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Research Article

Domination Numbers of Amalgamations of Cycles at Connected Subgraphs

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A set S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G)$ is adjacent to some vertex in S. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. Let G_1 and G_2 be disjoint graphs, H_1 be a subgraph of G_1, H_2 be a subgraph of G_2, and f be an isomorphism from H_1 to H_2. The *amalgamation (the glued graph)* of G_1 and G_2 at H_1 and H_2 with respect to f is the graph $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ obtained by forming the disjoint union of G_1 and G_2 and then identifying H_1 and H_2 with respect to f. In this paper, we determine the domination numbers of the amalgamations of two cycles at connected subgraphs.

1. Introduction

Studying on several graph parameters is an interesting topic in graph theory. The domination number is one of the most importance parameter which was introduced from 1958 by Berge [1], called as "coefficient of external stability." In 1962, Ore [2] studied the same concept and used the name "dominating set" and "domination number" for a graph. In 1977, Cockayane and Hedetneimi [3] gave a survey of the results about dominating sets and used the notation $\gamma(G)$ for the domination number of a graph. In 1998, a text devoted to this subject was introduced by Haynes et al. [4]. Over 2000 articles on graph domination numbers have been studied extensively (see, for example, [2, 3, 5–16]), in particular, the study of the domination number of product graphs such as the Cartesian product of two cycles [12], the cross product of two paths [8], and the lexicographic product of two graphs [16]. It is natural to investigate the domination number of the amalgamation of two graphs, especially, the domination number of the amalgamation of two paths or two cycles.

Let G_1 and G_2 be disjoint graphs and $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that $H_1 \cong H_2$. Let f be an isomorphism from H_1 to H_2. The *amalgamation (the glued graph)* of G_1 and G_2 at H_1 and H_2 with respect to f is the graph $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ obtained by forming the disjoint union of G_1 and G_2 and then identifying H_1 and H_2 with respect to f. Equivalently, $G = G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ is the graph such that

$$V(G) = (V(G_1) - V(H_1)) \cup (V(G_2) - V(H_2)) \cup \{(v, f(v)) | v \in V(H_1)\}$$
$$E(G) = E(G_1 - V(H_1)) \cup E(G_2 - V(H_2)) \cup \{\{u, (v, f(v))\} | \{u, v\} \in E(G_1)\}$$

$$\cup \{\{u, (v, f(v))\} | \{u, f(v)\} \in E(G_2)\}$$
$$\cup \{\{(u, f(u)), (v, f(v))\} | \{u, v\} \in E(G_1) \ or \ \{f(u), f(v)\} \in E(G_2)\}. \quad (1)$$

Note that if $\{u, v\} \in E(G_1)$ and $\{f(u), f(v)\} \in E(G_2)$, then $\{u, v\} \in E(H_1)$ and $\{f(u), f(v)\} \in E(H_2)$. As an example, Figure 1 illustrates the amalgamation $G_1 \underset{H_1 \cong_f H_2}{\lhd\rhd} G_2$ with respect to the isomorphism $f: H_1 \longrightarrow H_2$ defined by $f(v_1) = u_1, f(v_2) = u_2$, and $f(v_3) = u_4$. The amalgamation or the glued graph of two graphs was defined from 2003 by Uiyyasathian [17] for solving the maximal-clique partition problem. In 2006, Promsakon and Uiyyasathian [18] gave an upper bound of the chromatic number of glued graphs in terms of the chromatic numbers of their original graphs. Here, we are interested in finding the domination number of an amalgamation of two cycles at connected subgraphs.

2. Basic Definitions and Results

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, we write $H \subseteq G$, and we say that G contains H. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a *proper subgraph* of G. For a vertex v of a graph G, a *neighbor* of v is a vertex adjacent to v in G. The *neighborhood (or open neighborhood)* $N(v)$ of v is the set of neighbors of v. The *closed neighborhood* $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its *closed neighborhood*. Therefore, v dominates $1 + \deg(v)$ vertices. For a set S of vertices of a graph G, the *closed neighborhood* $N[S]$ is defined as $N[S] = \cup_{v\in S} N[v]$. A set S of vertices of a graph G is said to *dominate* the vertices in $N[S]$. A set S of vertices of a graph G is a *dominating set* of G if every vertex of G is dominated by some vertex in S, i.e., every vertex in $V(G) - S$ is adjacent to some vertex in S. A *minimum dominating set* in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the *domination number* of G, and is denoted by $\gamma(G)$. A dominating set of a graph G with minimum cardinality is called a $\gamma-$ *set* of G.

Since the cardinality of the vertex set of a graph G is finite, the number of dominating sets of G with minimum cardinality is finite too. This gives, for a given graph G of order n, the domination number can have a value from the following range: $1 \leq \gamma(G) \leq n$. In particular, $\gamma(G) = 1$ if and only if $\Delta = n - 1$, where Δ is the maximum degree of G. Let P_n denote a path of order n such that $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{\{v_i, v_{i+1}\} | \ i = 1, 2, \ldots, n-1\}$. Let C_n denote a cycle of order n ($n \geq 3$) such that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{\{v_i, v_{i+1}\} | \ i = 1, 2, \ldots, n\}$, where + is the addition modulo n. It is easy to obtain that $\gamma(C_n) = \gamma(P_n) = \lceil n/3 \rceil$, where $\lceil x \rceil$ is the least integer greater than or equal to x.

A graph G is *isomorphic* to a graph H if there is a bijection $f: V(G) \longrightarrow V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. If such a function exists, it is called an *isomorphism* from G to H and written by $G \cong H$. A graph *automorphism* is simply an isomorphism from a graph to itself. Let $Iso(G, H)$ denote the set of all isomorphisms from a graph G to a graph H and $Aut(G)$ denote the set of all automorphisms on a graph G. It is easy to see that if H is a connected subgraph of C_n, then either $H \cong P_1, P_2, \ldots, P_{n-1}, P_n$ or C_n. Moreover, $Aut(P_n, P_n) = \{f_1, f_2\}$ such that $f_1(v_i) = v_i$ and $f_2(v_i) = v_{n-i+1}$ for all $i = 1, 2, \ldots, n$. Let H_1 be a connected proper subgraph of C_r and H_2 a connected proper subgraph of C_t such that $V(C_r) = \{v_1, v_2, \ldots, v_r\}$, $E(C_r) = \{\{v_i, v_{i+1}\} | \ i = 1, 2, \ldots, r\}$, $V(C_t) = \{u_1, u_2, \ldots, u_t\}$, and $E(C_t) = \{\{u_i, u_{i+1}\} | \ i = 1, 2, \ldots, t\}$. It follows that if $H_1 \cong H_2$, then H_1 and H_2 are paths of order s for some $s \leq \min\{r, t\}$ such that $V(H_1) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+s}\}$ and $V(H_2) = \{u_{j+1}, u_{j+2}, \ldots, u_{j+s}\}$, for some $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, t\}$. We thus get $Iso(H_1, H_2) = \{f_1, f_2\}$ such that $f_1(v_{i+q}) = u_{j+q}$ and $f_2(v_{i+q}) = u_{j+(s-q)+1}$, for all $q = 1, 2, \ldots, s$. Figures 2 and 3 illustrate $C_r \underset{H_1 \cong_{f_1} H_2}{\lhd\rhd} C_t$ and $C_r \underset{H_1 \cong_{f_2} H_2}{\lhd\rhd} C_t$, respectively.

It is easily seen that $C_r \underset{H_1 \cong_{f_1} H_2}{\lhd\rhd} C_t \cong C_r \underset{H_1 \cong_{f_2} H_2}{\lhd\rhd} C_t$. Moreover, if $H_1, H_1' \subseteq C_r$ and $H_1 \cong H_1'$, then $C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong C_r \underset{H_1' \cong_{f'} H_2'}{\lhd\rhd} C_t$ for all $f \in Iso(H_1, H_2)$ and $f' \in Iso(H_1', H_2')$. This implies the following lemma.

