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Research Article

On Solution of Nonlinear Integral and Fractional Differential Equations via Discontinuous Nonlinear Contractions

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The aim of this research work is to demonstrate the survival of exactly one common fixed point for nonlinear contractions without assuming containment of range spaces of an involved pair of mappings, commutativity, and continuity (or any of their weaker forms). It is interesting to note that these findings involve different techniques of proof and comparatively fewerassumptions related to underlying mappings to find the existence of common fixed points of certain mappings. In this way, such investigations are generalizations and improvements over the several prominent recent results of the existing literature. For the application part, we solve the system of nonlinear integral equations of Fredholm and Volterra type and fractional differential equation of Caputo type to validate our results. Moreover, we validate the main conclusion with the help of an example.

1. Introduction and Preliminaries

The Banach contraction principle [1] provides a significant technique to guarantee the existence of the fixed point. In the last 100 years, it has been explored by several researchers in diverse directions (see, [2, 3]). Wongyat and Sintunavarat [4] applied fixed point results to find the solution of nonlinear Fredholm integral equations and Volterra integral equations together with nonlinear fractional differential equations of Caputo type. Boyd and Wong [5] utilized a function that is described on the closure of the range of a metrically convex metric space, to generalize the conclusion of Banach [1]. However, the involved mapping must be continuous in Banach [1] as well as Boyd and Wong's [5] results.

In 1962, a note on contractive mappings was presented by Rakotch [6]. Recently Nziku and Kumar [7] presented the fixed point results via contractive mappings in partial metric spaces. The respective authors also presented the Boyd and Wong variant of the fixed point theorem in the same metric settings (see [8]). Imdad and Kumar [9] extended the existing results by relaxing "continuity" and lightening the "commutativity" requirement besides increasing the number of involved maps from "two" to "four." Several other researchers extended these results in different directions. One can see the results proved in ([10–14]) and the references therein. In this continuation, most recently some interesting generalizations of these results were obtained by Karapinar et al. [15] and by Prasad ([16, 17]). In fact, mappings that shrink the distances have been of concern for several years.

Inspired by the fact that the majority of the phenomena evolving in the physical world are discontinuous, we extend the result of Boyd and Wong [5] to a pair of discontinuous nonlinear mappings without utilizing the containment of range spaces of involved pair of mappings, a weaker form of commutativity or continuity (or any of its variants) via the technique of Akkouchi [18]. Next, we utilize our conclusions to solve Fredholm and Volterra type nonlinear integral equations and an application for the existence and uniqueness of a solution for the nonlinear fractional differential equation of Caputo type too. Our conclusions generalize, improve and extend two celebrated and renowned results due to (Banach [1], Boyd, and Wong [5]) besides comparable other ones. It is interesting to mention here that the containment of range spaces of mappings under consideration, commutativity or its weaker form as well as continuity or its variants are essential for the survival of a common fixed point (see, Singh and Tomar [19] and Tomar and Karapinar [20]).

In this way, these investigations provide a different perspective to determine a common fixed point for nonlinear contraction without exploiting the continuity, commutativity (or any of its weaker forms), and containment of range spaces of an involved pair of discontinuous mappings. Furthermore, inspired by the reality that the nonlinear integral equations of Fredholm and Volterra and the fractional differential equation of Caputo types appear in numerous real-life problems. We resolve them to verify the genuineness and efficacy of the established results.

Definition 1 (see [21]). Let $\{\mu_n\}$ be a sequence in a metric space (X, ϱ). Then,

(i) $\{\mu_n\}$ is convergent to a point $\mu \in X$ if and only if $\lim_{n\longrightarrow\infty}\varrho(\mu_n, \mu) = 0$

(ii) $\{\mu_n\}$ is a Cauchy sequence if there exists $\varepsilon > 0$ so that $\varrho(\mu_n, \mu_m) < \epsilon$, $n, m > N$ for some integers $N \geq 0$, that is, $\lim_{n,m\longrightarrow+\infty}\varrho(\mu_n, \mu_m) = 0$

(iii) (X, ϱ) is complete if every Cauchy sequence $\{\mu_n\}$ converges to a point $\mu \in X$

Definition 2 (see [18]). Let Φ symbolize the set of continuous functions ϕ: $[0, \infty) \longrightarrow [0, \infty)$, so that

(i) $\phi(t) = 0$ if and only if $t = 0$ and

(ii) if $\{\phi(t_n)\}$ is a decreasing sequence, then all the sequences $\{t_n\}$ of elements in $[0, \infty)$ are bounded, that is, $\sup_n (t_n) < \infty$.

Noticeably, if either of the subsequent conditions holds for a continuous function ϕ: $[0, \infty) \longrightarrow [0, \infty)$ then it belongs to Φ;

(iii) ϕ is nondecreasing in $[0, \infty)$,

(iv) $\phi(t) \geq Mt^a$, $t > 0$, and $M, a > 0$.

Let us use the following notations in the subsequent sections:

(i) P: $= \{\varrho(\mu, \nu)$: $\mu, \nu \in X\}$,

(ii) Let χ: $\overline{\phi(P)} \longrightarrow [0, \infty)$ be an upper semicontinuous from the right on $\overline{\phi(P)}$ (the closure of $\phi(P)$) so that $\chi(t) < t$, $t \in \overline{\phi(P)}\backslash\{0\}$.

