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Research Article

Reliability Analysis of the Proportional Mean Departure Time Model

Mansour Shrahili and Mohamed Kayid

Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia

Correspondence should be addressed to Mohamed Kayid; drkayid@ksu.edu.sa

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In this article, the mean lifetime of an individual whose lives lost based on a function of the time before which the individual has passed away is considered. The measure is used to construct a semi-parametric model called proportional mean departure time model. Examples are given and evidences are gathered to show that the model is a proper alternative for the proportional mean past lifetime model. Closure properties of the model concerning several stochastic orders and a number of reliability properties are established. Finally, the model is extended to entertain random amounts of the parameter and establish a proportional mean departure time frailty model. Further stochastic properties using several stochastic orders are developed in the context of the frailty model.

1. Introduction

Let X be a non-negative random variable (r.v.) having absolutely continuous cumulative distribution function (c.d.f.) F and probability density function (p.d.f.) f. Consider a situation where X is the time-to-failure of a lifespan and assume that it has been realized that the item has failed before the time point t. In literature, there has been growing interest in the study of reliability measures in reversed time and their applications. The amount of X, after knowing at the time t that a past failure has happened, is then the time of departure among lives lost before the time t. The mean departure time (m.d.t.) function of X is defined by

$$m_X(t) = E(X|X \le t), \quad = \frac{\int_0^t x f(x)\mathrm{d}x}{F(t)}, \quad = t - \frac{\int_0^t F(x)\mathrm{d}x}{F(t)}, \tag{1}$$

which is valid for all $t \ge 0$ for which $F(t) > 0$. The conditional random variable $X_{(t)} = (t - X|X \le t)$, where $(X|A)$ denotes the conditional random variable X given that the event A has happened, is well-known in the literature as the past lifetime or the inactivity time (see, e.g., Di Crescenzo and Longobardi [1] and Kayid and Ahmad [2]). The random variable $X_{(t)}$ is also called the revered residual life (see Nanda et al. [3]). The concept of mean past lifetime (m.p.l.) or mean inactivity time (m.i.t.) is closely related to the mean departure time function given in (1). The m.i.t (or the m.p.l.) function provides the expected time elapsed since the failure of a subject given that he/she has failed before the time of observation. The m.i.t. function is given by

$$M_X(t) = E\left(X_{(t)}\right) = \frac{\int_0^t F(x)\mathrm{d}x}{F(t)}, \tag{2}$$

where $F(t) > 0$. The inactivity time $X_{(t)}$ and the measure (2) have been very useful in science and engineering contexts. They have many applications in various disciplines such as reliability theory, survival analysis, risk theory, and actuarial studies, among others (cf. Ortega [4], Izadkhah and Kayid [5], Jayasinghe and Zeephongsekul [6], Kayid and Izadkhah [7], Kayid et al. [8], Kayid and Izadkhah [9], Bhattacharyya et al. [10], Balmert et al. [11], Khan et al. [12], and Kayid and Alrasheedi [13]). The inactivity time at random times is one of the important notions in reliability and queuing theory (see, e.g., Kayid et al. [14] and Kundu and Patra [15]).

The reversed hazard rate (r.h.r.) function of X, as another measure related to the inactivity time, is given by

$$\widetilde{h}_X(t) = \lim_{\delta \longrightarrow 0^+} \frac{1}{\delta} P(X < t - \delta | X \leq t) = \frac{f(t)}{F(t)}. \quad (3)$$

The m.d.t. function is connected with the r.h.r. and the m.i.t. function as follows:

$$\widetilde{h}_X(t) = \frac{m_X'(t)}{t - m_X(t)}, \quad (4)$$

where $m_X'(t) = (\mathrm{d}/\mathrm{d}t) m_X(t)$ and the m.d.t. function is related to the m.i.t. function as follows:

$$M_X(t) = t - m_X(t). \quad (5)$$

The m.i.t. function (2) characterizes the underlying c.d.f. uniquely. By (5), it follows that the m.d.t. function also determines the underlying distribution in a unique way. It is said that X_w is a weighted version of X which has p.d.f. $f_w(x) = w(x) f(x)/E[w(x)]$, where w is a non-negative function such that $E[w(X)] < +\infty$. As pointed out in Equations (2.3) in Sunoj and Maya [16] by taking the weight function $w(x) = x$, the c.d.f. of X can be recovered from the m.d.t. function m_X as

$$F(t) = \exp\left(-\int_t^{+\infty} \frac{m_X'(x)}{x - m_X(x)} \mathrm{d}x\right). \quad (6)$$

Let X and Y have m.i.t. functions M_X and M_Y, respectively. Asadi and Berred [17] constructed a model by holding the two m.i.t. functions in a proportional relation so that

$$M_Y(t) = \theta M_X(t). \quad (7)$$

For all $t \geq 0$ and $\theta > 0$ is the constant of proportionality. The r.v. X plays the role of the independent variable and the r.v. Y is the dependent variable whose distribution depends on θ and also it depends on the distribution of X. This model is called the proportional mean past lifetime (PMPL) model. There is one difficulty, as the authors believe and, further, mathematical strategies confirms it, with the PMPL model given in (7) which makes it somewhat controversial to use this model in applied situations. The question is that to what extent the model (7) is useful to model lifetime events. Repeatedly encountered in survival analysis applications, consider a situation where X and Y with c.d.f.s F and G are supported in $[0, +\infty)$ and also X and Y have finite means. Let us assume that the identity in (7) holds true. Then, since $E(X) = \int_0^{+\infty} \overline{F}(x)\mathrm{d}x < +\infty$ and also $E(Y) = \int_0^{+\infty} \overline{G}(x)\mathrm{d}x < +\infty$, thus

$$\begin{aligned} \theta &= \lim_{t \longrightarrow +\infty} \frac{F(t)}{G(t)} \frac{\int_0^t G(x)\mathrm{d}x}{\int_0^t F(x)\mathrm{d}x}, \\ &= \lim_{t \longrightarrow +\infty} \frac{F(t)}{G(t)} \frac{t - \int_0^t \overline{G}(x)\mathrm{d}x}{t - \int_0^t \overline{F}(x)\mathrm{d}x}, \\ &= \lim_{t \longrightarrow +\infty} \frac{tF(t)}{tG(t)} = 1, \end{aligned} \quad (8)$$

which shows that X and Y are identical in distribution. The conclusion is that every set of non-negative r.v.s (X, Y) with finite means having unbounded supports which satisfies Equation (7) has to have identical components in distribution and θ is not a parameter in this case. Therefore, the model (7) cannot be useful to model lifetime events on $[0, +\infty)$ which obviously surrounds the applicability of the model (7). Finding an alternative model of (7) is, therefore, necessary as there are many cases in which the lifetime data are left censored and modeling data in the framework of past failures is needed.

This article aims to introduce an appropriate alternative model for the proportional mean past lifetime model. We show by some examples that the new model is applicable for modeling random lifetimes with unbounded supports. The role of the parameter of the model which is added to the original distribution is to extend the existing model and provided flexibility to adjust the mean departure time function of life spans. The theory of stochastic orders is used to provide some comparison results to address the question of whether the reliability of a device is either improved or deteriorated under the setup of the model. Throughout the article, we will not use the terms "increasing" and "decreasing" in the strict case and will take these properties equivalently as "non-decreasing" and "non-increasing" behaviors, respectively.

The organization of the article will be in the following order. In Section 2, we introduce and illustrate the proportional mean departure time model with some examples. In Section 3, we consider the closure properties of the model with respect to some well-known stochastic orders and also a number of reliability classes of lifetime distributions. In Section 4, the model is extended to the case where the parameter of the model is a random variable and some stochastic ordering properties are investigated. Finally, in Section 5, we conclude the article with some illustrative statements to describe our contribution and also we will add some remarks on possible future studies.

2. Proportional Mean Departure Time Model

In this section, we introduce and describe the PMDT model and study the advantages of this model over the PMPL model. Further distributional properties of the model are investigated. We present some examples which fulfill the PMDT model.

Definition 1. Let us assume that X and Y have m.d.t. functions m_X and m_Y, respectively. Then, it is said that X and Y satisfy the PMDT model whenever

$$m_Y(t) = \theta m_X(t). \quad (9)$$

For all $t \geq 0$ and θ is a positive parameter. The parameter θ is called the departure index.

The departure index θ in Equation (9) represents the magnitude of departure time of individuals whose lives lost before t with lifetime Y, in average, relative to the departure time of individuals whose lives lost before t with lifetime X.