Lemma 1. *Let $H_1 \subseteq C_r$, $H_1' \subseteq C_r$, $H_2 \subseteq C_t$, and $H_2' \subseteq C_t$ be connected such that $H_1 \cong H_1' \cong H_2 \cong H_2'$. Then $\gamma(C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t) = \gamma(C_r \underset{H_1' \cong_{f'} H_2'}{\lhd\rhd} C_t)$ for all $f \in Iso(H_1, H_2)$ and $f' \in Iso(H_1', H_2')$.*

The next lemma gives the domination number of $C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$ for the case $H_1 = C_r$.

Lemma 2. *If $H_1 = C_r$, then $r = t$ and $\gamma(C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t) = \lceil r/3 \rceil$.*

Proof. Let $H_1 = C_r$ and $G = C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$. Since $H_1 \cong H_2$, it follows easily that $r = t$ and $G \cong C_r$. We thus get $\gamma(G) = \gamma(C_r) = \lceil r/3 \rceil$.

We now turn to the case $H_1 \neq C_r$. So, $H_1 \cong H_2 \cong P_s$, for some $s \leq \min\{r, t\}$. Assume, without loss of generality, that $\min\{r, t\} = r$, i.e., $3 \leq r \leq t$. For simplicity of notation, we write G instead of $C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t$. Based on the result of Lemma 1, from now on, we can assume that $V(H_1) = \{v_1, v_2, \ldots, v_s\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_s\}$, and the isomorphism $f: H_1 \longrightarrow H_2$ is defined by $f(v_i) = u_i$, for all $i = 1, 2, \ldots, s$. So, $(v_1, u_1), (v_2, u_2), \ldots, (v_s, u_s) \in V(G)$. Now, we consider $\gamma(G)$ with $H_1 \cong P_1$. □

Lemma 3. *If $H_1 \cong P_1$, then $\gamma(G) = \lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil + 1$.*

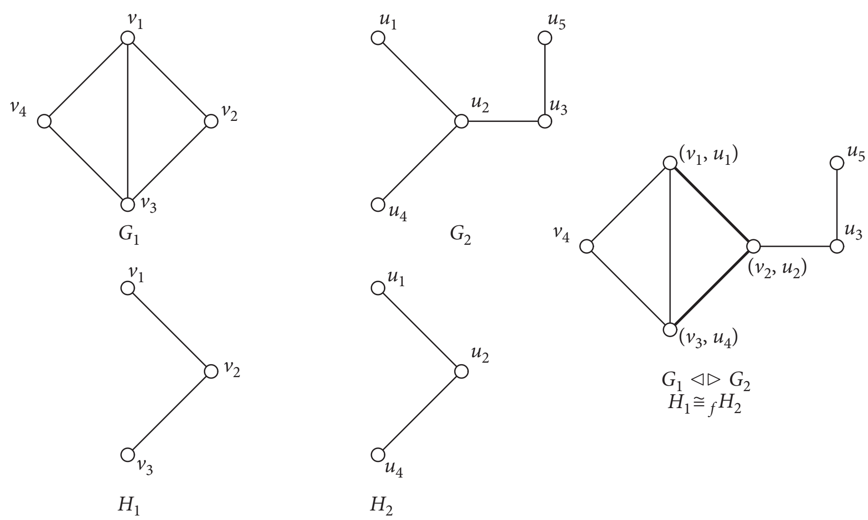

FIGURE 1: An amalgamation of G_1 and G_2 at H_1 and H_2 with respect to f.

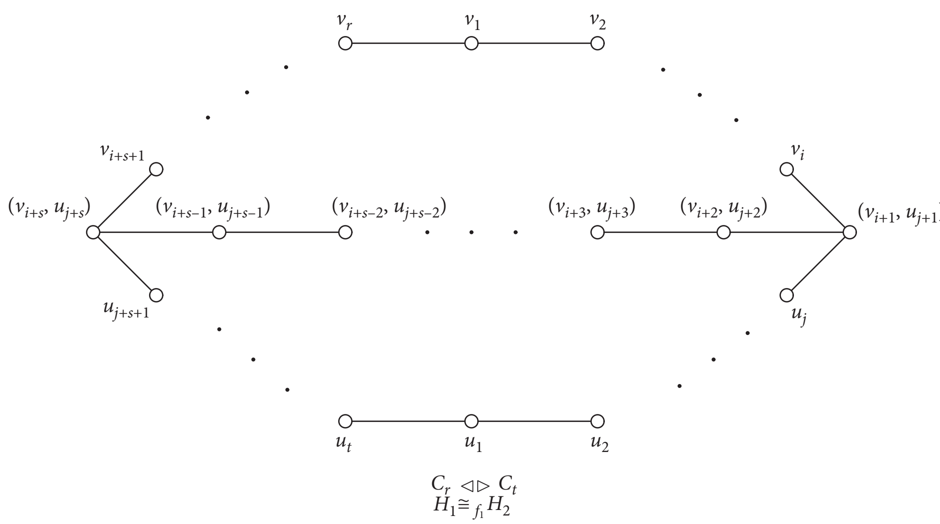

FIGURE 2: The amalgamation of C_r and C_t at connected proper subgraphs H_1 and H_2 with respect to f_1.

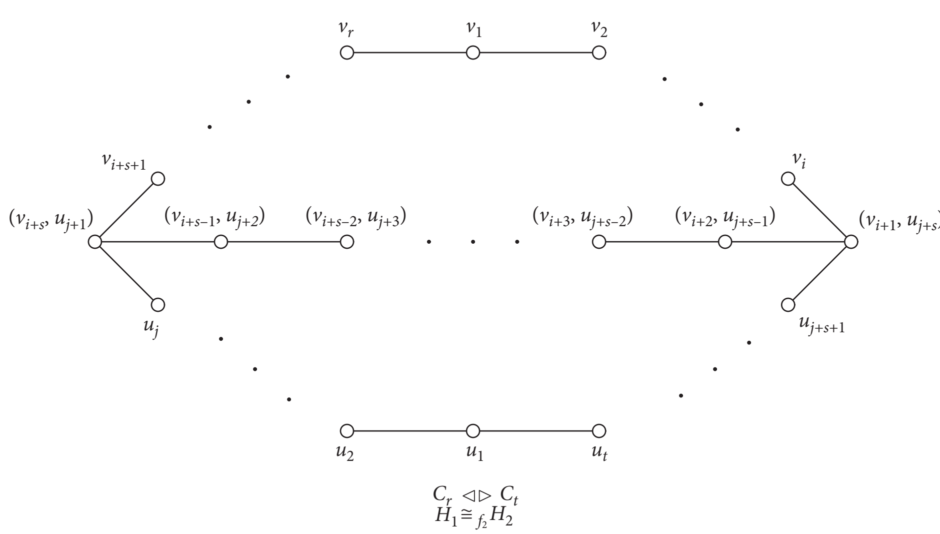

FIGURE 3: The amalgamation of C_r and C_t at connected proper subgraphs H_1 and H_2 with respect to f_2.

Proof. Suppose that $H_1 \cong P_1$. Then, $(v_1, u_1) \in V(G)$ (see Figure 4(a)).

Let $S = \{(v_1, u_1)\} \cup \{v_i| \ i \equiv 1 \pmod 3, 3 \le i \le r-1\} \cup \{u_i| \ i \equiv 1 \pmod 3, 3 \le i \le t-1\}$. We check at once that S is a dominating set of G and $|S| = \lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil + 1$. This gives $\gamma(G) \le \lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil + 1$. Let S' be a $\gamma-$ set of G. Thus, $(v_1, u_1) \in S'$ since otherwise S' is not a $\gamma-$ set of G. In order to dominate the vertices in $G - N[\{(v_1, u_1)\}]$, then $S - \{(v_1, u_1)\}$ must contain at least $\lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil$ vertices. This gives $\gamma(G) = |S'| \ge \lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil + 1$. It follows that $\gamma(G) = \lceil (r-3)/3 \rceil + \lceil (t-3)/3 \rceil + 1$.

