Research Article

Finite Time Prescribed Performance Control for Uncertain Second-Order Nonlinear Systems

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Received 30 November 2021; Accepted 18 January 2022; Published 16 February 2022

Academic Editor: Peijun Wang

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In this article, we discuss the finite time stability problem for second-order systems with an uncertain nonlinear function. A finite time performance function with the sinusoidal function is constructed, and the constrained problem of the original system is transformed into the stability problem of the equivalent system. Combining prescribed performance control and fuzzy logic systems, an effective control method is proposed. The simulation results also prove that the method we adopted is effective.

1. Introduction

Second-order nonlinear systems [1–6] have a variety of potentials in the real world, such as the robotic manipulator, horizontal platform system, navigation system, and gyro system. At present, many control methods [6–14] have been proposed for second-order nonlinear systems. For example, Zhao et al. [7] developed an output feedback control method for the second-order nonlinear system with uncertain parameters. For uncertain second-order nonlinear systems, a saturated controller was proposed in [8], which can overcome the system uncertainty and external disturbance. In [9], an adaptive fuzzy distributed control scheme was proposed, which realized the consensus tracking consistent of the second-order multiagent system. An adaptive PID control method was developed in [10], which makes the output error of the system tend to zero and ensures the stability of the system. In order to deal with the fault regulation problem of a class of second-order nonlinear systems, Van [11] proposed a high-order terminal sliding mode control method. It should be pointed out that the above control methods can only ensure that the tracking error enters a small neighborhood of zero, but the neighborhood

and the time to reach the neighborhood cannot be set in advance.

Recently, many researchers have been studying the prescribed performance control (PPC) method [12–18]. The idea is to transform the restriction problem of an original system into the stability problem of an equivalent system by using the performance function and transformation function. However, the traditional PPC method cannot solve the presetting time problem. Therefore, in recent years, different types of finite time performance functions were proposed. In [19], a finite time performance function with the exponential function was proposed, and the designed PPC method realized that the tracking error converges to the predefined zone in presetting time. Using the same finite time performance function in [19], Tran and Ho [20] studied the PPC strategy of uncertain horizontal platform system. In [21], a finite time performance function was constructed by polynomials, based on the partial persistent excitation condition, the tracking error of the strict feedback system can approach the predefined zone in presetting time and unknown functions can also be estimated accurately.

Inspired by the above work, this paper will construct a finite time performance function through the sinusoidal

function to investigate the stability of uncertain second-order nonlinear systems. The overall structure of this paper is as follows. In Section 2, some preliminaries are presented. The finite time PPC method and its stability analysis are investigated in Section 3. Section 4 gives an example for simulation. Finally, the conclusion is given in Section 5.

2. System Descriptions and Problem Formulations

Consider the following uncertain second-order nonlinear system:

$$\ddot{x}(t) = f(t, x) + g(t, x)u(t), \quad (1)$$

where $x(t) \in \mathbb{R}$ is the system state, $x = [x(t), \dot{x}(t)]^T$, $f(t, x)$ is an unknown nonlinear function, and $u(t) \in \mathbb{R}$ is the control input. Let $x_d(t) \in \mathbb{R}$ be a desired signal. The following assumptions are provided for later discussion.

Assumption 1. States $x(t)$, $\dot{x}(t)$, $x_d(t)$, $\dot{x}_d(t)$, and $\ddot{x}_d(t)$ are measurable.

Assumption 2. The nonlinear function $f(t, x)$ is unknown but bounded.

Assumption 3. $g(t, x)$ is known, and $g(t, x) \neq 0$ for all $t \geq 0$ and x .

Define the tracking error $e(t) = x(t) - x_d(t)$; the aim of this paper is to limit the tracking error $e(t)$ within the preset boundary through the finite time prescribed performance control method.

In order to make the tracking error $e(t)$ meet the steady-state performance and transient performance, the following constraint condition is designed:

$$-\rho(t) < e(t) < \rho(t), \quad (2)$$

where $\rho(t)$ is a finite time performance function and defined as

$$\rho(t) = \begin{cases} a_0 \sin^3\left(\frac{\pi}{2T}(T-t)\right) + a_1, & 0 \leq t \leq T, \\ a_1, & T < t, \end{cases} \quad (3)$$

where a_0 and a_1 are preset positive parameters and T is the preset time. Usually, the tracking error e is transformed into an equivalent expression using the transformation function. In this article, the transformation function is defined as follows:

$$\Gamma(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}. \quad (4)$$

Define $e(t) = \rho(t)\Gamma(z)$; if $z(t)$ is bounded, then $\Gamma(z)$ satisfies $-1 < \Gamma(z) < 1$, which implies that the tracking error $e(t)$ is limited within $(-\rho(t), \rho(t))$. Therefore, the following work focuses on the boundedness of $z(t)$.

Remark 1. Similar to performance function (3), we can also use the cosine function to design a finite time performance function as

$$\rho(t) = \begin{cases} a_0 \cos^3\left(\frac{\pi}{2T}(T-t)\right) + a_1, & 0 \leq t \leq T, \\ a_1, & T < t, \end{cases} \quad (5)$$

where parameters α_0 and α_1 are the same as those in (3).

Remark 2. The design of the transformation function is also important for the PPC method, and transformation function (4) can also be changed as

$$z = \frac{e(t)}{p(t)}. \quad (6)$$

In order to ensure $|z| < 1$, the construction of the barrier Lyapunov function needs to be considered.