Akkouchi [18] proved the following theorem by generalizing the contraction condition due to Boyd and Wong [5] and applying the control function χ on the right-hand side of the inequality as follows:

Theorem 1 (see [18]). *Let (M, ϱ) be a complete metric space and let P: $= \{\varrho(\mu, \nu)$: $\mu, \nu \in M\}$. Let $\phi \in \Phi$ and χ: $\overline{\phi(P)} \longrightarrow [0, \infty)$ be an upper semicontinuous from the right on $\overline{\phi(P)}$ and satisfies $\chi(t) < t$ for all $t \in \overline{\phi(P)}\backslash\{0\}$ where $\overline{\phi(P)}$ denotes the closure of $\phi(P)$. Let T be the mapping of M satisfying the following contractive condition:*

$$\phi(\varrho(T\mu, T\nu)) \leq \chi(\phi(\varrho(\mu, \nu))), \forall \mu, \nu \in M. \tag{1}$$

Then, T has a unique fixed point z and $\varrho(T^n\mu, z) \longrightarrow 0$ for all $\mu \in M$ as $n \longrightarrow \infty$.

2. Main Results

Firstly, we state and prove the main result and then validate it with an illustrative example.

Theorem 2. *Let $\mathcal{S}$ and $\mathcal{T}$ be self-mappings of a complete metric space (X, ϱ) so that:*

$$\phi(\varrho(\mathcal{S}\mu, \mathcal{T}\nu)) \leq \chi(\phi(M(\mu, \nu))). \tag{2}$$

$\mu, \nu \in X$, where $M(\mu, \nu) = \max\{\varrho(\mu, \nu), \varrho(\mu, \mathcal{S}\mu), \varrho(\nu, \mathcal{T}\nu), \varrho(\mu, \mathcal{T}\nu) + \varrho(\nu, \mathcal{S}\mu)/2\}$. Then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point. Further, any fixed point of $\mathcal{S}$ is also a fixed point of $\mathcal{T}$ and vice-versa.

Proof. Suppose $\mu_0 \in X$. For a positive integer n, define a sequence $\{\mu_n\}$ by $\mu_{n+1} = \mathcal{S}\mu_n$ and $\mu_{n+2} = \mathcal{T}\mu_{n+1}$. If $\mu_n = \mu_{n+1}$, then μ_n is a fixed point of $\mathcal{S}$. Similarly, if $\mu_{n+1} = \mu_{n+2}$, then μ_{n+1} is a fixed point of $\mathcal{T}$.

Let $\mu_n \neq \mu_{n+1}$ and $a_n = \varrho(\mu_n, \mu_{n+1})$, $\{\mu_n\} \in X$, that is, $\phi(a_{n+1}) = \phi(\varrho(\mu_{n+1}, \mu_{n+2}))$. Utilizing the nonlinear contraction (2), we get

$$\begin{aligned}\phi(\varrho(\mu_{n+1}, \mu_{n+2})) &= \phi(\varrho(\mathcal{S}\mu_n, \mathcal{T}\mu_{n+1})) \leq \chi(\phi(M(\mu_n, \mu_{n+1})))\\ &= \chi\left(\phi\left(\max\left\{\varrho(\mu_n, \mu_{n+1}), \varrho(\mu_n, \mathcal{S}\mu_n), \varrho(\mu_{n+1}, \mathcal{T}\mu_{n+1}), \frac{\varrho(\mu_n, \mathcal{T}\mu_{n+1}) + \varrho(\mu_{n+1}, \mathcal{S}\mu_n)}{2}\right\}\right)\right)\\ &= \chi\left(\phi\left(\max\left\{\varrho(\mu_n, \mu_{n+1}), \varrho(\mu_n, \mu_{n+1}), \varrho(\mu_{n+1}, \mu_{n+2}), \frac{\varrho(\mu_n, \mu_{n+2}) + \varrho(\mu_{n+1}, \mu_{n+1})}{2}\right\}\right)\right).\end{aligned} \tag{3}$$

Since,

$$\frac{\varrho(\mu_n,\mu_{n+2})+\varrho(\mu_{n+1},\mu_{n+1})}{2}\leq\frac{\varrho(\mu_n,\mu_{n+1})+\varrho(\mu_{n+1},\mu_{n+2})+\varrho(\mu_{n+1},\mu_{n+1})}{2}$$
$$=\frac{\varrho(\mu_n,\mu_{n+1})+\varrho(\mu_{n+1},\mu_{n+2})}{2},\tag{4}$$
$$\phi(\varrho(\mu_{n+1},\mu_{n+2}))=\phi(\varrho(\mathcal{S}\mu_n,\mathcal{T}\mu_{n+1}))$$
$$\leq\chi(\phi(\max\{\varrho(\mu_n,\mu_{n+1}),\varrho(\mu_{n+1},\mu_{n+2})\})).$$

If $\max\{\varrho(\mu_n,\mu_{n+1}),\varrho(\mu_{n+1},\mu_{n+2})\}=\varrho(\mu_{n+1},\mu_{n+2})$. Then,

$$\phi(\varrho(\mu_{n+1},\mu_{n+2}))=\phi(\varrho(\mathcal{S}\mu_n,\mathcal{T}\mu_{n+1}))\leq\chi(\phi(\varrho(\mu_{n+1},\mu_{n+2})\}))<\phi(\varrho(\mu_{n+1},\mu_{n+2})),\tag{5}$$

which is a contradiction.