Now, we consider $\gamma(G)$ with $H_1 \cong P_r$ and define sets S and S' as follows:

$$S = \{(v_i, u_i) | i \equiv 1 \pmod 3, 1 \le i \le r\} \cup \{u_i | i \equiv 2 \pmod 3, r+1 \le i \le t\},$$

$$S' = \{(v_i, u_i) | i \equiv 1 \pmod 3, 1 \le i \le r\} \cup \{u_i | i \equiv 1 \pmod 3, r+1 \le i \le t\}.$$

Note that $|S| = \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil$ and $|S'| = \lceil t/3 \rceil$. □

Lemma 4. *Let $H_1 \cong P_r$. Then,*

$$\gamma(G) = \begin{cases} \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil & \text{if } r \equiv 0 \pmod 3 \text{ and } t \equiv 1 \pmod 3, \\ \lceil t/3 \rceil & \text{otherwise.} \end{cases} \tag{2}$$

Proof. Suppose that $H_1 \cong P_r$. Then, $(v_1, u_1), \ldots, (v_r, u_r) \in V(G)$ (see Figure 5(a)).

If $r = t$, then G is the graph obtained from C_t by joining v_1 and v_t with a new edge. It follows easily that $\gamma(G) = \gamma(C_t) = \lceil t/3 \rceil$. For $r \ne t$, we consider two cases. □

Case 1. $r \equiv 0 \pmod 3$ and $t \equiv 1 \pmod 3$. We check at once that the set S defined above is a dominating set of G. Thus, $\gamma(G) \le \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil$. We now prove that $\gamma(G) \ge \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil$. Let S_1 be a $\gamma-$ set of G. In order to dominate (v_1, u_1), S_1 must contain at least one vertex of $\{(v_1, u_1), (v_2, u_2), (v_r, u_r), u_t\}$. There are two possible cases: either $(v_1, u_1) \in S_1$ or $(v_r, u_r) \in S_1$ since otherwise S_1 is not a $\gamma-$ set of G. We give the proof only for the case $(v_1, u_1) \in S_1$; the proof of the case $(v_r, u_r) \in S_1$ is similar. Consider the vertices of $G - N[\{(v_1, u_1)\}] \cong P_{r-3} \cup P_{t-r-1}$. In order to dominate $(v_3, u_3), \ldots, (v_{r-1}, u_{r-1})$ and $u_{r+1}, \ldots, u_{t-1}$, there are at least $\lceil (r-3)/3 \rceil + \lceil (t-r-1)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_{r-1}, u_{r-1})\} \cup \{u_{r+1}, \ldots, u_t\}$ in $S_1 - \{(v_1, u_1)\}$. This gives $|S_1| \ge 1 + \lceil (r-3)/3 \rceil + \lceil (t-r-1)/3 \rceil$, so $\gamma(G) \ge \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil$. Hence, $\gamma(G) = \lceil r/3 \rceil + \lceil (t-r-1)/3 \rceil$.

Case 2. $r \equiv 0 \pmod 3$ or $t \equiv 1 \pmod 3$. We see at once that the set S' defined above is a dominating set of G. Thus, $\gamma(G) \le \lceil t/3 \rceil$. We now prove that $\gamma(G) \ge \lceil t/3 \rceil$. Let S_2 be a $\gamma-$ set of G. We consider two subcases.

Case 2.1. $r \equiv 0 \pmod 3$ and $t \equiv 0 \pmod 3$ and $r \equiv 1 \pmod 3$ and $t \equiv a \pmod 3$, for some $a \in \{0, 1, 2\}$, or $r \equiv 2 \pmod 3$ and $t \equiv a \pmod 3$, for some $a \in \{0, 2\}$.

Case 2.1.1 $(v_1, u_1) \in S_2$ or $(v_r, u_r) \in S_2$. If $(v_1, u_1) \in S_2$ and $t > r+1$, then, in order to dominate $(v_3, u_3), \ldots, (v_{r-1}, u_{r-1})$ and $u_{r+2}, \ldots, u_{t-1}$, there are at least $\lceil (r-3)/3 \rceil + \lceil (t-r-2)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_r, u_r)\} \cup \{u_{r+1}, \ldots, u_t\}$ in $S_2 - \{(v_1, u_1)\}$. It follows that $|S_2| \ge 1 + \lceil (r-3)/3 \rceil + \lceil (t-r-2)/3 \rceil = \lceil t/3 \rceil$. If $(v_1, u_1) \in S_2$ and $t = r+1$, then $\{u_{r+2}, \ldots u_{t-1}\} = \varnothing$. So, in order to dominate the vertices in $G - N[\{(v_1, u_1)\}]$, there are at least $\lceil (r-3)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_r, u_r)\}$ in $S_2 - \{(v_1, u_1)\}$. This gives $|S_2| \ge 1 + \lceil (r-3)/3 \rceil = \lceil t/3 \rceil$. Similarly, if $(v_r, u_r) \in S_2$, then $|S_2| \ge \lceil t/3 \rceil$. Thus, $\gamma(G) \ge \lceil t/3 \rceil$.

Case 2.1.2. $(v_1, u_1) \notin S_2$ and $(v_r, u_r) \notin S_2$. Clearly, $\gamma(G) = |S_2| \ge \gamma(C_t) = \lceil t/3 \rceil$.

Case 2.2. $r \equiv 0 \pmod 3$ and $t \equiv 2 \pmod 3$ or $r \equiv 2 \pmod 3$ and $t \equiv 1 \pmod 3$.

Case 2.2.1. either $(v_1, u_1) \in S_2$ or $(v_r, u_r) \in S_2$. If $(v_1, u_1) \in S_2$, then, in order to dominate $(v_3, u_3), \ldots, (v_r, u_r), u_{r+1}, \ldots, u_{t-1}$ in G, there are at least $\lceil (t-3)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_r, u_r), u_{r+1}, \ldots, u_t\}$, in $S_2 - \{(v_1, u_1)\}$. It follows that $|S_2| \ge 1 + \lceil (t-3)/3 \rceil = \lceil t/3 \rceil$. Similarly, if $(v_r, u_r) \in S_2$, then $|S_2| \ge \lceil t/3 \rceil$. Thus, $\gamma(G) \ge \lceil t/3 \rceil$.

Case 2.2.2. $(v_1, u_1) \in S_2$ and $(v_r, u_r) \in S_2$. If $r > 3$ and $t > r+1$, then, in order to dominate $(v_3, u_3), \ldots, (v_{r-2}, u_{r-2})$ and $u_{r+2}, \ldots, u_{t-1}$, there are at least $\lceil (r-4)/3 \rceil + \lceil (t-r-2)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_{r-1}, u_{r-1})\} \cup \{u_{r+1}, \ldots, u_t\}$ in S_2. Hence, $|S_2| \ge 2 + \lceil (r-4)/3 \rceil + \lceil (t-r-2)/3 \rceil = \lceil t/3 \rceil$. If $r = 3$ and $t > r+1$, then there are at least $\lceil (t-r-2)/3 \rceil$ vertices of $\{u_{r+1}, \ldots, u_t\}$ in S_2. The result is $|S_2| \ge 2 + \lceil (t-r-2)/3 \rceil = \lceil t/3 \rceil$. If $r > 3$ and $t = r+1$, then there are at least $\lceil (r-4)/3 \rceil$ vertices of $\{(v_2, u_2), \ldots, (v_{r-1}, u_{r-1})\}$. We thus get $|S_2| \ge 2 + \lceil (r-4)/3 \rceil = \lceil t/3 \rceil$. Clearly, if $r = 3$ and $t = r+1$, then $t = 4$ and $|S_2| \ge 2 = \lceil t/3 \rceil$. Thus, $\gamma(G) \ge \lceil t/3 \rceil$.