The parameter θ is, therefore, a conditional deterioration rate of Y with respect to X. Unlike the m.i.t function which is not originally an increasing function of time, the m.d.t. function is monotonically increasing in time, i.e., it certainly holds that for all $t_1 \leq t_2 \in \mathbb{R}^+$, $m_X(t_1) \leq m_X(t_2)$. If we consider the ratio of two m.d.t. functions as a parameter then a new model, which we call it proportional mean departure time (PMDT) model, is constructed. The assumption that the ratio of two increasing functions which are a reliability characteristic associated with two different distributions coincides with a horizontal line is a more appropriate assumption than the same for two arbitrary reliability measures. In a bivariate setup, when the sample $X_1, \ldots, X_n$ copies of X and the sample $Y_1, \ldots, Y_m$ copies of Y is available then the model (9) may be used and the parameter θ is estimated by the estimation of m.d.t. functions of Y divided by the estimation of m.d.t. function of X. The model (7) in situations where it is applicable (when X and Y have finite supports) may not be proper in the two sample setting. This is because the shapes of m.i.t. functions in the model (7) have to be the same since θ does not depend on time and in spite of that the set on samples on X and Y may not exhibit similar shapes as the distributions of X and Y may not belong to the same class of lifetime distributions, e.g., the increasing mean inactivity time (IMIT) class. Therefore, as the m.d.t. functions of X and Y are always increasing independent of the distributions of X and Y, thus the model (9) may be preferable.

Note that $m_X(t) = E(X|X \leq t)$ is the expected time at which individuals whose deaths happened before the time t departed. The time t may be considered as the first time one realizes that a failure or death has occurred in the past. In the PMDT model, the ratio $m_Y(t)/m_X(t)$ is independent of the observation time t which induces that the m.d.t. function of X relative to the m.d.t. function of Y is independent of the process of observation of past failures. From a mathematical perspective,

$$\frac{m_Y(t_1)}{m_X(t_1)} = \frac{m_Y(t_2)}{m_X(t_2)}, \quad \text{for all } t_i \in \mathbb{R}_+, \ i = 1, 2. \tag{10}$$

Indicating that relative mean departure time functions remain unchanged during time. Specifically, when r.v.s X and Y have finite means, the choices of $t_1 = t$ and $t_2 = +\infty$ in (10) lead to

$$\frac{m_Y(t)}{m_X(t)} = \frac{E(Y)}{E(X)} = \theta, \quad \text{for all } t > 0. \tag{11}$$

By appealing to (6) for r.v.s X and Y satisfying the PMDT model one has

$$G(t) = \exp\left\{-\int_t^{+\infty} \frac{\theta m_X'(x)}{x - \theta m_X(x)} \mathrm{d}x\right\}, \quad t \geq 0. \tag{12}$$

For an unspecified m_X associated with a general distribution, the formula (12) may not provide a closed form for the c.d.f. of the r.v. Y relative to the c.d.f. of X. It is not hard to verify that (12) is a valid c.d.f. if the following conditions are satisfied:

(i) For all $x > 0$, $\theta \leq x/m_X(x)$.

(ii) $\int_t^{+\infty} (\theta m_X'(x)/x - \theta m_X(x))\mathrm{d}x < \infty$, for all $0 < t < +\infty$.

(iii) $\int_0^{+\infty} \theta m_X'(x)/x - \theta m_X(x)\mathrm{d}x = +\infty$.

Denote by $\widehat{X}$ and $\widehat{Y}$, two r.v.s with p.d.f.s

$$\widehat{f}(t) = \frac{tf(t)}{E(X)} \text{ and } \widehat{g}(t) = \frac{tg(t)}{E(Y)}, \tag{13}$$

where $t > 0$ and the expectations are assumed to exist and they are finite. The r.v. $\widehat{X}$ (resp. $\widehat{Y}$) is said to be the length-biased version of X (resp. Y). We will denote by $\widehat{F}$ and $\widehat{G}$ the c.d.f.s of $\widehat{X}$ and $\widehat{Y}$, respectively. The r.h.r. functions of X and Y are given by $\widetilde{h}_X(t) = f(t)/F(t)$ and $\widetilde{h}_Y(t) = g(t)/G(t)$. The m.d.t. function appears in the expression of c.d.f. of the length-biased distribution and also it appears in the expression of reversed hazard rate (r.h.r.) function of the length-biased distribution. Therefore, the PMDT model can be characterized by the relationship between the ratios of c.d.f.s of the length-biased distributions and the ratio of the c.d.f.s of the underlying distributions. The model can also be characterized via the relationship between the ratios of the r.h.r. functions of the length-biased distributions and also the r.h.r. functions of the underlying distributions.

The concept of length-biased distribution as a typical weighted distribution has been very useful in survival analysis, etiologic studies, and marketing research (see, e.g., Wang [18], Simon [19], and Nowell and Stanley [20]). Length-biased sampling arises when a component which is already in use is sampled at a fixed time and then allowed to fail (see Scheaffer [21]).

Proposition 1. *The random lifetimes X and Y satisfy the PMDT model if, and only if, (i) or (ii) below holds:*

(i) $\widetilde{h}_{\widehat{X}}(t)/\widetilde{h}_{\widehat{Y}}(t) = \theta(\widetilde{h}_X(t)/\widetilde{h}_Y(t))$, *for all* $t > 0$.

(ii) $\widehat{F}(t)/\widehat{G}(t) = F(t)/G(t)$ *for all* $t > 0$.

Proof. First, we prove the assertion (i). By Equations (2.5) in Sunoj and Maya [16], for all $t > 0$,

$$\begin{aligned} \widetilde{h}_{\widehat{X}}(t) &= \frac{t\widetilde{h}_X(t)}{m_X(t)}, \text{ and} \\ \widetilde{h}_{\widehat{Y}}(t) &= \frac{t\widetilde{h}_Y(t)}{m_Y(t)}. \end{aligned} \tag{14}$$

Therefore, it is straightforward that $m_Y(t) = \theta m_X(t)$, for all $t > 0$, if, and only if, $\widetilde{h}_{\widehat{X}}(t)/\widetilde{h}_{\widehat{Y}}(t) = \theta\widetilde{h}_X(t)/\widetilde{h}_Y(t)$, for all $t > 0$ which is equivalent to (i). By applying the identities in Equations (2.4) in Sunoj and Maya [16], for all $t > 0$ we have

$$\begin{aligned} \widehat{F}(t) &= \frac{m_X(t)}{m_X(+\infty)} F(t), \text{and} \\ \widehat{G}(t) &= \frac{m_Y(t)}{m_Y(+\infty)} G(t). \end{aligned} \tag{15}$$

From (10), it follows that $m_Y(t) = \theta m_X(t)$, for all $t \geq 0$, if, and only if,

$$\frac{\widehat{G}(t)}{G(t)} = \frac{m_Y(t)}{m_Y(+\infty)} = \frac{m_X(t)}{m_X(+\infty)} = \frac{\widehat{F}(t)}{F(t)}. \tag{16}$$

For all $t \geq 0$. This is also equivalent to (ii) and hence the proof is completed.

In survival analysis, the length-biased samples frequently come up where random sampling from X and Y with respective target distributions F and G cannot be conducted. In such situations, the available data follow the associated length-biased distributions $\widehat{F}$ and $\widehat{G}$. Proposition 1 illustrates that in the setup of the PMDT model, the ratio of c.d.f.s G and F (or the ratio of the r.h.r. functions $\widetilde{h}_Y$ and $\widetilde{h}_X$) of the original distributions could be estimated under length-biased sampling as well as that estimated under random sampling. If the objective is the estimation of G using length-biased samples on Y and F is either known or predetermined, then Proposition 1 is again useful to present an estimator of G.

We present an example to introduce two distributions fulfilling the PMDT model. □

Example 1. Suppose that T is a non-negative r.v. with a differentiable c.d.f. K which is independent of X and Y such that $K(0) = 0$. Let us assume that X and Y are two non-negative r.v.s such that $F(0) = G(0) = 0$ with finite means and respective r.h.r. functions

$$\widetilde{h}_X(t) = \frac{K'(t)}{t/E[Y] - K(t)}, \text{and}$$
$$\widetilde{h}_X(t) = \frac{K'(t)}{t/E[Y] - K(t)}, \tag{17}$$

where $K(t) < \min\{1/E[X], 1/E[Y]\}t$, for all $t > 0$. Then, the r.v.s $T_X = \max\{X, T\}$ and $T_Y = \max\{Y, T\}$ have length-biased distributions associated with the underlying distributions of X and Y, respectively. It is plain to show that T_X and T_Y have c.d.f.s $\widehat{F}(t) = F(t)K(t)$ and $\widehat{G}(t) = G(t)K(t)$, respectively. Hence, $\widehat{F}(t)/F(t) = \widehat{G}(t)/G(t) = K(t)$, for all $t \geq 0$. By Proposition 1 (ii) it thus follows that X and Y satisfy the PMDT model. For example, let T be an exponential r.v. with c.d.f. $K(t) = 1 - \exp(-\lambda t)$ and let X and Y satisfy $E(X) < 1/\lambda$ and $E(Y) < 1/\lambda$. Then, the denominators in the r.h.r. functions given in (17) are positive and X and Y satisfy the PMDT model.