So $\max\{\varrho(\mu_n,\mu_{n+1}),\varrho(\mu_{n+1},\mu_{n+2})\}=\varrho(\mu_n,\mu_{n+1})$. Hence,

$$\phi(\varrho(\mu_{n+1},\mu_{n+2}))=\phi(\varrho(\mathcal{S}\mu_n,\mathcal{T}\mu_{n+1}))\leq\chi(\phi(\varrho(\mu_n,\mu_{n+1})\}))<\phi(\varrho(\mu_n,\mu_{n+1})),\tag{6}$$

that is, the sequence $\{\phi(a_n)\}$ is decreasing and bounded below in $[0,\infty)$ for $n\geq 0$.

Let a be the limit point of $\{\phi(a_n)\}$. Obviously $a\in[0,\infty)$. We assert that $a=0$. Instead, if $a>0$, taking $\limsup_{n\longrightarrow\infty}$ in inequality (6) we get,

$$0<a\leq\limsup_{n\longrightarrow\infty}\chi(\phi(a_n))\leq\chi(a)<a,\tag{7}$$

which is a contradiction of the presumption on the mapping χ, that is, $a=0$. Hence, $\lim_{n\longrightarrow\infty}\phi(a_n)=\lim_{n\longrightarrow\infty}\phi(\varrho(\mu_n,\mu_{n+1}))=0$.

Clearly, $\lim\phi(a_n)=\phi(\lim a_n)=0$. Utilizing the definition of ϕ,

$$\lim_{n\longrightarrow\infty}a_n=\lim_{n\longrightarrow\infty}\varrho(\mu_n,\mu_{n+1})=0.\tag{8}$$

Next, we assert that a sequence $\{\mu_n\}$ is Cauchy. Instead, if $\{\mu_n\}$ is not a Cauchy sequence in X, then there exists $\epsilon>0$ and a sequence of integers $m(k),n(k)$ so that,

$$\varrho\left(\mu_{n(k)},\mu_{m(k)}\right)\geq\epsilon,$$
$$n(k)>m(k)\geq k,k\geq 0.\tag{9}$$

If $m(k)$ is the smallest integer, so that inequality (9) holds,

$$\varrho\left(\mu_{(k)},\mu_{m(k)-1}\right)<\epsilon.\tag{10}$$

For $n(k)>m(k)$ we obtain,

$$\varrho\left(\mu_{n(k)},\mu_{m(k)}\right)\leq\varrho\left(\left(\mu_{n(k)},\mu_{m(k)-1}\right)+\varrho\left(\mu_{m(k)-1},\mu_{m(k)}\right),\right.$$
$$\leq\varrho\left(\mu_{n(k)},\mu_{m(k)-1}\right)+a_{n(k)}.\tag{11}$$

As $k\longrightarrow\infty$, considering (8) we get,

$$\varrho\left(\mu_{n(k)},\mu_{m(k)}\right)\longrightarrow\epsilon.\tag{12}$$

By elementary computations we have,

$$\varrho\left(\mu_{n(k)-1},\mu_{m(k)-1}\right)\longrightarrow\epsilon.\tag{13}$$

From the nonlinear contraction (2) we obtain,

$$\phi\left(\varrho\left(\mu_{n(k)},\mu_{m(k)}\right)\right)=\phi\left(\varrho\left(\mathcal{S}\mu_{n(k)-1},\mathcal{T}\mu_{m(k)-1}\right)\right)\leq\chi\left(\phi\left(M\left(\mu_{n(k)-1},\mu_{m(k)-1}\right)\right)\right),\tag{14}$$

where

$$\begin{aligned} M\left(\mu_{n(k)-1},\mu_{m(k)-1}\right) &= \max\left\{\varrho\left(\mu_{n(k)-1},\mu_{m(k)-1}\right),\varrho\left(\mu_{n(k)-1},\mathcal{S}\mu_{n(k)-1}\right),\varrho\left(\mu_{m(k)-1},\mathcal{T}\mu_{m(k)-1}\right),\frac{\varrho\left(\mu_{n(k)-1},\mathcal{T}\mu_{m(k)-1}\right)+\varrho\left(\mu_{m(k)-1},\mathcal{S}\mu_{n(k)-1}\right)}{2}\right\} \\ &= \max\left\{\varrho\left(\mu_{n(k)-1},\mu_{m(k)-1}\right),\varrho\left(\mu_{n(k)-1},\mu_{n(k)}\right),\varrho\left(\mu_{m(k)-1},\mu_{m(k)}\right),\frac{\varrho\left(\mu_{n(k)-1},\mu_{m(k)}\right)+\varrho\left(\mu_{m(k)-1},\mu_{n(k)}\right)}{2}\right\}. \end{aligned} \tag{15}$$

As $k \longrightarrow \infty$, utilizing (12) and (13), inequality (14) transforms to,

$$0 < \phi(\varepsilon) \le \chi(\phi(\varepsilon)) < \phi(\varepsilon). \tag{16}$$

Since $\chi(t) < t$, $w \in Xt > 0$ and $\phi(\varepsilon) \in \overline{\phi(P)}\backslash\{0\}$ then inequality (16) is not valid. As a result, the sequence $\{\mu_n\}$ is a Cauchy sequence and $\lim_{n,m\longrightarrow\infty}\varrho(\mu_n,\mu_m) = 0$.