Case 2.2.3. $(v_1, u_1) \notin S_2$ and $(v_r, u_r) \notin S_2$. Clearly, $\gamma(G) = |S_2| \ge \gamma(C_t) = \lceil t/3 \rceil$.

Lemma 5. *If $H_1 \cong P_2$, then*

$$\gamma(G) = \begin{cases} \left\lceil \dfrac{r}{3} \right\rceil + \left\lceil \dfrac{t-3}{3} \right\rceil & \text{if } r \equiv 0 \pmod 3 \text{ and } t \equiv 0 \pmod 3, \\[2ex] \left\lceil \dfrac{t+r-2}{3} \right\rceil & \text{otherwise.} \end{cases} \tag{3}$$

Proof. Suppose that $H_1 \cong P_2$. It is easy to check that $G = C_r \underset{H_1 \cong_f H_2}{\lhd\rhd} C_t \cong C_r \underset{H_1' \cong_{f'} H_2'}{\lhd\rhd} C_{t'}$ where $H_1' \cong P_r$ and $t' = t + r - 2$ (see Figures 4(b) and 5(a)). By Lemma 4, the result holds.

Next, we will give the domination number of G such that $H_1 \cong P_3$ $(r > 3)$. We define three sets S_0, S_1, and S_2 as follows:

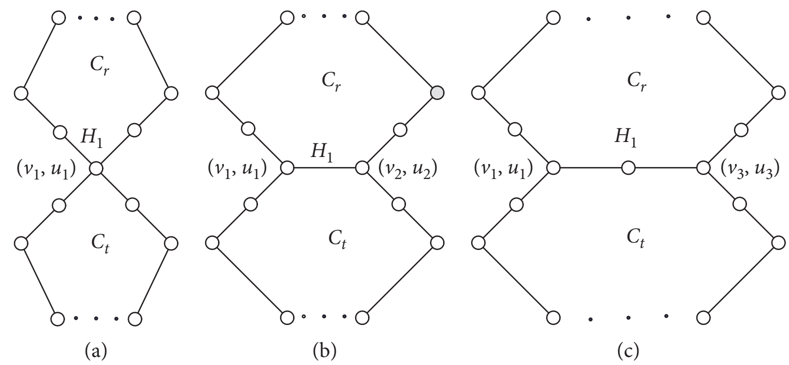

Figure 4: $C_r \underset{H_1\cong_f H_2}{\lhd\rhd} C_t$ with $H_1 \cong P_1$, $H_1 \cong P_2$, and $H_1 \cong P_3$.

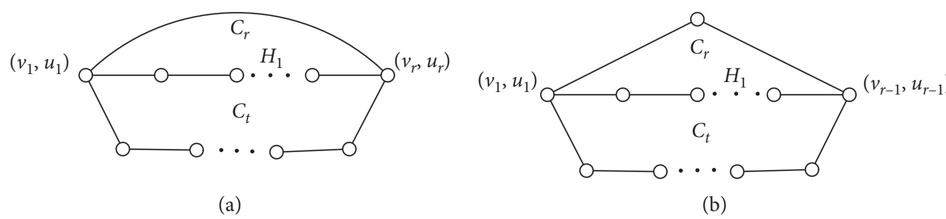

Figure 5: $C_r \underset{H_1\cong_f H_2}{\lhd\rhd} C_t$ with $H_1 \cong P_r$ and $H_1 \cong P_{r-1}$.

$S_0 = \{(v_1, u_1)\} \cup \{u_i | i \equiv 1 \pmod 3, 4 \le i \le t\} \cup \{v_i | i \equiv 2 \pmod 3, 4 \le i \le r\}$.

$S_1 = \{(v_1, u_1)\} \cup \{u_i | i \equiv 2 \pmod 3, 4 \le i \le t\} \cup \{v_i | i \equiv 1 \pmod 3, 4 \le i \le r\}$.

$S_2 = \{(v_1, u_1)\} \cup \{u_i | i \equiv 0 \pmod 3, 4 \le i \le t\} \cup \{v_i | i \equiv 0 \pmod 3, 4 \le i \le r\}$.

Note that if $t \equiv j \pmod 3$, for some $j \in \{0, 1, 2\}$, then $|S_j| = \lceil (r + t - 7)/3 \rceil + 1$. □

Lemma 6. *If $H_1 \cong P_3$ and $r > 3$, then $\gamma(G) = \lceil (r + t - 7)/3 \rceil + 1$.*

Proof. Suppose that $H_1 \cong P_3$. Then, $(v_1, u_1), (v_2, u_2), (v_3, u_3) \in V(G)$ (see Figure 4(c)). Note that if $t \equiv j \pmod 3$, for some $j \in \{0, 1, 2\}$, then the set S_j defined above is a dominating set of G, so $\gamma(G) \le \lceil (r + t - 7)/3 \rceil + 1$. We now prove that $\gamma(G) \ge \lceil (r + t - 7)/3 \rceil + 1$. Let S be a $\gamma-$ set of G. If $r = 4$, then, in order to dominate vertices $(v_1, u_1), (v_2, u_2), (v_3, u_3), u_4, \ldots, u_t$, there are at least $\lceil t/3 \rceil$ vertices of $\{(v_1, u_1), (v_2, u_2), (v_3, u_3), u_4, \ldots, u_t\} \cup \{v_4\}$ in S. Thus, $\gamma(G) = |S| \ge \lceil t/3 \rceil = \lceil (4 + t - 7)/3 \rceil + 1$. For $r \ge 5$, we consider two cases. □

Case 1. $(v_1, u_1) \in S$. In this case, we consider two subcases.

Case 1.1. $(v_3, u_3) \in S$. Consider the vertices in $G - N[\{(v_1, u_1), (v_3, u_3)\}] \cong P_{r-5} \cup P_{t-5}$. We see that there are at least $\lceil (r - 5)/3 \rceil + \lceil (t - 5)/3 \rceil$ vertices in $S - \{(v_1, u_1), (v_3, u_3)\}$. This gives $|S| \ge \lceil (r - 5)/3 \rceil + \lceil (t - 5)/3 \rceil + 2 \ge \lceil (r + t - 7)/3 \rceil + 1$.

Case 1.2. $(v_3, u_3) \notin S$. Consider the vertices in $G - N[\{(v_1, u_1)\}] \cong P_{r+t-7}$. We see at once that there are at least $\lceil (r + t - 7)/3 \rceil$ vertices in $S - \{(v_1, u_1)\}$. It follows that $|S| \ge \lceil (r + t - 7)/3 \rceil + 1$.

Case 2. $(v_1, u_1) \notin S$. In order to dominate (v_1, u_1), one vertex of $\{(v_2, u_2), v_r, u_t\}$ must be in S.

Case 2.1. $(v_2, u_2) \in S$. Consider the vertices in $G - N[\{(v_2, u_2)\}]$. In order to dominate $v_4, \ldots, v_r$ and $u_5, \ldots, u_t$, there are at least $\lceil (r - 3)/3 \rceil + \lceil (t - 4)/3 \rceil$ vertices in $S - \{(v_2, u_2)\}$. It follows that $|S| \ge \lceil (r - 3)/3 \rceil + \lceil (t - 4)/3 \rceil + 1 \ge \lceil (r + t - 7)/3 \rceil + 1$.

Case 2.2. $v_r \in S$. If $(v_2, u_2) \in S$, then this case is the same as Case 2.1. If $(v_2, u_2) \notin S$, then, in order to dominate $u_t, \ldots, u_4, (v_3, u_3), v_4, \ldots, v_{r-2}$ in $G - N[\{v_r\}]$, $S - \{v_r\}$ contains at least $\lceil (t + r - 7)/3 \rceil$ vertices. This gives $|S| \ge \lceil (r + t - 7)/3 \rceil + 1$.

Case 2.3. $u_t \in S$. By the same argument as in Case 2.2, we also get $|S| \ge \lceil (r + t - 7)/3 \rceil + 1$.