The following result determines that a limiting property of the r.h.r. functions of two r.v.s with the PMDT model is a characteristic for equality in distribution of the two r.v.s.

Theorem 1. *Let X and Y having finite means and r.h.r. functions $\widetilde{h}_X$ and $\widetilde{h}_Y$, respectively, and denote* $\lim_{t\longrightarrow 0^+} t\widetilde{h}_X(t) = l_1$, $\lim_{t\longrightarrow 0^+} t\widetilde{h}_Y(t) = l_2$, $\lim_{t\longrightarrow +\infty} t\widetilde{h}_X(t) = l_3$, *and* $\lim_{t\longrightarrow +\infty} t\widetilde{h}_Y(t) = l_4$.

(i) If $l_1 = l_2$ then, X and Y have PMDT functions if, and only if, they are equal in distribution.

(ii) If $l_3 = l_4$ then, X and Y have PMDT functions if, and only if, they are equal in distribution.

(iii) If $l_1(l_2+1)/l_2(l_1+1) \neq l_3(l_4+1)/l_4(l_3+1)$ then, X and Y do not satisfy the PMDT model.

Proof. We only prove the assertion (i) and the assertion (iii). The assertion (ii) can be proved similarly as done for assertion (i). From (9), $\theta = m_Y(t)/m_X(t)$, for all $t > 0$, which is independent of t. Denote by M_X and M_Y, the m.i.t. functions of X and Y, respectively. One has

$$\begin{aligned}\theta &= \lim_{t\longrightarrow 0^+} \frac{m_Y(t)}{m_X(t)} = \lim_{t\longrightarrow 0^+} \frac{t - M_Y(t)}{t - M_X(t)} \\ &= \frac{1 - \lim_{t\longrightarrow 0^+}\{M_Y(t)/t\}}{1 - \lim_{t\longrightarrow 0^+}\{M_X(t)/t\}}.\end{aligned} \tag{18}$$

Using L'Hopital's rule,

$$\begin{aligned}\lim_{t\longrightarrow 0^+} \frac{M_X(t)}{t} &= \lim_{t\longrightarrow 0^+} \frac{\int_0^t F(x)\mathrm{d}x}{tF(t)} \\ &= \lim_{t\longrightarrow 0^+} \frac{F(t)}{tf(t) + F(t)} = \frac{1}{\lim_{t\longrightarrow 0^+} t\widetilde{h}_X(t) + 1}.\end{aligned} \tag{19}$$

By a similar method, we also have

$$\lim_{t\longrightarrow 0^+} \frac{M_Y(t)}{t} = \frac{1}{\lim_{t\longrightarrow 0^+} t\widetilde{h}_Y(t) + 1}. \tag{20}$$

We conclude that $\theta = 1$ and this means that $m_Y(t) = m_X(t)$, for all $t > 0$, and since the m.d.t. function uniquely determines the distribution it follows that $F(t) = G(t)$, for all $t > 0$. It can also be shown under the condition of assertion (iii) that there is no value of θ to fulfill the PMDT model.

Remark 1. The results of Theorem 1 (i) and Theorem 1 (ii) show that if either the right limits of length-biased r.h.r. functions given by $t\widetilde{h}_X(t)$ *and* $t\widetilde{h}_Y(t)$ at the point 0 or the limits of $t\widetilde{h}_X(t)$ and $t\widetilde{h}_Y(t)$ *at* $+\infty$ do not depend on F and G, respectively, then the PMDT model is not a meaningful model. For instance, if X has an exponential distribution with mean $1/\lambda$ and also Y has an exponential distribution with mean $1/\eta$ then by the L'Hopital's rule,

$$l_1 = \lim_{t\longrightarrow 0^+} t\widetilde{h}_X(t) = \lim_{t\longrightarrow 0^+} \frac{\lambda t}{e^{\lambda t} - 1} = 1, \tag{21}$$

and analogously,

$$l_2 = \lim_{t\longrightarrow 0^+} t\widetilde{h}_Y(t) = \lim_{t\longrightarrow 0^+} \frac{\eta t}{e^{\eta t} - 1} = 1. \tag{22}$$

We can also observe that

$$l_3 = \lim_{t \longrightarrow +\infty} t\widetilde{h}_X(t) = \lim_{t \longrightarrow +\infty} \frac{\lambda t}{e^{\lambda t} - 1} = 0, \text{ and}$$
$$l_4 = \lim_{t \longrightarrow +\infty} t\widetilde{h}_Y(t) = \lim_{t \longrightarrow +\infty} \frac{\eta t}{e^{\eta t} - 1} = 0. \tag{23}$$

Therefore, $l_1 = l_2$ and also $l_3 = l_4$ thus according to the result of Theorem 1 (i) and Theorem 1 (ii), if X and Y satisfy the PMDT model then X and Y are equal in distribution, i.e., $\lambda = \eta$.

3. Closure Properties With Respect to Some Reliability Classes and Stochastic Orders

In this section, sufficient conditions to get the closure property of the PMDT model with respect to the reversed hazard rate (mean inactivity time) order and also sufficient conditions to establish the closure property of the model with respect to four reliability classes related to the inactivity time will be presented. Closure properties of models in reliability and survival analysis have attracted the attention of many researchers in the recent past decades (see, e.g., Crescenzo [22], Abouammoh and Qamber [23], Nanda et al. [24], Nanda and Das [25], Kayid et al. [26], and Jarrahiferiz et al. [27] among others).

Below the definition of three stochastic orders are given (see Shaked and Shanthikumar [28] and Ahmad and Kayid [29]). We will use the convention $a/0 = +\infty$ for $a > 0$ when a statement on the monotonicity of a ratio is made.

Definition 2. Suppose that X and Y are two non-negative r.v.s with absolutely continuous c.d.f.s F and G, p.d.f.s f and g, r.h.r. functions $\widetilde{h}_X$ and $\widetilde{h}_Y$, and m.i.t. functions M_X and M_Y, respectively. Then, it is said that X is smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) whenever $g(t)/f(t)$ is increasing in $t \geq 0$.

(ii) reversed hazard rate order (denoted by $X \leq_{rhr} Y$) whenever $\widetilde{h}_X(t) \leq \widetilde{h}_Y(t)$, for all $t > 0$, or equivalently if, $G(t)/F(t)$ is increasing in $t > 0$.

(iii) mean inactivity time order (denoted by $X \leq_{mit} Y$) whenever $M_X(t) \geq M_Y(t)$, for all $t \geq$, or equivalently if $\int_0^t G(x)\mathrm{d}x / \int_0^t F(x)\mathrm{d}x$ is increasing in $t > 0$.

It has been proved that $X \leq_{lr} Y \Rightarrow X \leq_{rhr} Y \Rightarrow X \leq_{mit} Y$. The following result establishes a necessary and sufficient condition on the parameter θ in the model (9) to get X is smaller than Y in terms of the r.h.r. (or the m.i.t.) order.

Theorem 2. *Let X and Y satisfy the PMDT model. $X \leq_{rhr} (\leq_{mit}) Y$ if, and only if, $\theta \geq 1$.*

Proof. It is enough to show that $\theta \geq 1$ implies $X \leq_{rhr} Y$, and also that $X \leq_{mit} Y$ yields $\theta \geq 1$. From (4)

$$\begin{aligned}\widetilde{h}_Y(t) - \widetilde{h}_X(t) &= \frac{m_Y'(t)}{t - m_Y(t)} - \frac{m_X'(t)}{t - m_X(t)}\\ &= \frac{\theta m_X'(t)}{t - \theta m_X(t)} - \frac{m_X'(t)}{t - m_X(t)}, \quad \text{for all}\, t > 0,\end{aligned} \tag{24}$$

which is non-negative for $\theta \geq 1$ since $m_X'(t) > 0$, for all $t > 0$ because $m_X(t) = E(X|X \leq t)$ is increasing in $t \geq 0$ and also it is readily proved that $\theta m_X'(t)/(t - \theta m_X(t))$ is increasing in θ, and thus $\theta m_X'(t)/(t - \theta m_X(t)) \geq m_X'(t)/(t - m_X(t))$, for any $t \geq 0$. This demonstrates that $\theta \geq 1$ gives $\widetilde{h}_Y(t) \geq \widetilde{h}_X(t)$, for all $t > 0$, i.e., $X \leq_{rhr} Y$. It remains to prove that $X \leq_{mit} Y$ provides that $\theta \geq 1$. From (5), we can get

$$\begin{aligned}M_Y(t) &= t - m_Y(t),\\ &= t - \theta m_X(t),\\ &= t - \theta(t - M_X(t)),\\ &= (1 - \theta) + \theta M_X(t), \quad \text{for all}\, t > 0.\end{aligned} \tag{25}$$

Thus, if $M_X(t) \geq M_Y(t)$, for all $t > 0$, then $\theta = (t - M_Y(t))/(t - M_X(t)) \geq 1$, for all $t > 0$, which completes the proof.