Since X is complete, we select a point $w \in X$ so that $\lim_{n\longrightarrow\infty}\varrho(\mu_n, w) = 0$. Next, we assert that w is a fixed point of $\mathcal{T}$. Instead, if $w \neq \mathcal{T}w$.

From inequality (2), we attain

$$\phi(\varrho(\mu_{n+1},\mathcal{T}w)) = \phi(\varrho(\mathcal{S}\mu_n,\mathcal{T}w)) \le \chi(\phi(M(\mu_n,w))), \tag{17}$$

where

$$\begin{aligned} M(\mu_n,w) &= \max\left\{\varrho(\mu_n,w),\varrho(\mu_n,\mathcal{S}\mu_n),\varrho(w,\mathcal{T}w),\frac{\varrho(\mu_n,\mathcal{T}w)+\varrho(w,\mathcal{S}\mu_n)}{2}\right\} \\ &= \max\left\{\varrho(\mu_n,w),\varrho(\mu_n,\mu_{n+1}),\varrho(w,\mathcal{T}w),\frac{\varrho(\mu_n,\mathcal{T}w)+\varrho(w,\mu_{n+1})}{2}\right\}. \end{aligned} \tag{18}$$

Taking the limit as $n \longrightarrow \infty$ we attain,

$$M(\mu_n,w) \longrightarrow \varrho(w,\mathcal{T}w). \tag{19}$$

Taking the limit as $n \longrightarrow \infty$ in the inequality (17), we obtain

$$\phi(\varrho(w,\mathcal{T}w)) \le \chi(\phi(\varrho(w,\mathcal{T}w))) < \phi(\varrho(w,\mathcal{T}w)), \tag{20}$$

which is a contradiction. Thus $w = \mathcal{T}w$.

Now, we assert that w is a unique common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Instead, if $\mathcal{S}$ and $\mathcal{T}$ have two common fixed points μ_0 and ν_0 in X and $\mu_0 \neq \nu_0$. Then, from inequality (2), for all $\mu_0, \nu_0 \in X$ we obtain,

$$\phi(\varrho(\mathcal{S}\mu_0,\mathcal{T}\nu_0)) = \phi(\varrho(\mu_0,\nu_0)) \le \chi(\phi(M(\mu_0,\nu_0))), \tag{21}$$

where

$$\begin{aligned} M(\mu_0,\nu_0) &= \max\left\{\varrho(\mu_0,\nu_0),\varrho(\mu_0,\mathcal{S}\mu_0),\varrho(\nu_0,\mathcal{T}\nu_0),\frac{\varrho(\mu_0,\mathcal{T}\nu_0)+\varrho(\nu_0,\mathcal{S}\mu_0)}{2}\right\} \\ &= \max\left\{\varrho(\mu_0,\nu_0),\varrho(\mu_0,\mu_0),\varrho(\nu_0,\nu_0),\frac{\varrho(\mu_0,\nu_0)+\varrho(\nu_0,\mu_0)}{2}\right\} \\ &= \varrho(\mu_0,\nu_0). \end{aligned} \tag{22}$$

Thus, inequality (21) transforms to,

$$\begin{aligned} \phi(\varrho(\mathcal{S}\mu_0,\mathcal{T}\nu_0)) &= \phi(\varrho(\mu_0,\nu_0)) \le \chi(\phi(\varrho(\mu_0,\nu_0))) \\ &< \phi(\varrho(\mu_0,\nu_0)), \end{aligned} \tag{23}$$

which is a contradiction and hence $\mu_0 = \nu_0$. As a result, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

On contrary, to the hypothesis, if μ_0 is a fixed point of $\mathcal{S}$ and $\mu_0 \neq \mathcal{T}\mu_0$, utilizing inequality (2), for $\mu_0 \in X$ we obtain,

$$\phi(\varrho(\mathcal{S}\mu_0,\mathcal{T}\mu_0)) = \phi(\varrho(\mu_0,\mathcal{T}\mu_0)) \le \chi(\phi(M(\mu_0,\mathcal{T}\mu_0))), \tag{24}$$

where

$$\begin{aligned} M(\mu_0,\mathcal{T}\mu_0) &= \max\left\{\varrho(\mu_0,\mathcal{T}\mu_0), \varrho(\mu_0,\mathcal{S}\mu_0), \varrho(\mathcal{T}\mu_0,\mathcal{T}^2\mu_0), \frac{\varrho(\mu_0,\mathcal{T}^2\mu_0)+\varrho(\mathcal{T}\mu_0,\mathcal{S}\mu_0)}{2}\right\} \\ &= \max\left\{\varrho(\mu_0,\mathcal{T}\mu_0), \varrho(\mu_0,\mu_0), \varrho(\mathcal{T}\mu_0,\mathcal{T}^2\mu_0), \frac{\varrho(\mu_0,\mathcal{T}^2\mu_0)+\varrho(\mathcal{T}\mu_0,\mu_0)}{2}\right\} \\ &= \varrho(\mu_0,\mathcal{T}\mu_0). \end{aligned} \tag{25}$$

Thus, inequality (24) transforms to,

$$\begin{aligned} \phi(\varrho(\mathcal{S}\mu_0,\mathcal{T}\mu_0)) = \phi(\varrho(\mu_0,\mathcal{T}\mu_0)) &\le \chi(\phi(\varrho(\mu_0,\mathcal{T}\mu_0))) \\ &< \phi(\varrho(\mu_0,\mathcal{T}\mu_0)), \end{aligned} \tag{26}$$

which is a contradiction, that is, $\mu_0 = \mathcal{T}\mu_0$. Similarly, we may demonstrate that a fixed point of $\mathcal{T}$ is also a fixed point of $\mathcal{S}$. □

Remark 1. By setting $\mathcal{T} = \mathcal{S}$, one gets the following corollary as an extended variant of Theorem 1.2 [18].