Lemma 7. *If $H_1 \cong P_{r-1}$, then $\gamma(G) = \lceil (t - 3)/3 \rceil + 1$.*

Proof. Suppose that $H_1 \cong P_{r-1}$. It is easy to check that $G = C_r \underset{H_1\cong_f H_2}{\lhd\rhd} C_t \cong C_r \underset{H_1'\cong_{f'} H_2'}{\lhd\rhd} C_{t'}$, where $H_1' \cong P_3$ and $t' = t - r + 4$ (see Figures 4(c) and 5(b)). By Lemma 6, the result holds.

By Lemma 3–7, we know the domination number of G, for all r such that $3 \le r \le 5$. We also know the domination number of G, for all H_1 such that $H_1 \cong P_1, P_2, P_3, P_{r-1}, P_r$. We now consider the case $r \ge 6$ and $H_1 \cong P_4, P_5, \ldots, P_{r-2}$.

For $n, k, m \in \mathbb{N} - \{1, 2\}$, let us denote by $C(m,k,n)$ a graph with $V(C(m,k,n)) = \{1, x\} \cup \{2_m, 3_m, \ldots, m_m\} \cup \{2_k, 3_k, \ldots, k_k\} \cup \{2_n, 3_n, \ldots, n_n\}$ and $E(C(m,k,n)) = \cup_{q\in\{m,k,n\}} \{\{1, 2_q\}, \{2_q, 3_q\}, \ldots, \{(q-1)_q, q_q\}, \{q_q, x\}\}$. Figure 6 illustrates $C(m,k,n)$. □

Lemma 8. *If $H_1 \cong P_s$, for some $s \in \{4, 5, \ldots, r-2\}$ and $r \geq 6$, then $G \cong C(m,k,n)$ where $r = m + k$, $s = k + 1$, and $t = n + k$.*

Proof. Suppose that $H_1 \cong P_s$, for some $s \in \{4, 5, \ldots, r-2\}$ and $r \geq 6$. Let $r = m + k$, $s = k + 1$, and $t = n + k$. Define $f\colon V(G) \longrightarrow V(C(m,k,n))$ by

$$f(g) = \begin{cases} 1 & \text{if } g = (v_1, u_1), \\ i_k & \text{if } g = (v_i, u_i) \text{ and } 2 \leq i \leq s-1, \\ (m-i+1)_m & \text{if } g = v_{s+i} \text{ and } 1 \leq i \leq r-s, \\ (n-i+1)_n & \text{if } g = u_{s+i} \text{ and } 1 \leq i \leq t-s, \\ x & \text{if } g = (v_s, u_s). \end{cases} \quad (4)$$

It is easy to check that f is an isomorphism. Then, $G \cong C(m,k,n)$. □

3. Domination Numbers of Amalgamations of Cycles at Connected Subgraphs

In this section, we calculate the domination number $\gamma(C(m,k,n))$. Then, by Lemma 8, we thus get the domination number $\gamma(G)$ for the case $r \geq 6$ and $H_1 \cong P_4, P_5, \ldots, P_{r-2}$. The following two lemmas provide upper bounds of $\gamma(C(m,k,n))$.

Lemma 9. *Let $m, k, n \in \mathbb{N} - \{1, 2\}$ and $S = \{1, x\} \cup \cup_{q\in\{m,k,n\}} \{a_q | a \equiv 1 \pmod 3, 3 \leq a \leq q-1\}$. Then, S is a dominating set of $C(m,k,n)$ and $|S| = 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.*

Proof. Let $v \in V(C(m,k,n)) - S$. If $v = 2_q$, for some $q \in \{m,k,n\}$, then there is $1 \in S$ such that $\{1, v\} \in E(C(m,k,n))$. If $v = q_q$, for some $q \in \{m,k,n\}$, then there is $x \in S$ such that $\{x, v\} \in E(C(m,k,n))$. If $v = u_q$, for some $q \in \{m,k,n\}$, such that $u \equiv 0 \pmod 3$ and $u \notin \{m,k,n\}$, then $u+1 \equiv 1 \pmod 3$ and thus $(u+1)_q \in S$. It follows that there is $(u+1)_q \in S$ such that $\{v, (u+1)_q\} \in E(C(m,k,n))$. If $v = u_q$, for some $q \in \{m,k,n\}$, such that $u \equiv 2 \pmod 3$ and $u \notin \{2, m, k, n\}$, then $u-1 \equiv 1 \pmod 3$ and thus $(u-1)_q \in S$. Hence, there is $(u-1)_q \in S$ such that $\{v, (u-1)_q\} \in E(C(m,k,n))$. Therefore, S is a dominating set of $C(m,k,n)$. It is easily seen that $|S| = 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$. □

Lemma 10. *Let $n, k, m \in \mathbb{N} - \{1, 2\}$ and $S = \{1\} \cup \cup_{q\in\{m,k,n\}} \{a_q | a \equiv 1 \pmod 3, 3 \leq a \leq q\}$. If $m, k, n \equiv 0 \pmod 3$ and $z \equiv 1 \pmod 3$, for some $z \in \{m,k,n\}$, then S is a dominating set of $C(m,k,n)$ and $|S| = 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$.*

Proof. Suppose that $m, k, n \equiv 0 \pmod 3$ and $z \equiv 1 \pmod 3$, for some $z \in \{m,k,n\}$. Without loss of generality, we can assume that $m \equiv 1 \pmod 3$. We will prove that S is a dominating set of $C(m,k,n)$. There are four cases for k and n. We give the proof only for the case $k \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$; the proofs of the other cases are similar. Let $v \in V(C(m,k,n)) - S$. If $v = 2_q$, for some $q \in \{m,k,n\}$, then there is $1 \in S$ such that $\{1, v\} \in E(C(m,k,n))$. If $v = u_q$, for some $q \in \{m,k,n\}$ such that $u \equiv 0 \pmod 3$, then $u+1 \equiv 1 \pmod 3$ and so $(u+1)_q \in S$. Thus, there is $(u+1)_q \in S$ such that $\{v, (u+1)_q\} \in E(C(m,k,n))$. If $v = u_q$, for some $q \in \{m,k,n\}$ such that $u \equiv 2 \pmod 3$ and $u \neq 2$, then $u-1 \equiv 1 \pmod 3$, and thus, $(u-1)_q \in S$. Consequently, there is $(u-1)_q \in S$ such that $\{v, (u-1)_q\} \in E(C(m,k,n))$. If $v = x$, then there is $m_m \in S$ such that $\{m_m, v\} \in E(C(m,k,n))$. Therefore, S is a dominating set of $C(m,k,n)$. It is easy to check that $|S| = 1 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

The following lemmas about γ−set will be used in Theorem 1 to determine lower bounds of $\gamma(C(m,k,n))$. □

Lemma 11. *Let $n, k, m \in \mathbb{N} - \{1, 2\}$ such that $y \equiv 0 \pmod 3$, for all $y \in \{m,k,n\}$ and $z \equiv 1 \pmod 3$, for some $z \in \{m,k,n\}$, and let S be a γ−set of $C(m,k,n)$. Then, the following hold:*

(1) $|S \cap \{1, x\}| \leq 1$.

(2) If $1 \in S$, then $q_q \in S$ for some $q \in \{m,k,n\}$ such that $q \equiv 1 \pmod 3$.

(3) If $x \in S$, then $2_q \in S$ for some $q \in \{m,k,n\}$ such that $q \equiv 1 \pmod 3$.

Proof. Without loss of generality, we can assume that $m \equiv 1 \pmod 3$.

(1) There are four cases for k and n. We give the proof only for the case $k \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$; the proofs of the other cases are similar. On the contrary, suppose that $|S \cap \{1, x\}| = 2$. This gives $1, x \in S$, and thus, every vertex in $N[\{1, x\}] = \{1, 2_m, 2_k, 2_n, x, m_m, k_k, n_n\}$ is dominated by S. Obviously, $C(m,k,n) - N[\{1, x\}] \cong P_{m-3} \cup P_{k-3} \cup P_{k-3}$ (see Figure 7).