The following example shows that the reversed hazard rate order in Theorem 2 cannot be replaced by the likelihood ratio order which is stronger than the reversed hazard rate order and, therefore, it is stronger than the mean inactivity time order. It is said that a non-negative r.v. T has Lomax distribution with parameters $\alpha > 0$ and $\lambda > 0$ whenever it has survival function (s.f.) $\overline{H}(t) = 1/(1 + (t/\lambda))^{\alpha}$ and we write $T \sim L(\alpha, \lambda)$.

Example 2. Let $X \sim L(2, 1)$ and also let Y have c.d.f.

$$G_\theta(t) = \left(\frac{(t + 2)^{1-(\theta/2)} t^{\theta/2}}{t + 2 - \theta}\right)^{2/(2-\theta)}, \quad t \geq 0,\, 0 < \theta < 2. \tag{26}$$

It can be seen that θ is in fact a departure index parameter, i.e., $m_Y(t) = \theta m_X(t)$, for all $t \geq 0$. Denoted by g_θ, the p.d.f. of Y, and also note that $f(t) = g_1(t)$ is the p.d.f. of X. We can observe that

$$\frac{g_{3/2}(t)}{g_1(t)} = \frac{48t^2(1 + t)^3}{(1 + 2t)^5}, \tag{27}$$

is not increasing in t because $g_{3/2}(0)/g_1(0) = 0$, $g_{3/2}(1)/g_1(1) = 128/81$ and $g_{3/2}(t)/g_1(t) = 3/2$ when $t \longrightarrow +\infty$, that is, $g_{3/2}(0)/g_1(0) < g_{3/2}(1)/g_1(1) > g_{3/2}(+\infty)/g_1(+\infty)$. Therefore, $X \nleq_{lr} Y$ but according to Theorem 2, since $\theta = 3/2$ thus $X \leq_{rhr} (\leq_{mit}) Y$. This means that the result of Theorem 2 cannot be strengthened to the case when likelihood ratio order is used instead for stochastic comparison of X and Y.

In Section 4, in Theorem 7, we present sufficient conditions under which it is concluded that $X \leq_{lr} Y$. Below, the definitions of some reliability classes of lifetime distributions are given.

Definition 3. Suppose that X is a non-negative r.v. with absolutely continuous c.d.f. F, the r.h.r. function $\tilde{h}_X$ and the m.i.t. function M_X. Then, it is said that X has

(i) a decreasing reversed hazard rate (denoted as $X \in$ DRHR) distribution when $\tilde{h}_X(t)$ is decreasing in $t > 0$ (see Ahmad and Kayid [29]).

(ii) an increasing mean inactivity time (denoted as $X \in$ IMIT) distribution when $M_X(t)$ is increasing in $t > 0$ (see Kayid and Ahmad [2]).

(iii) a decreasing proportional reversed hazard rate (denoted as $X \in$ DPRHR) distribution when $\Pr h_X(t) = t\tilde{h}_X(t)$ is decreasing in $t > 0$ (see Oliveira and Torrado [30]).

(iv) a strong mean inactivity time (denoted as $X \in$ SIMIT) distribution when $M_X(t)/t$ is increasing in $t > 0$ (see Kayid and Izadkhah [7]).

The foregoing reliability classes are connected as follows:

$$\begin{array}{ccc} X \in \text{DPRHR} & \Rightarrow & X \in \text{SIMIT} \\ \Downarrow & & \Downarrow \\ X \in \text{DRHR} & \Rightarrow & X \in \text{IMIT} \end{array}. \tag{28}$$

According to the PMDT model, we assume that $m_Y(t) = \theta m_X(t)$, for all $t > 0$ in which θ is a positive parameter (see Equation (8)). We investigate whether the reliability properties of the DPRHR, the SIMIT, the DRHR, and the IMIT of X are inherited by the same reliability properties of Y and vice versa. The next technical lemma is useful to establish closure properties with respect to the foregoing reliability classes.

Lemma 1. *Let $X \in$ SIMIT such that $0 < \theta < 1$ (resp. $\theta > 1$) and assume that m_X is differentiable. Then the function γ defined by*

$$\gamma(t,\theta) = \frac{\theta\left(t - m_X(t)\right)}{t - \theta m_X(t)}, \tag{29}$$

is an increasing (resp. a decreasing) non-negative function in $t > 0$.

Proof. We want to prove that $\partial/\partial t\gamma(t,\theta) \geq (\leq) 0$, for all $t > 0$ and for any $\theta \in (0,1)$ $(\theta \in (1,\infty))$, we can get

$$\begin{aligned} \frac{\partial}{\partial t}\gamma(t,\theta) &\overset{\text{sgn}}{=} \theta\left(1 - m_X'(t)\right)\left(t - \theta m_X(t)\right) \\ &\quad - \theta\left(1 - \theta m_X'(t)\right)\left(t - m_X(t)\right), \\ &\overset{\text{sgn}}{=} m_X(t) + \theta t m_X'(t) - t m_X'(t) - \theta m_X(t), \\ &= \left(m_X(t) - t m_X'(t)\right) - \theta\left(m_X(t) - t m_X'(t)\right), \\ &= (1-\theta)\left(m_X(t) - t m_X'(t)\right), \\ &\overset{\text{sgn}}{=} (1-\theta)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{t}{m_X(t)}\right), \end{aligned} \tag{30}$$

where $a \overset{\text{sgn}}{=} b$ means that a and b have the same sign. On the other hand, when $M_X(t)$ is the m.i.t. function of X, one has

$$\frac{t}{m_X(t)} = \frac{1}{1 - \left(M_X(t)/t\right)}, \quad \text{for all } t > 0, \tag{31}$$

and thus X is SIMIT if, and only if, $t/m_X(t)$ is increasing in $t > 0$, which holds if, and only if, $\mathrm{d}/\mathrm{d}t\,(t/m_X(t)) \geq 0$, for all $t > 0$. Hence, if X is SIMIT and $\theta < 1$, then $\gamma(t,\theta)$ is increasing in $t > 0$ and if X is SIMIT and $\theta > 1$, then $\gamma(t,\theta)$ is decreasing in $t > 0$. The proof now is completed.

The class of SIMIT distributions includes many standard distributions, for example, if $X \sim U(0,1)$, $X \sim \text{Beta}(2,2)$, or $X \sim \text{Beta}(1,3)$ then $M_X(t)/t = 1/2$, $M_X(t)/t = (2-t)/(6-4t)$, and $M_X(t)/t = (3-t)/(6-3t)$ which are increasing functions over $t \in (0,1]$ and thus X has a SIMIT distribution.

Theorem 3. *Let X and Y satisfy the PMDT model as given in (9). Then,*

(i) For all $\theta > 0$, $X \in$ SIMIT if, and only if, $Y \in$ SIMIT.

(ii) For any $\theta < 1$, if $X \in$ IMIT, then $Y \in$ IMIT.

(iii) For any $\theta > 1$, if $X \in$ SIMIT and $X \in$ DRHR, then $Y \in$ DRHR.

(iv) For any $\theta > 1$, if $X \in$ DPRHR, then $Y \in$ DPRHR.

Proof. Let us prove the assertion (i). We can see that

$$\begin{aligned} \frac{M_Y(t)}{t} &= \frac{t - E(Y|Y \leq t)}{t}, \\ &= 1 - \frac{m_Y(t)}{t}, \\ &= 1 - \theta\frac{m_X(t)}{t}, \\ &= 1 - \theta\frac{t - M_X(t)}{t}, \\ &= \overline{\theta} + \theta\frac{M_X(t)}{t}, \quad \text{for all } t > 0, \end{aligned} \tag{32}$$

where $\overline{\theta} = 1 - \theta \in (-\infty, 1)$. It can be concluded now that $M_X(t)/t$ is increasing in $t > 0$ or equivalently X has a SIMIT distribution, if, and only if, $M_Y(t)/t$ is increasing in $t > 0$ or equivalently Y has a SIMIT distribution. To prove the assertion (ii), note that from the proof of assertion (i), we have

$$M_Y(t) = \overline{\theta}t + \theta M_X(t), \quad \text{for all } t > 0. \tag{33}$$

Thus, it suffices to prove that $\mathrm{d}/\mathrm{d}tM_X(t) \geq 0$, for all $t > 0$ implies that $\mathrm{d}/\mathrm{d}tM_Y(t) \geq 0$, for all $t > 0$. For $0 < \theta < 1$ observe that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}M_Y(t) &= \overline{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}M_X(t), \\ &\geq \frac{\mathrm{d}}{\mathrm{d}t}M_X(t), \\ &\geq 0, \quad \text{for all } t > 0. \end{aligned} \tag{34}$$