Corollary 1. *Let $\mathcal{S}$ be a self-mapping of a complete metric space (X,ϱ) so that:*

$$\phi(\varrho(\mathcal{S}\mu,\mathcal{S}\nu)) \le \chi(\phi(M(\mu,\nu))). \tag{27}$$

$\mu,\nu \in X$, where $M(\mu,\nu) = \max\{\varrho(\mu,\nu), \varrho(\mu,\mathcal{S}\mu), \varrho(\nu,\mathcal{S}\nu), \varrho(\mu,\mathcal{S}\nu)+\varrho(\nu,\mathcal{S}\mu)/2\}$. Then, $\mathcal{S}$ has a unique fixed point.

Remark 2. By setting $\mathcal{T} = I$, the identity map, one gets the subsequent corollary:

Corollary 2. *Let $\mathcal{S}$ be a self-mapping of a complete metric space (X,ϱ) so that*

$$\phi(\varrho(\mathcal{S}\mu,\nu)) \le \chi(\phi(M(\mu,\nu))). \tag{28}$$

$\mu,\nu \in X$, where $M(\mu,\nu) = \max\{\varrho(\mu,\nu), \varrho(\mu,\mathcal{S}\mu), \varrho(\mu,\nu)+\varrho(\nu,\mathcal{S}\mu)/2\}$. Then, $\mathcal{S}$ has a unique fixed point.

Remark 3. It is interesting to notice that, Theorem 2, Corollaries 1 and 2 continue to be valid if we substitute, besides $M(\mu,\nu) = \varrho(\mu,\nu)$ retaining the rest of the assumptions. In this transformation Theorem 2 and Corollary 2 are still generalizations and extensions over the existing results, however, Corollary 1 reduces to Theorem 1.4 [18] due to Akkouchi.

Remark 4. We also point out the fact that, if we set $\phi = I$, an identity map, $\chi(t) = kt$ for $k \in [0,1)$ and $t \ge 0$, in light of the above remark in Corollary 1, Corollary 1 reduces to the Banach contraction principle [1].

Remark 5. If we compare Theorem 2 with Theorem 2 in [6], one can see that our results are more generalized than that of Rakotch [6]. Theorem 2 is proved for two maps while Theorem 2 [6] is proved for one map. Also in our case, the continuity condition is relaxed.

We provide an example to authenticate the genuineness of the obtained conclusion.

Example 1. Let $X = [0,\infty)$ be equipped with the usual metric. Noticeably, (X,ϱ) is a complete metric space. Define two self-mappings $\mathcal{S}$ and $\mathcal{T}$ on X such that

$$\mathcal{S}\mu = \begin{cases} \dfrac{\mu}{3}, & \text{if } \mu \in [1,\infty), \\ 0, \text{if} & \mu \in [0,1), \end{cases} \qquad \mathcal{T}\nu = \begin{cases} \dfrac{\nu}{2}, \text{if} & \nu \in [1,\infty), \\ 0, \text{if} & \nu \in [0,1). \end{cases} \tag{29}$$

For $\mu,\nu \in X$ and $t \in [a,b]$. Define a function $\phi\colon P \longrightarrow [0,\infty)$ as:

$$\phi(\varrho(\mu,\nu)) = |\mu| + |\nu|. \tag{30}$$

Clearly, ϕ is nondecreasing and $\phi(t) = 0$ if and only if $t = 0$ and so, $\phi \in \Phi$. Define a function $\chi\colon [0,\infty) \longrightarrow [0,\infty)$ by $\chi(t) = t/2, t \ge 0$. Now, we shall show that $\mathcal{S}$ and $\mathcal{T}$ validate the nonlinear contractive assumption (2) of Theorem 2 in light of Remark 3. Consider the subsequent two cases:

Case 1. If $\mu \in [0,\infty)$ and $\nu \in [1,\infty)$, then we obtain

$$\begin{aligned}\phi(\varrho(\mathcal{S}\mu,\mathcal{T}\nu)) &= |\mathcal{S}\mu| + |\mathcal{T}\nu| \\ &= \frac{\mu}{3} + \frac{\nu}{2} \\ &\le \frac{(\mu+\nu)}{2} \\ &= \chi(\phi(\varrho(\mu,\nu))).\end{aligned} \tag{31}$$

Case 2. If $\mu \in [0,\infty)$ and $\nu \in [1,\infty)$, then we obtain

$$\begin{aligned}\phi(\varrho(\mathcal{S}\mu,\mathcal{T}\nu)) &= |\mathcal{S}\mu| + |\mathcal{T}\nu| \\ &= \frac{\mu}{3} + 0 \\ &\le \frac{(\mu+\nu)}{2} \\ &= \chi(\phi(\varrho(\mu,\nu))).\end{aligned} \tag{32}$$

Thus, for $\mu,\nu \in X$,

$$\phi(\varrho(\mathcal{S}\mu,\mathcal{T}\nu)) \le \chi(\phi(\varrho(\mu,\nu))), \tag{33}$$

that is, $\mathcal{S}$ and $\mathcal{T}$ satisfy the nonlinear contractive assumption (2). Therefore, all the assumptions of Theorem 2 are validated(taking $M(\mu,\nu) = \varrho(\mu,\nu)$) and $\mu = 0$ is a unique common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Noticeably, Example 1 demonstrates that inequality (2) is not a linear contraction which marks the supremacy of Theorem 2 over analogous renowned conclusions.