In order to dominate the vertices in $C(m,k,n) - N[\{1, x\}]$, then $S - \{1, x\}$ must contain at least $\lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$ vertices. This gives $|S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil = 2 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. From Lemma 10, we know that $C(m,k,n)$ contains a dominating set of order $1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. This contradicts the fact that S is a γ− set of $C(m,k,n)$.

(2) Let $1 \in S$. By (1), $x \notin S$. Suppose, by contrary, $q_q \notin S$, for all $q \in \{m,k,n\}$ such that $q \equiv 1 \pmod 3$. In order to dominate x, then there exists $z_z \in S - \{1\}$, for some $z \in \{m,k,n\}$ such that $z \neq m$ and $z \equiv 2 \pmod 3$. There are two cases for z. We give

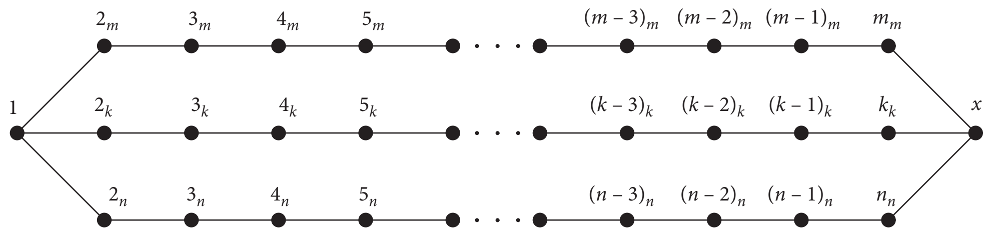

Figure 6: $C(m,k,n)$.

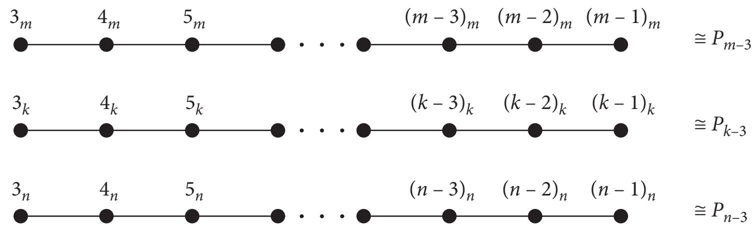

Figure 7: $C(m,k,n) - N[\{1,x\}]$.

the proof only for the case $z = n$; the proof of the case $z = k$ is similar. Since $1, n_n \in S$, every vertex in $N[\{1, n_n\}] = \{1, 2_m, 2_k, 2_n, (n-1)_n, n_n, x\}$ is dominated by S. It is clear that $C(m,k,n) - N[\{1,n_n\}] \cong P_{m-2} \cup P_{k-2} \cup P_{n-4}$ (see Figure 8).

In order to dominate the vertices in $C(m, k,n) - N[\{1,n_n\}]$, then $S - \{1,n_n\}$ must contain at least $\lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-4)/3 \rceil$ vertices. We thus get $|S| \geq 2 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-4)/3 \rceil = 2 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. From Lemma 10, we know that $C(m,k,n)$ contains a dominating set of order $1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. This contradicts the fact that S is a $\gamma-$ set of $C(m,k,n)$.

(3) The proof is similar to that for 2. □

Lemma 12. *Let $n,k,m \in \mathbb{N} - \{1,2\}$ and let S be a $\gamma-$ set of $C(m,k,n)$ and $|S \cap \{1,x\}| = 0$. Then, the following hold:*

(1) If $m,k,n \equiv 1 \pmod 3$, then either $2_m, m_m \in S$, $2_k, k_k \in S$, or $2_n, n_n \in S$.

(2) If $m \equiv 2 \pmod 3$ and $k,n \equiv 1 \pmod 3$, then $2_m, m_m \in S$.

(3) If $k \equiv 2 \pmod 3$ and $m,n \equiv 1 \pmod 3$, then $2_k, k_k \in S$.

(4) If $n \equiv 2 \pmod 3$ and $m,k \equiv 1 \pmod 3$, then $2_n, n_n \in S$.

Proof. We give the proofs only for (1) and (2); the proofs of (3) and (4) are similar to that of (2).

(1) Let $m,k,n \equiv 1 \pmod 3$. Suppose, by contrary, that neither $2_m, m_m \in S$, $2_k, k_k \in S$ nor $2_n, n_n \in S$. So, in order to dominate 1 and x, we get $2_q, z_z \in S$ for some $q, z \in \{m,k,n\}$ such that $q \neq z$. There are nine cases for q and z. We give the proof only for the case $q = m$ and $z = k$; the proofs of the other cases are similar. Since $2_m, k_k \in S$, every vertex in $N[\{2_m, k_k\}] = \{1, 2_m, 3_m, (k-1)_k, k_k, x\}$ is dominated by S. We check at once that $C(m,k,n) - N[\{2_m, k_k\}] \cong P_{m-3} \cup P_{k-3} \cup P_{n-1}$ (see Figure 9).

In order to dominate the vertices in $C(m,k,n) - N[\{2_m, k_k\}]$, then $S - \{2_m, k_k\}$ must contain at least $\lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-1)/3 \rceil$ vertices. This gives $|S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-1)/3 \rceil = 2 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. From Lemma 10, we know that $C(m,k,n)$ contains a dominating set of order $1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. This contradicts the fact that S is a $\gamma-$ set of $C(m,k,n)$.

(2) Let $m \equiv 2 \pmod 3$ and $k,n \equiv 1 \pmod 3$. On the contrary, suppose that $2_m, m_m \notin S$. In order to dominate 1 and x, then $2_q, z_z \in S$, for some $q, z \in \{m,k,n\}$ such that $q \neq z$ or $q = z \in \{k,n\}$. If $q \neq z$, then the proof is similar to that for 1. Let $q = z \in \{k,n\}$. There are two cases for q and z. We give the proof only for the case $q = z = k$; the proof of the case $q = z = n$ is similar. Since $2_k, k_k \in S$, every vertex in $N[\{2_k, k_k\}] = \{1, 2_k, 3_k, (k-1)_k, k_k, x\}$ is dominated by S. If $k \geq 5$, then $C(m,k,n) - N[\{2_k, k_k\}] \cong P_{m-1} \cup P_{k-5} \cup P_{n-1}$ (see Figure 10).

In order to dominate the vertices in $C(m, k,n) - N[\{2_k, k_k\}]$, then $S - \{2_k, k_k\}$ must contain at least $\lceil (m-1)/3 \rceil + \lceil (k-5)/3 \rceil + \lceil (n-1)/3 \rceil$ vertices. We thus get $|S| \geq 2 + \lceil (m-1)/3 \rceil + \lceil (k-5)/3 \rceil + \lceil (n-1)/3 \rceil = 3 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. If $k < 5$, then $k = 4$. It is easily seen that $C(m,k,n) - N[\{2_k, k_k\}] \cong P_{m-1} \cup P_{n-1}$. In order to dominate the vertices in $C(m, k,n) - N[\{2_k, k_k\}]$, then $S - \{2_k, k_k\}$ must contain at least $\lceil (m-1)/3 \rceil + \lceil (n-1)/3 \rceil$ vertices. This gives $|S| \geq 2 + \lceil (m-1)/3 \rceil + \lceil (n-1)/3 \rceil = 2 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-$

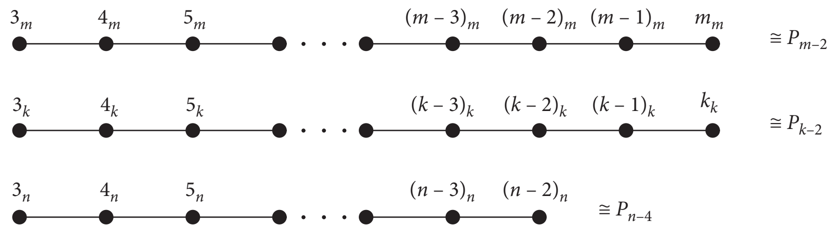

Figure 8: $C(m,k,n) - N[\{1,n_n\}]$.