The proof of assertion (ii) is thus complete. To prove the assertion (iii), notice that from (4) the r.h.r. function of X is written as $\tilde{h}_X(t) = m_X'(t)/t - m_X(t)$ and further the r.h.r. function of Y so that Y and X satisfy the PMDT model given in (9), can be written as

$$\begin{aligned}\tilde{h}_Y(t) &= \frac{\theta m_X'(t)}{t-\theta m_X(t)},\\ &= \frac{\theta(t-m_X(t))}{t-\theta m_X(t)}\frac{m_X'(t)}{t-m_X(t)},\\ &= \gamma(t,\theta)\tilde{h}_X(t), \quad \text{for all } t>0.\end{aligned} \tag{35}$$

From Lemma 1 we know that if $X \in$ SIMIT, then for $\theta > 1$, $\gamma(t,\theta)$ is a non-negative decreasing function in $t>0$ and since $X \in$ DRHR thus $\tilde{h}_X(t)$ is also a non-negative decreasing function in $t>0$ and so is $\tilde{h}_Y(t) = \gamma(t,\theta)\tilde{h}_X(t)$. The proof of this assertion is also complied. It remains to prove the last assertion. Let us write

$$\begin{aligned}\Pr h_Y(t) &= \frac{\theta t m_X'(t)}{t-\theta m_X(t)},\\ &= \frac{\theta(t-m_X(t))}{t-\theta m_X(t)}\frac{t m_X'(t)}{t-m_X(t)},\\ &= \gamma(t,\theta)\Pr h_X(t), \quad \text{for all } t>0.\end{aligned} \tag{36}$$

By assumption, $X \in$ DPRHR which further implies that $X \in$ SIMIT and also we have by assumption that $\theta > 1$. Hence, Lemma 1 concludes that $\gamma(t,\theta)$ is a non-negative decreasing function in $t>0$. Since $X \in$ DPRHR thus $\Pr h_X(t)$ is also a non-negative decreasing function in $t>0$ and so is $\Pr h_Y(t) = \gamma(t,\theta)\Pr h_X(t)$. The proof of assertion (iii) is obtained.

Recently, some authors have shown their interest in stochastic comparisons of random lifetimes according to reversed average intensity (r.a.i.) function (see, for instance, Rezaei and Khalaf [31], Kundu and Ghosh [32], and Buono et al. [33]). The r.a.i. function of X is given by $L_X(t) = t\tilde{h}_X(t)/\int_t^{+\infty}\tilde{h}_X(x)\mathrm{d}x$. Below is the definition of r.a.i. stochastic order.

Definition 4. The r.v.s X and Y with respective r.a.i. functions L_X and L_Y satisfy the r.a.i. order (denoted by $X \le_{rai} Y$) whenever $L_X(t) \ge L_Y(t)$, for all $t>0$, or equivalently if, $\int_t^{+\infty}\tilde{h}_Y(x)\mathrm{d}x/\int_t^{+\infty}\tilde{h}_X(x)\mathrm{d}x$ is increasing in $t>0$.

In the next result, we build the r.a.i. order between X and Y which satisfy the PMDT model given in (9) under some sufficient conditions.

Theorem 4. *Let X have a SIMIT distribution and let $0<\theta<1$, then $X \le_{rai} Y$.*

Proof. We can see that $X \le_{rai} Y$, if, and only if, for all $t>0$,

$$\int_t^{+\infty}\left(\tilde{h}_Y(x)\tilde{h}_X(t) - \tilde{h}_Y(t)\tilde{h}_X(x)\right)\mathrm{d}x \ge 0, \tag{37}$$

which holds equivalently if,

$$\int_t^{+\infty}\tilde{h}_X(x)\tilde{h}_X(t)(\gamma(x,\theta)-\gamma(t,\theta))\mathrm{d}x \ge 0, \quad \text{for all } t>0. \tag{38}$$

It is known from Lemma 1 that if X is SIMIT, then for every $0<\theta<1$, $\gamma(x,\theta) \ge \gamma(t,\theta)$, for all $x \ge t > 0$. Thus the inequality in (38) holds true and hence the proof.

Remark 2. The result of Theorem 2 and Theorem 3 can be developed by symmetry to get the desired closure properties in the reversed direction. Note that if the m.d.t. function of Y is proportional to the m.d.t. function of X according to the identity $m_Y(t) = \theta m_X(t)$, $t>0$, $\theta>0$, then the m.d.t. function of X is also proportional to the m.d.t. function of X since $m_X(t) = \theta^* m_Y(t)$, $t>0$ where $\theta^* = 1/\theta$ which is positive. The result of Theorem 7 is thus translated to get

$$Y \le_{rhr}(\le_{mit})X, \quad \text{if and only if}, \ \theta \le 1. \tag{39}$$

The result of Theorem 3 can be used to conclude that.

(ii)′ For any $\theta>1$, if $Y \in$ IMIT, then $X \in$ IMIT.

(iii)′ For any $\theta>1$, if $Y \in$ SIMIT and $Y \in$ DRHR, then $X \in$ DRHR.

(iv)′ For any $\theta>1$, if $Y \in$ DPRHR, then $X \in$ DPRHR. The result we presented in Theorem 4 can also be accompanied with the following implication:

$$Y \le_{rai} X \ \text{when}, Y \ \text{is SIMIT and} \ \theta > 1. \tag{40}$$

The following example illustrates an application of Theorem 4.

Example 3. Suppose that X has c.d.f. $F(t) = t(t+2)/(1+t)^2$, $t \ge 0$. *Let* Y have c.d.f. $G_\theta(t) = ((t+2)^{1-(\theta/2)}t^{\theta/2}/t+2-\theta)^{2/(2-\theta)}$, $t \ge 0$, $0<\theta<2$, $\theta \ne 2$. Note that $F(t) = G_1(t)$. In Example 8, it was shown that $m_Y(t) = \theta m_X(t)$. It can be seen that X has an m.i.t. function $M_X(t) = t(t+1)/t+2$, thus $M_X(t)/t = (t+1)/(t+2)$ is increasing in $t>0$, so X has a SIMIT distribution. Notice that from Theorem 3 (i), since X is SIMIT, thus Y is also SIMIT. It can be observed that $\int_t^{+\infty}\tilde{h}_Y(x)\mathrm{d}x = -\ln(G_\theta(t))$, for all $t>0$ and for any $0<\theta<2$, $\theta \ne 2$. Thus, by Definition 12,$X \le_{rai} Y$ holds whenever, the ratio $\ln(G_\theta(t))/\ln(G_1(t))$ is increasing in $t>0$ and also, by symmetry, $Y \le_{rai} X$ provided that $\ln(G_\theta(t))/\ln(G_1(t))$ is decreasing in $t>0$. Figure 1 presents the plot of $\ln(G_\theta(t))/\ln(G_1(t))$ for values $\theta = 4/3, 3/4, 3/2, 2/3$. It is realized that for $\theta = 2/3$ and $\theta = 3/4$ which are smaller than one, the ratio is increasing which fulfills the result of Theorem 4, i.e., $X \le_{rai} Y$. Furthermore, it is seen in the plot that for $\theta = 3/2$ and $\theta = 4/3$ which are greater than one, the ratio is decreasing, i.e., $Y \le_{rai} X$ and thus the claim given after assertion $(iv)'$ in Remark 2 is validated.

4. The Model with Random Departure Index

In recent past decades, frailty models have been frequently used in survival analysis to handle the influence of the covariates on the lifetime variable (see, e.g., Hougaard [34] and Hanagal [35]). In this section, the PMDT model with

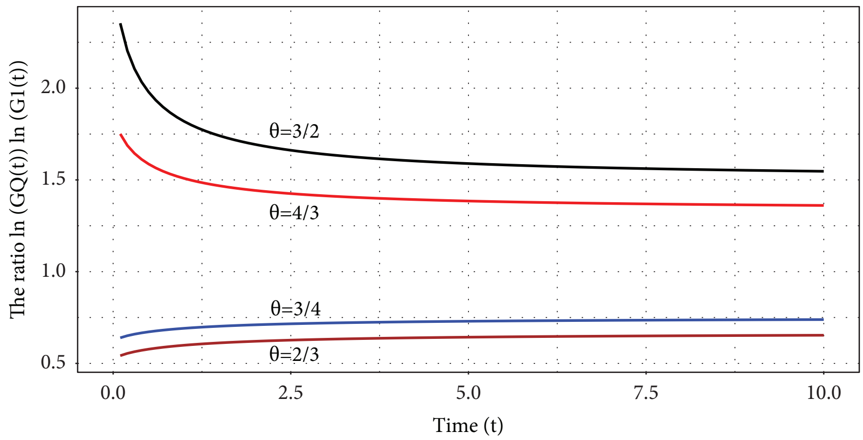

FIGURE 1: The plot of the function $\ln(G_\theta(t))/\ln(G_1(t))$ in Example 3 for $\theta = 4/3, 3/4, 3/2, 2/3$.

random departure index is considered. By introducing the PMDT model with random effects, a novel frailty model is produced where we develop the model given earlier in (9) to the case where θ is an observation of an r.v. Θ. The r.v. Θ is non-negative.