3. Applications

The aim of this section is to provide some applications validating our main results in the previous section. In this way, we demonstrate the existence and uniqueness of a solution for various problems.

3.1. Existence of a Solution for Nonlinear Integral Equations. The theory of systems of nonlinear integral equations is a substantial field of mathematics, having various applications in numerous branches of physics, biology, chemistry, engineering, and other fields related to real-life problems. Nowadays, several authors explored distinct results of this theory (see for instance, [4, 22, 23]) in the settings of metric spaces. Based on this fact, we present a unique solution for the systems of nonlinear Fredholm and Volterra integral equations by utilizing the main result in light of Remark 3.

Theorem 3. *Consider the nonlinear Fredholm integral equation*

$$\mu(t) = \alpha(t) + \int_a^b \mathcal{K}(t,s,\mu(s))\mathrm{d}s. \tag{34}$$

Let $\mu,\nu \in C[a,b]$, $a,b \in \mathbb{R}$ with $a<b, \alpha\colon [a,b] \longrightarrow \mathbb{R}$ and $\mathcal{K}\colon [a,b]\times[a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous mappings. Assume that

$$|\mathcal{K}(t,s,\mu(s))| + |\mathcal{K}(t,s,\nu(s))| \le \frac{\chi\left(\sup_{s\in[a,b]}|\mu(s)| + \sup_{s\in[a,b]}|\nu(s)|\right) - 2\alpha(t)}{b-a}, \tag{35}$$

$t,s \in [a,b]$. Then, the integral equation (34) has a unique solution.

Proof. Let $X = C[a,b]$. Define self-mappings $\mathcal{S}$ and $\mathcal{T}$ on X, such that

$$\begin{aligned}\mathcal{S}\mu(t) &= \alpha(t) + \int_a^b \mathcal{K}(t,s,\mu(s))\mathrm{d}s, \\ \mathcal{T}\nu(t) &= \alpha(t) + \int_a^b \mathcal{K}(t,s,\nu(s))\mathrm{d}s,\end{aligned} \tag{36}$$

$\mu,\nu \in X$ and $s,t \in [a,b]$. Let $\varrho\colon X\times X \longrightarrow [0,\infty)$ be the metric defined as $\varrho(\mu,\nu) = \sup_{t\in[a,b]}|\mu(t)-\nu(t)|$, $\mu,\nu \in X$. Then, (X,ϱ) is a complete metric space. For $t \in [a,b]$, define a function $\phi\colon P \longrightarrow [0,\infty)$ as follows:

$$\phi(\varrho(\mu,\nu)) = \sup_{t\in[a,b]}|\mu(t)| + \sup_{t\in[a,b]}|\nu(t)|. \tag{37}$$

Noticeably, ϕ is nondecreasing and $\phi(t) = 0$ if and only if $t = 0$ and so, $\phi \in \Phi$.

Now, we assert that $\mathcal{S}$ and $\mathcal{T}$ verify the nonlinear contractive assumption (2). For $\mu,\nu \in X$ and $s,t \in [a,b]$,

$$\begin{aligned}|\mathcal{S}\mu(t)| + |\mathcal{T}\nu(t)| &= \left|\alpha(t) + \int_a^b \mathcal{K}(t,s,\mu(s))\mathrm{d}s\right| + \left|\alpha(t) + \int_a^b \mathcal{K}(t,s,\nu(s))\mathrm{d}s\right| \\ &\le |\alpha(t)| + \left|\int_a^b \mathcal{K}(t,s,\mu(s))\mathrm{d}s\right| + |\alpha(t)| + \left|\int_a^b \mathcal{K}(t,s,\nu(s))\mathrm{d}s\right| \\ &\le 2|\alpha(t)| + \int_a^b |\mathcal{K}(t,s,\mu(s))|\mathrm{d}s + \int_a^b |\mathcal{K}(t,s,\nu(s))|\mathrm{d}s \\ &= 2|\alpha(t)| + \int_a^b (|\mathcal{K}(t,s,\mu(s)| + |\mathcal{K}(t,s,\nu(s))|)\mathrm{d}s \\ &\le 2|\alpha(t)| + \int_a^b \left(\frac{\chi\left(\sup_{s\in[a,b]}|\mu(s)| + \sup_{s\in[a,b]}|\nu(s)|\right) - 2\alpha(t)}{b-a}\right)\mathrm{d}s \\ &= 2|\alpha(t)| + \frac{1}{b-a}\int_a^b [\chi(\phi(\mu,\nu)) - 2\alpha(t)]\mathrm{d}s, = \chi(\phi(\mu,\nu)),\end{aligned} \tag{38}$$

that is,

$$\sup_{t\in[a,b]}|\mathcal{S}\mu(t)| + \sup_{t\in[a,b]}|\mathcal{T}\nu(t)| \le \chi(\phi(\mu,\nu)). \tag{39}$$

Hence, for $\mu,\nu \in X$, we obtain

$$\phi(\varrho(\mathcal{S}\mu,\mathcal{T}\nu)) \le \chi(\phi(\mu,\nu)). \tag{40}$$

In this way, $\mathcal{S}$ and $\mathcal{T}$ satisfy the contractive assumption (highlighted by Remark 3) on the setting $M(\mu,\nu) = \varrho(\mu,\nu)$. As a result, all the suppositions of Theorem 2 are validated. Hence $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point which is a unique solution (34).