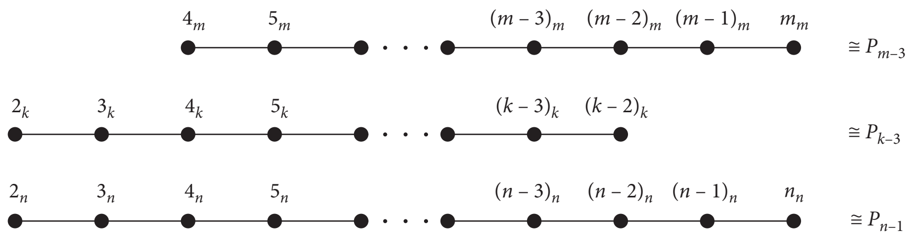

Figure 9: $C(m,k,n) - N[\{2_m,k_k\}]$.

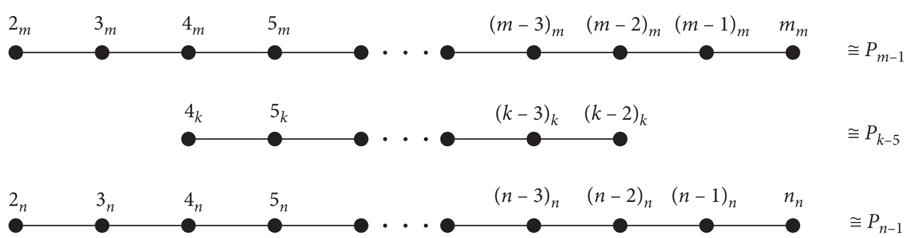

Figure 10: $C(m,k,n) - N[\{2_k,k_k\}]$.

$2)/3\rceil$. From Lemma 10, we know that $C(m,k,n)$ contains a dominating set of order $1+\lceil (m-2)/3\rceil + \lceil (k-2)/3\rceil + \lceil (n-2)/3\rceil$. This contradicts the fact that S is a $\gamma-$ set of $C(m,k,n)$. □

Lemma 13. *Let $n,k,m \in \mathbb{N} - \{1,2\}$ and let S be a $\gamma-$ set of $C(m,k,n)$. Then, $|S\cap\{1,x\}| = 1$ if one of the following holds:*

(1) $m \equiv 1 \pmod 3$ and $k,n \equiv 2 \pmod 3$.

(2) $k \equiv 1 \pmod 3$ and $m,n \equiv 2 \pmod 3$.

(3) $n \equiv 1 \pmod 3$ and $m,k \equiv 2 \pmod 3$.

Proof. We give the proof only for the case $m \equiv 1 \pmod 3$ and $k,n \equiv 2 \pmod 3$; the proofs of the other cases are similar. Suppose, by contrary, that $|S\cap\{1,x\}| \neq 1$. By Lemma 11 (1), $|S\cap\{1,x\}| = 0$. We thus get $1,x \notin S$. So, in order to dominate 1 and x, we get $2_q, z_z \in S$, for some $q,z \in \{m,k,n\}$. There are two cases: $q \neq z$ and $q = z$. Following in a same manner as the proof of Lemma 12 (2), we can obtain $|S| \geq 1+\lceil (m-2)/3\rceil + \lceil (k-2)/3\rceil + \lceil (n-2)/3\rceil$, which contradicts the fact that S is a γ–set of $C(m,k,n)$. □

Theorem 1. *Let $n,k,m \in \mathbb{N} - \{1,2\}$, $D = \{m,k,n\}$, and $C = C(m,k,n)$. Then,*

$$\gamma(C) = \begin{cases} 2+\left\lceil \dfrac{m-3}{3}\right\rceil + \left\lceil \dfrac{k-3}{3}\right\rceil + \left\lceil \dfrac{n-3}{3}\right\rceil & \text{if } y \equiv 0 \pmod 3 \text{ for some } y \in D \text{ or } m,k,n \equiv 2 \pmod 3, \\ 1+\left\lceil \dfrac{m-2}{3}\right\rceil + \left\lceil \dfrac{k-2}{3}\right\rceil + \left\lceil \dfrac{n-2}{3}\right\rceil & \text{if } m,k,n \equiv 0 \pmod 3 \text{ and } y \equiv 1 \pmod 3 \text{ for some } y \in D. \end{cases} \quad (5)$$

Case 1. $y \equiv 0 \pmod 3$, for some $y \in D$ or $m,k,n \equiv 2 \pmod 3$. By Lemma 9, $\gamma(C) \leq 2+\lceil (m-3)/3\rceil + \lceil (k-3)/3\rceil + \lceil (n-3)/3\rceil$. We next prove that $\gamma(C) \geq 2+\lceil (m-3)/3\rceil + \lceil (k-3)/3\rceil + \lceil (n-3)/3\rceil$. Let S be a γ–set of C.

Case 1.1. $y \equiv 0 \pmod 3$ for some $y \in D$. Without loss of generality, we can assume that $m \equiv 0 \pmod 3$. In order to dominate 1 and x, there are sixteen cases, $u,v \in S$, for some $u \in \{1,2_m,2_k,2_n\}$ and $v \in \{x,m_m,k_k,n_n\}$. We give the proofs only for the

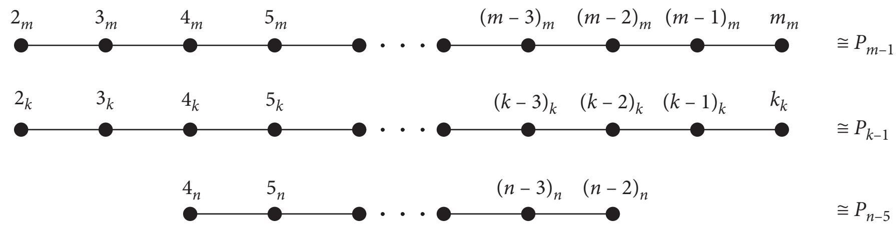

Figure 11: $C(m,k,n) - N[\{2_n, n_n\}]$.

four cases $1, x \in S$, $1, m_m \in S$, $2_m, m_m \in S$, and $2_m, k_k \in S$; the proofs of the other cases are similar.

Case 1.1.1. $1, x \in S$. Then, every vertex in $N[\{1, x\}] = \{1, 2_m, 2_k, 2_n, x, m_m, k_k, n_n\}$ is dominated by S. Thus, $C - N[\{1, x\}] \cong P_{m-3} \cup P_{k-3} \cup P_{k-3}$, so $\gamma(C) = |S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

Case 1.1.2. $1, m_m \in S$. It follows that every vertex in $N[\{1, m_m\}] = \{1, 2_m, 2_k, 2_n, (m-1)_m, m_m, x\}$ is dominated by S. If $m \geq 4$, then it is easy to check that $C - N[\{1, m_m\}] \cong P_{m-4} \cup P_{k-2} \cup P_{n-2}$. This gives $|S| \geq 2 + \lceil (m-4)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. For $m = 3$, $C - N[\{1, m_m\}] \cong P_{k-2} \cup P_{n-2}$. It follows that $|S| \geq 2 + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. Since $m \equiv 0 \pmod 3$, $\gamma(C) = |S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

Case 1.1.3. $2_m, m_m \in S$. Then, every vertex in $N[\{2_m, m_m\}] = \{1, 2_m, 3_m, (m-1)_m, m_m, x\}$ is dominated by S. We can see that $C - N[\{2_m, m_m\}] \cong P_{m-5} \cup P_{k-1} \cup P_{n-1}$ if $m \geq 5$, and $C - N[\{2_m, m_m\}] \cong P_{k-1} \cup P_{n-1}$ otherwise. It follows that $|S| \geq 2 + \lceil (m-5)/3 \rceil + \lceil (k-1)/3 \rceil + \lceil (n-1)/3 \rceil$ if $m \geq 5$, and $|S| \geq 2 + \lceil (k-1)/3 \rceil + \lceil (n-1)/3 \rceil$ otherwise. Since $m \equiv 0 \pmod 3$, $\gamma(C) = |S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

Case 1.1.4. $2_m, k_k \in S$. We thus get every vertex in $N[\{2_m, k_k\}] = \{1, 2_m, 3_m, (k-1)_k, k_k, x\}$ is dominated by S. Since $C - N[\{2_m, k_k\}] \cong P_{m-3} \cup P_{k-3} \cup P_{n-1}$, $\gamma(C) = |S| \geq 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

Case 1.2. $m, k, n \equiv 2 \pmod 3$. In order to dominate 1 and x, there are sixteen cases as above. In the same manner, we can prove that $\gamma(C) = 2 + \lceil (m-3)/3 \rceil + \lceil (k-3)/3 \rceil + \lceil (n-3)/3 \rceil$.