Let us take m_0 as the baseline m.d.t. function and further consider Θ as the random departure index with p.d.f. k and c.d.f. K. The r.v. X with the p.d.f. f, c.d.f. F, and m.d.t. function m_0 is considered as the reference variable. Let us consider a randomly drawn individual from population for which $\Theta = \theta$. Moving forward with this observation, the m.d.t. function is given by $m(t|\theta) = \theta m_0(t)$, for all $t > 0$, according to the PMDT model. In spirit of (12), the c.d.f. of Y with m.d.t. function $m(t|\theta)$ is obtained as

$$G(t|\theta) = \exp\left(-\int_t^{+\infty} \frac{\theta m_0'(x)}{x - \theta m_0(x)} \mathrm{d}x\right). \tag{41}$$

From which the corresponding p.d.f. is derived as

$$g(t|\theta) = \frac{\theta m_0'(t)}{t - \theta m_0(t)} \exp\left(-\int_t^{+\infty} \frac{\theta m_0'(x)}{x - \theta m_0(x)} \mathrm{d}x\right). \tag{42}$$

Given a predetermined distribution function to be a choice for the c.d.f. of X, the family of distributions generated by (41) provides a way to add a parameter to the family of distributions of X. In the context of statistical inference, the arisen model could be examined using real data sets in different scenarios to model lifetime events. For a given value of θ, since $m(t|\theta) = \theta m_0(t)$ is a mean departure time function thus $m(t|\theta) \leq t$, for all $t > 0$, thus $\theta \leq t/m_0(t)$ for all $t \geq 0$. Therefore, $g(t|\theta) = 0$, for all $\theta > t/m_0(t)$ which means that when F is predetermined and, therefore, $t/m_0(t)$ is fixed then for a value θ satisfying $\theta > t/m_0(t)$ the PMDT model does not hold. This consideration may be useful before proceeding to do a statistical inference on the model. For example, prior to fitting the PMDT model to a real data set, we might want to test whether $\theta \leq t/m_0(t)$, for all $t \geq 0$.

To integrate the effect of a random variable Θ which is random, we have to consider the unconditional r.v. Y^* with a mixture distribution according to (13) which has s.f.

$$G^*(t) = \int_0^{+\infty} \exp\left(-\int_t^{+\infty} \frac{\theta m_0'(x)}{x - \theta m_0(x)} \mathrm{d}x\right) k(\theta)\mathrm{d}\theta. \tag{43}$$

With underlying p.d.f.

$$g^*(t) = \int_0^{+\infty} \frac{\theta m_0'(t)}{t - \theta m_0(t)} \exp\left(-\int_t^{+\infty} \frac{\theta m_0'(x)}{x - \theta m_0(x)} \mathrm{d}x\right) k(\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{44}$$

Remark 3. We may notice that in the PMDT model, in the fixed level of the departure index parameter, the values of θ and the c.d.f. F cannot be independently determined. The identity $m(t|\theta) = \theta m_0(t)$ is meaningful whenever $\theta \leq t/m_0(t)$, for all $t > 0$. It is straightforward that $t/m_0(t)$ is a functional of F and can be written as $\theta_t(F) = t/m_0(t)$. For instance, when $\theta = 3$ and F satisfying the inequality $\theta_t(F) < 3$ for some choices of t in the support of F, it is concluded that the PMDT model does not hold. In spite of that, when $\theta \in (0, 1]$ the selection of F does not depend on the choice of θ. This is because the inequality $\theta \leq 1 \leq \theta_t(F)$, holds for all $t \geq 0$ and for all lifetime distributions F with m.d.t. function m_0 without any further consideration.

The random pair (Y^*, Θ) is assumed to follow the joint p.d.f $g(y, \theta)$ and joint c.d.f. $G(y, \theta)$. In the case θ is a realization of the r.v. Θ, the m.d.t. function may be written as

$$\begin{aligned} m(t|\theta) &= E(Y^*|Y^* > t, \Theta = \theta), \\ &= \int_0^t \frac{y g(y, \theta)}{\partial G(t, \theta)/\partial \theta} \mathrm{d}y, \\ &= \int_0^t \frac{y g(y|\theta)}{G(t|\theta)} \mathrm{d}y, \\ &= t - \int_0^t \frac{G(y|\xi)}{G(t|\theta)} \mathrm{d}y, \end{aligned} \tag{45}$$

where $g(y|\theta)$ and $G(y|\theta)$ are the conditional p.d.f. and the conditional c.d.f. of Y given $\Theta = \theta$, respectively, as given in (42) and (41). If we denote by T an r.v. with conditional p.d.f. $g(t|\theta)$ for a realization θ of Θ, then, according to (45) one has $m(t|\theta) = m_T(t) = t - M_T(t)$ where m_T and M_T are, respectively, the m.d.t. function and m.i.t. function of T. Therefore, in view of (43), Y^* follows a mixture of distributions with proportional mean departure time functions with respect to the mixing distribution $K(\theta)$ for the random departure index Θ. In fact, when we are uncertain about the amount of θ in the model, we consider it to be in a dynamical state. Frequencies of values of θ in different intervals construct an empirical probability distribution for θ which converges to K as the observations on θ increase. The c.d.f. (43) takes an average of distributions, having proportional mean departure time functions, with fixed level of the departure index parameter θ with respect to the c.d.f. K as a mixing distribution.In a dynamic population, the departure index parameter varies from one individual to another and that is the value of Θ. Let us assume that an individual enters our investigation randomly. The amount of θ for this randomly chosen subject is considered to be a realization of Θ. The lifetime of this individual then follows the c.d.f. (43).

The amount the function $G(t|\theta) = P(Y^* \leq t|\Theta = \theta)$ takes is the probability for failure before time t for an individual with fixed departure index θ. In statistical Bayesian analysis, an inference strategy is done using the conditional likelihood function of an unknown parameter given data which follow a mixture model. Here, we present the density function of Θ given a single observation on Y^*. Specifically, given that $Y^* = t$, the density of Θ is obtained as

$$\begin{aligned} k(t|\theta) &= \frac{g(t|\theta)k(\theta)}{\int_0^\infty g(t|\theta)k(\theta)\mathrm{d}\theta}, \\ &= \frac{\theta m_0'(t)/t - \theta m_0(t)\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta m_0(x)\mathrm{d}x\right)k(\theta)}{\int_0^\infty \theta m_0'(t)/t - \theta m_0(t)\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta m_0(x)\mathrm{d}x\right)k(\theta)\mathrm{d}\theta}. \end{aligned} \tag{46}$$

The density function of Θ among individuals whose lives lost prior to time t is

$$\begin{aligned} k(\theta|Y^* \leq t) &= \frac{G(t|\theta)k(\theta)}{\int_0^{+\infty} G(t|\theta)k(\theta)\mathrm{d}\theta}, \\ &= \frac{k(\theta)\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta m_0(x)\mathrm{d}x\right)}{\int_0^\infty k(\theta)\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta m_0(x)\mathrm{d}x\right)\mathrm{d}\theta}. \end{aligned} \tag{47}$$

In applied probability, there are always methods to infer on population without data. By means of concepts arisen by theory of stochastic orderings we can partially infer on θ conditional on the events $Y^* \leq t$ and $Y^* = t$. The shapes the density function of $\Theta|Y^* = t$ and density function of $\Theta|Y^* = t$ have are complicated. Therefore, in the Bayesian setting, the likelihood equation to derive the maximum likelihood estimations of θ probably fails. For this reason, it may be more appropriate to investigate some stochastic ordering properties in terms of the posterior distribution of Θ among individuals with a certain departure time t and also the posterior distribution of Θ among the individuals whose lives lost before time t. The likelihood ratio order is utilized here to build a stochastic ordering property. The likelihood ratio order is stronger than the hazard rate order and that is stronger than the usual stochastic order (cf. Shaked and Shanthikumar [28]).

Theorem 5. *Let Y^* denote an r.v. with c.d.f. (43). Then,*

(i) For any $t > 0$, $(\Theta|Y^* \leq t) \leq_{lr} (\leq_{hr})[\leq_{st}](\Theta|Y^* = t)$.

(ii) For all $t_2 \geq t_1 > 0$, $(\Theta|Y^* \leq t_2) \leq_{lr} (\leq_{hr})[\leq_{st}]$ $(\Theta|Y^* \leq t_1)$.

