$$

Presently, utilizing the reality $(Rew)^{1/\eta} < Re(w^{1/\eta})$ for $0 < Re(w)$ and $1 \leq \eta$, we get (85), which is required.

Note that, by offering specific values to parameters in Theorem 3, we attain certain fascinating results from [15]. $\square$

Corollary 4. Assumes that $\alpha > 0, \varrho > 0, -1 \leq \mathcal{D} < \mathcal{C} \leq 1$, and $p = 1$. Furthermore, if $f \in \mathcal{A}$ is presented as in (2), which satisfies

$$(1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} < \frac{1 + \mathcal{C}z}{1 + \mathcal{D}z}, \quad (95)$$

Then, for $\eta \geq 1$, one can have

$$\begin{aligned} & Re\left(\frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z}\right)^{1/\eta} \\ & > \left(\frac{[\varrho + 1]_q}{\alpha q^{\varrho}} \int_0^1 s^{([\varrho + 1]_q / \alpha q^{\varrho}) - 1} \frac{1 - \mathcal{C}s}{1 - \mathcal{D}s} d_q s\right)^{1/\eta}. \end{aligned} \quad (96)$$

Corollary 5. Assume that $\alpha > 1, \varrho > 1, \mathcal{C} = 1 - 2\beta$, $0 \leq \beta < 1, \mathcal{D} = 1$ and $p = 1$. Furthermore, if $f \in \mathcal{A}$ is presented as in (2), which satisfies

$$\begin{aligned} & (1 - \alpha) \frac{\mathfrak{R}_{(p,q)}^{\varrho} f(z)}{z} + \alpha \frac{\mathfrak{R}_{(p,q)}^{\varrho+1} f(z)}{z} \\ & < \frac{1 + (1 - 2\beta)z}{1 - z}, \end{aligned} \quad (97)$$

then, for $\eta \geq 1$, we obtain

$$\operatorname{Re}\left(\frac{\mathfrak{R}_q^{\varrho} f(z)}{z}\right)^{1/\eta} > \left((2\beta - 1)s^{([q+1]_q)/\alpha q^{\varrho}} + \frac{2(1-\beta)[q+1]_q}{\alpha q^{\varrho}} \int_0^1 \frac{s^{([q+1]_q)/\alpha q^{\varrho}-1}}{1+s} d_q s\right)^{1/\eta}. \quad (98)$$

4. Conclusion

The essential explanation behind this investigation work is to use (p, q) -calculus and inspected the generalized (p, q) -Ruscheweyh differential operator. At the present time, we utilized (p, q) -Ruscheweyh differential operator and examined some new subfamilies of $\mathcal{S}$ and study some essential properties, for example, inclusion and subordination properties. Our results sum up some charming known consequences. The way is open for researchers to investigate all the more right currently related areas.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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