Utilizing the almost identical procedure as presented in Theorem 3, we have the subsequent conclusion for the system of nonlinear Volterra integral equations. □

Theorem 4. *Consider the nonlinear Volterra integral equation*

$$\mu(t) = \alpha(t) + \int_a^t \mathcal{K}(t,s,\mu(s))\mathrm{d}s. \tag{41}$$

Let $\mu,\nu \in [a,b]$, $a,b \in \mathbb{R}$ with $a<b$, $\alpha\colon [a,b] \longrightarrow \mathbb{R}$ and $\mathcal{K}\colon [a,b]\times[a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous mappings. Assume that

$$|\mathcal{K}(t,s,\mu(s))| + |\mathcal{K}(t,s,\nu(s))| \le \frac{\chi\left(\sup_{s\in[a,b]}|\mu(s)| + \sup_{s\in[a,b]}|\nu(s)|\right) - 2\alpha(t)}{b-a}, \tag{42}$$

$t,s \in [a,b]$. Then, the integral (41) has a unique solution.

Proof. The proof of this theorem follows as above. □

3.2. Existence of a Solution for Nonlinear Fractional Differential Equations. Fractional order differential equations are generalized and noninteger order differential equations. The theory of nonlinear fractional differential equations nowadays has gotten extensive attention. The main reason is due to the rapid development of the subject fractional calculus itself and of having various applications in numerous branches of science and technology.

The first definition of the fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of noninteger derivative and integral was mentioned already in 1695 by Leibniz and L'Hospital. In fact, first-order differential equations are considered an alternative model to integer differential equations. The definition of the Riemann–Liouville derivative was established by Riemann in 1876. Thereafter many applications of the fractional derivatives and integrals of this Riemann–Liouville type have been presented in various fields of science and technology. These include studies on controllability, thermoelasticity, vibration, diffusion processes, and other complex phenomena (see [24, 25]).

Recently, many researchers are focused on understanding the various properties of fractional derivatives and their usefulness in certain complex systems. This research direction hinges on the results presented by various researchers on complex systems of nonlocal elements, that is, a local operator of ordinary derivative leads to a nonlocal fractional operator. Along these research lines, it is important to study fractional-order models in terms of global optimization. At the beginning of the twentieth century, another definition of fractional derivative was proposed by Caputo in regard to a Riemann–Liouville fractional integral (see for instance, [26–30]). In continuation of it, we present an application for the existence and uniqueness of a solution for the nonlinear fractional differential equation of Caputo type:

Let us recall some basic definitions of fractional calculus (see [24, 25]). For a continuous function $g: [0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of g of order $\beta > 0$ is defined as follows:

$$ {}^{C}D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\mathrm{d}s . D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\mathrm{d}s, \tag{43} $$

where $n = [\beta] + 1$ with $[\beta]$ denoting the integer part of a positive real number β, and Γ is the gamma function. Consider the nonlinear fractional differential equation

$$ {}^{C}D^{\beta}(\mu(t)) = f(t,\mu(t)), \tag{44} $$

with integral boundary conditions

$$ \mu(0) = 0, \mu(1) = \int_0^{\eta} \mu(s)\mathrm{d}s, \tag{45} $$

where $1 < \beta \leq 2, 0 < \eta < 1, \mu \in C[0,1]$, and $f: [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function (see [21]). It is well known that if f is continuous, then (44) is immediately inverted as the very familiar integral equation

$$ \begin{aligned} \mu(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,\mu(s))\mathrm{d}s \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,\mu(s))\mathrm{d}s \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m,\mu(m))dm \right)\mathrm{d}s. \end{aligned} \tag{46} $$

Now, we prove the following existence theorem.

Theorem 5. *Consider the nonlinear fractional differential (44). and for each* $\mu, \nu \in C[0,1]$, *we have*

$$ |f(s,\mu(s))| + |f(s,\nu(s))| \leq \frac{\Gamma(\beta+1)}{5} \left(\sup_{t\in[0,1]} |\mu(t)| + \sup_{t\in[0,1]} |\nu(t)| \right), \tag{47} $$

for all $s \in [0,1]$. *Then, the nonlinear fractional differential equation of Caputo type (44) has a unique solution. Moreover, for each* $\mu_0 \in C[0,1]$, *the Picard iteration* $\{\mu_n\}$ *defined by*

$$ \begin{aligned} (\mu_n)(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,\mu_{n-1}(s))\mathrm{d}s \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,\mu_{n-1}(s))\mathrm{d}s \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m,\mu_{n-1}(m))dm \right)\mathrm{d}s, \end{aligned} \tag{48} $$

for all $n \in \mathbb{N}$ *converges to a unique solution of the nonlinear fractional differential equation of Caputo type (44).*