Proof. To prove the assertion (i), we have to demonstrate that $g(\Theta|Y^* \leq t)/g(\Theta|Y^* = t)$ is decreasing in $\theta > 0$, for all $t > 0$. By (46) and (47), we get

$$\begin{aligned} \frac{k(\theta|Y^* \leq t)}{k(\theta|t)} &= \frac{G(t|\theta)\int_0^\infty g(t|\theta)k(\theta)\mathrm{d}\theta}{g(t|\theta)\int_0^{+\infty} G(t|\theta)k(\theta)\mathrm{d}\theta}, \\ &= \frac{(t/\theta - m_0(t))\int_0^\infty (\theta k(\theta)/(t - \theta m_0(t)))\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta m_0(x)\mathrm{d}x\right)\mathrm{d}\theta}{\int_0^\infty k(\theta)\exp\left(-\int_t^{+\infty}\theta m_0'(x)/x - \theta v_0(x)\mathrm{d}x\right)\mathrm{d}\theta}, \end{aligned} \tag{48}$$

which is decreasing in $\theta > 0$, for all $t > 0$. To prove the assertion (ii), it suffices to establish that $g(\theta|Y^* \leq t)$ is RR_2 in $(\theta, t) \in \mathbb{R}^+ \times \mathbb{R}^+$. By Theorem 7.1 in Holland and Wang [36], we get the desired property if we show that $(\partial^2)/(\partial\theta\partial t)\ln(k(\theta|Y^* \leq t)) \leq 0$, for all $\theta > 0$ and for all $t > 0$. In the spirit of (19), we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t\,\partial\theta}\ln\left(\frac{G(t|\theta)k(\theta)}{\int_0^{+\infty}G(t|\theta)k(\theta)\mathrm{d}\theta}\right) &= \frac{\partial}{\partial\theta}\frac{\partial}{\partial t}\left(\ln\left(\frac{k(\theta)}{J(t)}\right)-\int_t^{+\infty}\frac{\theta m_0'(x)}{x-\theta m_0(x)}\mathrm{d}x\right),\\ &= \frac{\partial}{\partial\theta}\left(\frac{\theta m_0'(t)}{t-\theta m_0(t)}-\ln(J(t))\right),\\ &= \frac{-tm_0'(t)}{(\theta m_0(t)-t)^2}\leq 0,\quad \text{for all } t>0,\ \theta>0,\end{aligned} \tag{49}$$

in which $J(t)=\int_0^{+\infty}k(\theta)\exp(-\int_t^{+\infty}\theta m_0'(x)/x-\theta m_0(x)\mathrm{d}x)\mathrm{d}\theta$ is the normalizing constant for the conditional density (47) which is free of θ. The proof is complete.

From the identity $m(t|\theta)=\theta m_0(t)$ which holds for all $t>0$, when t approaches $+\infty$ we get $E(Y^*|\Theta=\theta)=\theta E(X)$. Therefore, by the iterated expectation rule, it is deduced that $E(Y^*)=E(\Theta)E(X)$. The next result reveals the relationship the m.d.t. function of Y^* has with the m.d.t. function of X. Let us denote by m_{Y^*} the m.d.t. function of Y^*. Recall that m_0 is the m.d.t. function of X.

Theorem 6. *Let Y^* follow the c.d.f. (43). Then,$m_{Y^*}(t)=E[\Theta G(t|\Theta)]/E[[G(t|\Theta)]m_0(t)$, for all $t>0$.*

Proof. The random variable Y^* follows the p.d.f. (44). The m.d.t. function of Y^* is obtained as

$$\begin{aligned}m_{Y^*}(t) &= \frac{\int_0^t yg^*(y)\mathrm{d}y}{G^*(t)},\\ &= \frac{\int_0^t yE[g(y|\Theta)]\mathrm{d}y}{E[G(t|\Theta)]},\\ &= \frac{E\left[\int_0^t yg(y|\Theta)\mathrm{d}y\right]}{E[G(t|\Theta)]},\\ &= \frac{E[m(t|\Theta)G(t|\Theta)]}{E[G(t|\Theta)]},\\ &= \frac{E[\Theta G(t|\Theta)]}{E[G(t|\Theta)]}m_0(t),\quad \text{for all } t>0.\end{aligned} \tag{50}$$

Hence the proof is completed.

By an application of the relationship revealed in Theorem 6, the closure property of the PMDT frailty model with respect to the mean inactivity time order is obtained. Note that by dividing (42) to (41), the r.h.r. function of T is delivered as

$$\tilde{h}(t|\theta)=\frac{g(t|\theta)}{G(t|\theta)}=\frac{\theta m_0'(t)}{(t-\theta m_0(t))}, \tag{51}$$

is increasing in θ, for all $t>0$. This further implies that

$$(Y^*|\Theta=\theta_1)\leq_{rhr}(Y^*|\Theta=\theta_2),\quad \text{for all}\quad \theta_1\leq\theta_2, \tag{52}$$

which concludes that $(Y^*|\Theta=\theta_1)\leq_{st}(Y^*|\Theta=\theta_2)$, for all $\theta_1\leq\theta_2$. That is,

$$G(t|\theta)\quad \text{is decreasing in } \theta>0,\quad \text{for all}\quad t>0. \tag{53}$$

By negative association concept, $\mathrm{Cov}(\Theta,G(t|\Theta))\leq 0$, for all $t>0$. From Theorem 6 we deduce

$$\begin{aligned}\frac{m_{Y^*}(t)}{m_0(t)} &= \frac{E[\Theta G(t|\Theta)]}{E[G(t|\Theta)]},\\ &= \frac{\mathrm{Cov}(\Theta,G(t|\Theta))+E(\Theta)E[G(t|\Theta)]}{E[G(t|\Theta)]},\\ &\leq E(\Theta),\quad \text{for all } t>0.\end{aligned} \tag{54}$$

Thus if $E(\Theta)\leq 1$, then $m_{Y^*}(t)\leq m_0(t)$, for all $t>0$, or equivalently, $M_{Y^*}(t)\geq M_X(t)$, for all $t\geq 0$, i.e., $Y^*\leq_{mit}X$. This result can be compared with the result of Theorem 2 in Section 3. In Theorem 7 (ii) another conclusion is achieved in this direction. The residual part of this article also deals with stochastic comparisons between X and Y^* in terms of stronger stochastic orders than the mean inactivity time order for which stronger assumptions than $E(\Theta)\leq(\geq)1$ need to be imposed.

We present some sufficient (and/or necessary) conditions in order to terminate that X is smaller than Y^* in terms of the likelihood ratio order and also that X is smaller than Y^* in terms of the reversed hazard rate order.

Theorem 7. *Let Y^* have p.d.f. (14) and c.d.f. (13). Suppose that m_0 is twice differentiable. Then,*

(i) If $P(\Theta\geq 1)=1$, then $X\leq_{lr}Y^*$ if, and only if, $m_0(t)/\sqrt{t}$ increases in $t>0$.

(ii) If $P(\Theta\geq 1)=1$, then $X\leq_{rhr}Y^*$ and thus $X\leq_{st}Y^*$ and $X\leq_{mit}Y^*$.

Proof. (i) The r.h.r. of Y^* given $\Theta=\theta$ is $\tilde{h}(t|\theta)=\theta m_0'(t)/(t-\theta m_0(t))$. From (44), since $P(\Theta\geq 1)=1$, thus $P(\Theta<1)=\int_0^1 k(\theta)\mathrm{d}\theta=0$. We know that $k(\theta)\geq 0$, for all $\theta<1$. Therefore, as a result in real analysis, $k(\theta)=0$ for all $\theta<1$ thus Y^* has p.d.f.

$$g^*(t)=\int_1^{+\infty}\tilde{h}(t|\theta)e^{-\int_t^{+\infty}\tilde{h}(x|\theta)\mathrm{d}x}k(\theta)\mathrm{d}\theta. \tag{55}$$

We also notice that X has p.d.f.

$$\begin{aligned}f(t) &= \int_1^{+\infty}\tilde{h}(t|1)e^{-\int_t^{+\infty}\tilde{h}(x|1)\mathrm{d}x}k(\theta)\mathrm{d}\theta\\ &= \tilde{h}(t|1)e^{-\int_t^{+\infty}\tilde{h}(x|1)\mathrm{d}x}.\end{aligned} \tag{56}$$

As a result, $g^*(t)/f(t) = E[\psi(t,\Theta)]$ where

$$\psi(t,\theta) = \frac{\tilde{h}(t|\theta)}{\tilde{h}(t|1)} e^{\int_t^{+\infty} (\tilde{h}(x|1) - \tilde{h}(x|\theta))\mathrm{d}x}. \quad (57)$$