Case 2. $m, k, n \equiv 0 \pmod 3$ and $y \equiv 1 \pmod 3$, for some $y \in D$. By Lemma 10, $\gamma(C) \leq 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. We next show that $\gamma(C) \geq 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. Without loss of generality, we can assume that $m \equiv 1 \pmod 3$. Let S be a γ-set of C. By Lemma 11 (1), $|S \cap \{1, x\}| \leq 1$. It follows that $|S \cap \{1, x\}| = 1$ or $|S \cap \{1, x\}| = 0$. Consider the following four cases.

Case 2.1. $k, n \equiv 1 \pmod 3$. If $|S \cap \{1, x\}| = 1$, then $1 \in S$ or $x \in S$. We give the proof only for the case $1 \in S$; the proof of the case $x \in S$ is similar. By Lemma 11 (2), $m_m \in S$, $k_k \in S$, or $n_n \in S$. Here, we will give the proof only for the case $m_m \in S$; the proofs of the other two cases are similar. Hence, $1, m_m \in S$. As in the proof of Case 1.1.2, $\gamma(C) = |S| \geq 2 + \lceil (m-4)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil = 1 + \lceil (m-1)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil = 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. If $|S \cap \{1, x\}| = 0$, then $2_m, m_m \in S$, $2_k, k_k \in S$, or $2_n, n_n \in S$ by Lemma 12 (1). We give the proof only for the case $2_m, m_m \in S$; the proofs of the other two cases are similar. As in the proof of Case 1.1.3, we get that if $m \geq 5$, then $\gamma(C) = |S| \geq 2 + \lceil (m-5)/3 \rceil + \lceil (k-1)/3 \rceil + \lceil (n-1)/3 \rceil = 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. If $m \leq 4$, then $m = 4$. It follows that $\gamma(C) = |S| \geq 2 + \lceil (k-1)/3 \rceil + \lceil (n-1)/3 \rceil = 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$.

Case 2.2. $k \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. If $|S \cap \{1, x\}| = 1$, then $1 \in S$ or $x \in S$. We give the proof only for the case $1 \in S$; the proof of the case $x \in S$ is similar. By Lemma 11 (2), $m_m \in S$ or $k_k \in S$. Here, we will give the proof only for the case $m_m \in S$; the proof of the case $k_k \in S$ is similar. Hence, $1, m_m \in S$. As in the proof of Case 2.1, $\gamma(C) = |S| \geq 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$. If $|S \cap \{1, x\}| = 0$, then $2_n, n_n \in S$ by Lemma 12 (4). Thus, every vertex in $N[\{2_n, n_n\}] = \{1, 2_n, 3_n, (n-1)_n, n_n, x\}$ is dominated by S. It is clear that $C - N[\{2_n, n_n\}] \cong P_{m-1} \cup P_{k-1} \cup P_{n-5}$ (see Figure 11). In order to dominate the vertices in $C - N[\{2_n, n_n\}]$, then $S - \{2_n, n_n\}$ must contain at least $\lceil (m-1)/3 \rceil + \lceil (k-1)/3 \rceil + \lceil (n-5)/3 \rceil$ vertices. This gives $|S| \geq 2 + \lceil (m-1)/3 \rceil + \lceil (k-1)/3 \rceil + \lceil (n-5)/3 \rceil = 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$.

Case 2.3. $k \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$. The proof is similar to that of Case 2.2.

Case 2.4. $k, n \equiv 2 \pmod 3$. By Lemma 13, $|S \cap \{1, x\}| = 1$. So, $1 \in S$ or $x \in S$. We give the proof only for the case $1 \in S$; the proof of the case $x \in S$ is similar. By Lemma 11 (2), $m_m \in S$. Following in a same manner as the proof of Case 2.1, we can obtain $\gamma(C) = |S| \geq 1 + \lceil (m-2)/3 \rceil + \lceil (k-2)/3 \rceil + \lceil (n-2)/3 \rceil$.

Summarizing, we get the domination number of amalgamations of cycles at connected subgraphs $C_r \underset{H_1 \cong_f H_2}{\triangleleft\triangleright} C_t$.

From Lemmas 2-7 and Theorem 1, we then get the following theorem.

Theorem 2. *Let $r, s, t \in \mathbb{N}$, $3 \leq r \leq t$, $G = C_r \underset{H_1 \cong_f H_2}{\triangleleft\triangleright} C_t$, and $D = \{a, b, c\}$ where $a = r - s + 1$, $b = s - 1$, and $c = t - s + 1$. Then*

$$\gamma(C) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil & \text{if } H_1 \cong C_r, \\ \left\lceil \frac{r-3}{3} \right\rceil + \left\lceil \frac{t-3}{3} \right\rceil + 1 & \text{if } H_1 \cong P_1, \\ \left\lceil \frac{r}{3} \right\rceil + \left\lceil \frac{t-3}{3} \right\rceil & \text{if } H_1 \cong P_2 \text{ and } r \equiv 0 \pmod 3 \text{ and } t \equiv 0 \pmod 3, \\ \left\lceil \frac{t+r-2}{3} \right\rceil & \text{if } H_1 \cong P_2 \text{ and } r \equiv 0 \pmod 3 \text{ or } t \equiv 0 \pmod 3, \\ \left\lceil \frac{r+t-7}{3} \right\rceil + 1 & \text{if } H_1 \cong P_3 \text{ and } r > 3, \\ \left\lceil \frac{t-3}{3} \right\rceil + 1 & \text{if } H_1 \cong P_{r-1}, \\ \left\lceil \frac{r}{3} \right\rceil + \left\lceil \frac{t-r-1}{3} \right\rceil & \text{if } H_1 \cong P_r \text{ and } r \equiv 0 \pmod 3 \text{ and } t \equiv 1 \pmod 3, \\ \left\lceil \frac{t}{3} \right\rceil & \text{if } H_1 \cong P_r \text{ and } r \equiv 0 \pmod 3 \text{ or } t \equiv 1 \pmod 3, \\ 2 + \left\lceil \frac{a-3}{3} \right\rceil + \left\lceil \frac{b-3}{3} \right\rceil + \left\lceil \frac{c-3}{3} \right\rceil & \text{if } H_1 \cong P_s, 4 \leq s \leq r-2, \text{ and } y \equiv 0 \pmod 3 \text{ for some } y \in D \text{ or } a, b, c \equiv 2 \pmod 3, \\ 1 + \left\lceil \frac{a-2}{3} \right\rceil + \left\lceil \frac{b-2}{3} \right\rceil + \left\lceil \frac{c-2}{3} \right\rceil & \text{if } H_1 \cong P_s, 4 \leq s \leq r-2, a, b, c \equiv 0 \pmod 3, \text{ and } y \equiv 1 \pmod 3 \text{ for some } y \in D. \end{cases} \quad (6)$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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