Proof. Let $X = C[0,1]$. Define self-mappings $\mathcal{S}$ and $\mathcal{T}$ on X, such that

$$ \begin{aligned} (\mathcal{S}\mu)(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,\mu(s))\mathrm{d}s \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,\mu(s))\mathrm{d}s \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m,\mu(m))dm \right)\mathrm{d}s, \\ (\mathcal{T}\nu)(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,\nu(s))\mathrm{d}s \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,\nu(s))\mathrm{d}s \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m,\nu(m))dm \right)\mathrm{d}s, \end{aligned} \tag{49} $$

for all $\mu, \nu \in X$ and $s, t \in [0,1]$. Let $\varrho: X \times X \longrightarrow [0,\infty)$ be the metric defined as $\varrho(\mu,\nu) = \sup_{t\in[a,b]}|\mu(t) - \nu(t)|$, $\mu, \nu \in X$. Then, (X, ϱ) is a complete metric space. For $t \in [a,b]$, define a function $\phi: P \longrightarrow [0,\infty)$ as follows:

$$\phi(\varrho(\mu,\nu)) = \sup_{t\in[a,b]}|\mu(t)| + \sup_{t\in[a,b]}|\nu(t)|. \quad (50)$$

Noticeably, ϕ is nondecreasing and $\phi(t) = 0$ if and only if $t = 0$ and so, $\phi \in \Phi$. Now, we assert that $\mathcal{S}$ and $\mathcal{T}$ verify the nonlinear contractive assumption (2). For $\mu, \nu \in X$ and $s, t \in [a,b]$

$$\begin{aligned}
&|(\mathcal{S}\mu)(t)| + |(\mathcal{T}\nu)(t)| \\
&\quad = \left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,\mu(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,\mu(s))ds\right. \\
&\qquad \left. + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left(\int_0^s (s-m)^{\beta-1} f(m,\mu(m))dm\right)ds\right| \\
&\qquad + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,\nu(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,\nu(s))ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left(\int_0^s (s-m)^{\beta-1} f(m,\nu(m))dm\right)ds\,| \\
&\quad \leq \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}(|f(s,\mu(s))| + |f(s,\nu(s))|)\mathrm{d}s \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}(|f(s,\mu(s))| + |f(s,\nu(s))|)\mathrm{d}s \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left|\int_0^s (s-m)^{\beta-1}(f(m,\mu(m)) + f(m,\nu(m)))dm\right|\mathrm{d}s \\
&\quad \leq \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left(\sup_{t\in[0,1]}|\mu(t)| + \sup_{t\in[0,1]}|\nu(t)|\right)\mathrm{d}s \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left(\sup_{t\in[0,1]}|\mu(t)| + \sup_{t\in[0,1]}|\nu(t)|\right)\mathrm{d}s \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\left|\int_0^s (s-m)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\left(\sup_{t\in[0,1]}|\mu(t)| + \sup_{t\in[0,1]}|\nu(t)|\right)dm\right|\mathrm{d}s \\
&\quad \leq \frac{\Gamma(\beta+1)}{5}\left(\sup_{t\in[0,1]}|\mu(t)| + \sup_{t\in[0,1]}|\nu(t)|\right) \times \sup_{t\in(0,1)}\frac{1}{\Gamma(\beta)}\left(\begin{array}{c}\displaystyle\int_0^t |t-s|^{\beta-1}\mathrm{d}s + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\mathrm{d}s \\ \displaystyle + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta\int_0^s |s-m|^{\beta-1}\mathrm{d}m\mathrm{d}s\end{array}\right) \\
&\quad \leq \chi(\phi(\mu,\nu)).
\end{aligned} \quad (51)$$

This implies that

$$\sup_{t\in[0,1]}|(\mathcal{S}\mu)(t)| + \sup_{t\in[0,1]}|(\mathcal{T}\nu)(t)| \le \chi(\phi(\mu,\nu)), \tag{52}$$

and so

$$\phi(\varrho(\mathcal{S}\mu,\mathcal{T}\nu)) \le \chi(\phi(\mu,\nu)), \tag{53}$$

for all $\mu,\nu \in X$.

It follows that $\mathcal{S}$ and $\mathcal{T}$ satisfies the contractive assumption on the setting $M(\mu,\nu) = \varrho(\mu,\nu)$. Therefore, all the suppositions of Theorem 2 are validated. Hence $\mathcal{S}$ and $\mathcal{T}$ have a unique solution of the nonlinear fractional differential equation of Caputo type (44). This completes the proof. □

4. Conclusion

This paper provides yet another view to find a common fixed point for nonlinear contraction without exploiting the continuity, commutativity (or any of its weaker forms), and containment of range spaces of an involved pair of discontinuous mappings. Further, inspired by the reality that nonlinear integral equations of Fredholm and Volterra appear in numerous real-life problems, we resolve them to verify the genuineness and efficacy of the established conclusions. The theory of nonlinear fractional differential equations nowadays has gotten extensive attention due to its various applications in numerous branches of science and technology. We present an application for the existence and uniqueness of a solution for the nonlinear fractional differential equation of Caputo type too. In this way, these investigations provide yet another variant of common fixed point results and explored the possibility of their applications too.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to the writing of this article. All authors read and approved the final manuscript.

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