It can be observed that $\psi(t,\theta)$ is non-decreasing in $t > 0$ if, and only if,

$$\frac{\tilde{h}'(t|\theta)}{\tilde{h}(t|\theta)} + \tilde{h}(t|\theta) \geq \frac{\tilde{h}'(t|1)}{\tilde{h}(t|1)} + \tilde{h}(t|1), \quad \text{for all } t > 0 \text{ and for all } \theta > 1. \quad (58)$$

We can write

$$\begin{aligned}\frac{\tilde{h}'(t|\theta)}{\tilde{h}(t|\theta)} + \tilde{h}(t|\theta) &= \frac{m_0''(t)}{m_0'(t)} - \frac{1-\theta m_0'(t)}{t-\theta m_0(t)} + \frac{\theta m_0'(t)}{t-\theta m_0(t)}, \\ &= \frac{m_0''(t)}{m_0'(t)} + \frac{2\theta m_0'(t) - 1}{t - \theta m_0(t)}.\end{aligned} \quad (59)$$

Now, we can verify that

$$\begin{aligned}&\left\{\frac{\tilde{h}'(t|\theta)}{\tilde{h}t|\theta} + \tilde{h}(t|\theta)\right\} - \left\{\frac{\tilde{h}'(t|1)}{\tilde{h}(t|1)} + \tilde{h}(t|1)\right\} \\ &= \frac{(\theta-1)\left(2t\sqrt{t}\ (\mathrm{d}/\mathrm{d}t) m_0(t)/\sqrt{t}\right)}{(t-\theta m_0(t))(t-m_0(t))}.\end{aligned} \quad (60)$$

Therefore, if $M(t|\theta)$ is the m.i.t. function of T with p.d.f. $g(t|\theta)$ as given in (42) then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{g^*(t)}{f(t)} = 2t\sqrt{t}\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0(t)}{\sqrt{t}} \int_1^{+\infty} \frac{(\theta-1)\psi(t,\theta)}{M(t|\theta)M(t|1)} k(\theta)\mathrm{d}\theta, \quad (61)$$

which is non-negative if, and only if, $m_0(t)/\sqrt{t}$ is non-decreasing in $t > 0$. This completes the proof of (i). Let us prove the assertion (ii) now. We obtain the r.h.r. function of Y^* as

$$\begin{aligned}\tilde{h}_{Y^*}(t) &= \frac{\int_0^{+\infty} \tilde{h}(t|\theta) e^{-\int_t^{+\infty}\tilde{h}(x|\theta)\mathrm{d}x} k(\theta)\mathrm{d}\theta}{\int_0^{+\infty} e^{-\int_t^{+\infty}\tilde{h}(x|\theta)\mathrm{d}x} k(\theta)\mathrm{d}\theta}, \\ &= \int_0^{+\infty} \tilde{h}(t|\theta) k^*(\theta|t)\mathrm{d}\theta, \\ &= E\left[\tilde{h}(t|\Theta^*)\right],\end{aligned} \quad (62)$$

where Θ^* is a non-negative r.v. with p.d.f.

$$k^*(\theta|t) = \frac{e^{-\int_t^{+\infty}\tilde{h}(x|\theta)\mathrm{d}x} k(\theta)}{\int_0^{+\infty} e^{-\int_t^{+\infty}\tilde{h}(x|\theta)\mathrm{d}x} k(\theta)\mathrm{d}\theta} = \frac{G(t|\theta)k(\theta)}{E[G(t|\Theta)]}. \quad (63)$$

We have

$$\frac{\partial^2}{\partial\theta^2}\tilde{h}(t|\theta) = \frac{2t m_0(t) m_0'(t)}{(t - \theta m_0(t))^3} \geq 0, \quad \text{for all } t > 0. \quad (64)$$

Therefore, $\tilde{h}(t|\theta)$ is convex in θ and hence by applying Jensen's inequality we get

$$\tilde{h}_{Y^*}(t) = E\left[\tilde{h}(t|\Theta^*)\right] \geq \tilde{h}(t|E(\Theta^*)), \quad \text{for all } t > 0. \quad (65)$$

From (53) $G(t|\theta)$ is, for all $t > 0$, decreasing in θ thus Θ and $G(t|\theta)$ are negatively associated. Since $P(\Theta \geq 1)$, thus

$$E(\Theta^*) = \frac{E[\Theta G(t|\Theta)]}{E[G(t|\Theta)]} \geq 1. \quad (66)$$

Denote by $\tilde{h}_X$, the r.h.r. function of X. Since $\tilde{h}(t|\theta)$ is increasing in θ for all $t > 0$ and, further, since $\tilde{h}(t|1) = \tilde{h}_X(t)$, thus by (65) we deduce that

$$\tilde{h}_{Y^*}(t) \geq \tilde{h}(t|E(\Theta^*)) \geq h(t|1) \geq \tilde{h}_X(t), \quad \text{for all } t \geq 0. \quad (67)$$

Keep in mind that M_X is the m.i.t. function of the r.v. X with m.d.t. function m_0. Then, $m_0(t)/\sqrt{t} = \sqrt{t} - M_X(t)/\sqrt{t}$ which has to be increasing in the setup of Theorem 7 (i). In Example 2, it thus follows that $m_0(t)/\sqrt{t} = \sqrt{t}/t + 2$ which is not an increasing function. Note that when Θ degenerates at $\theta = 3/2$ then Y^* in Theorem 7 equals in distribution with Y in Theorem 2. Therefore, since $m_0(t)/\sqrt{t}$ is not increasing thus according to Theorem 7 $X \nleq_{lr} Y$ as acknowledged in Example 2 in Section 3.

Remark 4. Stochastic orders considered in literature including the likelihood ratio order ($\leq_{lr}$), the reversed hazard rate order ($\leq_{rhr}$), and the mean inactivity time order ($\leq_{mit}$) are partial orders of distributions. There may be situations where distributions are not ordered. For example, in the context of Theorem 7, the ratio $g^*(t)/f(t)$ may not be monotonically increasing as its graph may exhibit a bathtub (B.T.) shape, an upside bathtub shape (U.B.T.) or even a roller coaster shape. From Equation (65) in Theorem 7, it can be found that when $P(\Theta \geq 1) = 1$, the shape the function $m_0(t)/\sqrt{t}$ has is analogous with the shape the function $g^*(t)/f(t)$ has. Consequently, a necessary and sufficient condition for $g^*(t)/f(t)$ to exhibit a certain behavior when $X \nleq_{lr} Y^*$ is that the function $m_0(t)/\sqrt{t}$ exhibits the same behavior.

5. Conclusion

The article has introduced a novel semi-parametric lifetime model requiring proportional mean departure time functions. The mean departure time function of a random lifetime Z given by $m_Z(t) = E(Z|Z \leq t)$ is closely related to the well-known mean inactivity time function or the mean past lifetime function of Z which is given by $M_Z(t) = E(t - Z|Z \leq t)$. By using limiting techniques together with a knowledge of distribution theory, it was shown that the proportional mean past lifetime model which has been frequently applied in literature is not a suitable model for using to model unbounded lifetime events. The newly introduced model which deals with past events and can be used to analysis events occurred in a reversed scale of time is a proper alternative for the proportional mean past lifetime model. The proposed model is described and a distribution theory on the model was accomplished. Necessary and sufficient conditions for two random variables to satisfy the new model were found. An example was given to illustrate the derived conditions provide a convenient tool to construct two random lifetimes with proportional mean departure time functions. Necessary and sufficient condition to have closure properties of the new model with respect to the reversed hazard rate order and the mean inactivity time order is shown to be the amount of the parameter to be greater or smaller than one. It was shown by a counterexample that the model does not have the closure property with respect to the likelihood ratio order. Closure property with respect to the reversed aging intensity order was built under an additional assumption. Letting $m_Y(t) = \theta m_X(t)$ hold for all $t \geq 0$ and $\theta > 0$ preservation of the reliability classes of IMIT, SIMIT, DPRHR, and DRHR under the transformation $X \longrightarrow Y$ and also $Y \longrightarrow X$ was inaugurated. We considered this problem as a closure property of the model. We have also extended the model to a more dynamical case when the departure index parameter considered as a random variable. In this case, the conditional distributions of the random departure index among individuals with either certain or partially certain lifetimes have been compared by stochastic orders. In the context of the new (frailty) model, we develop some closure properties with respect to the likelihood ratio order, the reversed hazard rate order and the mean inactivity time order.

In the future of this study, we will investigate the properties of the proportional mean departure time frailty model where the parameter θ is considered to be a random variable. The closure properties tracked in this article can be developed for the extended frailty model and moreover preservation properties of stochastic orders of random parameters in the new setting can be obtained. The procedure of estimation of the parameter of the model accompanied with related statistical inferences and also applying the model to real data sets considering the possibility of having lifetime data that are left censored can also be considered as useful study in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equally contributed to this study.

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