3. Control Design and Stability Analysis

Now, the first derivative of $e(t)$ can be obtained as

$$\dot{e}(t) = \dot{\rho}(t)\Gamma(z) + \rho(t)\frac{\partial\Gamma(z)}{\partial z}\dot{z}(t), \quad (7)$$

which, together with (7), one gets

$$\dot{z}(t) = \Pi_1(t) + \Lambda(t)(\dot{x}(t) - \dot{x}_d(t)), \quad (8)$$

where $\Pi_1(t) = -(\dot{\rho}(t)\Gamma(z)/\rho(t)(\partial\Gamma(z)/\partial z))$, $\Lambda(t) = (1/\rho(t)(\partial\Gamma(z)/\partial z))$. According to (3) and (4), one has

$$0 < a_1 \leq \rho(t) \leq \rho(0), 0 < \frac{\partial\Gamma(z)}{\partial z} = \frac{4}{(e^z - e^{-z})^2} \leq 1. \quad (9)$$

So, $\Lambda(t)$ satisfies the inequality $\Lambda(t) \geq (1/\rho(0)) > 0$. From (7), one gets

$$\begin{aligned} \ddot{e}(t) &= \ddot{\rho}(t)\Gamma(z) + 2\dot{\rho}(t)\frac{\partial\Gamma(z)}{\partial z}\dot{z}(t) + \rho(t)\frac{\partial^2\Gamma(z)}{\partial z^2}(\dot{z}(t))^2 \\ &\quad + \rho(t)\frac{\partial\Gamma(z)}{\partial z}\ddot{z}(t). \end{aligned} \quad (10)$$

Combining (1), (7), and (10), one obtains

$$\ddot{z}(t) = \Pi_2(t) + \Lambda(t)(f(t, x) + g(t, x)u(t) - \ddot{x}_d(t)), \quad (11)$$

where $\Pi_2(t) = -(\ddot{\rho}(t)\Gamma(z) + 2\dot{\rho}(t)(\partial\Gamma(z)/\partial z)\dot{z}(t) + \rho(t)(\partial^2\Gamma(z)/\partial z^2)(\dot{z}(t))^2)/\rho(t)(\partial\Gamma(z)/\partial z)$. In order to prove that $z(t)$ is bounded, we introduce a new variable $s(t)$ as

$$s(t) = \dot{z}(t) + c_1 z(t), \quad (12)$$

where c_1 is a positive design parameter.

Since $f(t, x)$ is unknown, we will employ fuzzy logic systems to estimate $f(t, x)$ in system (1). According to Lemma 2 in [19], there exists a fuzzy logic system (FLS) $\theta^{*T}w(x)$ such that

$$f(t, x) = \theta^{*T} w(x) + \varepsilon_f(x), \quad (13)$$

where θ^* is the ideal constant weight vector, $w(x)$ is the basis function vector, and $\varepsilon_f(x)$ is the bounded approximation error, i.e., there exists positive constant $\bar{\varepsilon}_f$ such as $|\varepsilon_f(x)| \leq \bar{\varepsilon}_f$. Let $\hat{f}(x, \hat{\theta}) = \hat{\theta}^T w(x)$, where $\hat{\theta}$ is the estimation of θ^* and $\tilde{\theta} = \theta^* - \hat{\theta}$ is the estimation error. In this paper, the controller $u(t)$ is designed as

$$u(t) = \frac{-\Lambda(t)\tilde{\theta}^T w(x) - \Pi_3(t) - c_2 s(t)}{\Lambda(t)g(t, x)}, \quad (14)$$

where $\Pi_3(t) = \Pi_2(t) - \Lambda(t)\ddot{x}_d(t) + c_1(\Pi_1(t) + \Lambda(t)\dot{x}(t) - \Lambda(t)\dot{x}_d(t))$ and c_2 is a positive constant. And choose the adaptive law of $\hat{\theta}$ as

$$\dot{\hat{\theta}} = \gamma_f(\Lambda(t)w(x)s(t) - \lambda_f \hat{\theta}), \quad (15)$$

where $\gamma_f > 0$ and $\lambda_f > 0$. Now, we give the main conclusions as follows.

Theorem 1. *Under given Assumptions 1–3, when initial conditions $|x(0)| < \rho(0)$ are satisfied, then controller (14) and the parameter adaptive law (15) can guarantee that all signals of the closed-loop system are bounded, which also means that the tracking error $e(t)$ is limited within constraint condition (2).*

Proof. Consider the Lyapunov function as follows:

$$V(t) = \frac{1}{2} \left[s^2(t) + \frac{1}{\gamma_f} \tilde{\theta}^T \tilde{\theta} \right]. \quad (16)$$

By taking the derivative of $V(t)$, we can get

$$\begin{aligned} \dot{V}(t) &= s(t)\dot{s}(t) + \frac{1}{\gamma_f} \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= s(t)(\ddot{z}(t) + c_1 \dot{z}(t)) + \frac{1}{\gamma_f} \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= s(t)(\Pi_3(t) + \Lambda(t)f(t, x) + \Lambda(t)g(t, x)u(t)) - \frac{1}{\gamma_f} \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= s(t)(\Pi_3(t) + \Lambda(t)\theta^{*T}w(x) + \Lambda(t)\varepsilon_f(x) + \Lambda(t)g(t, x)u(t)) - \frac{1}{\gamma_f} \tilde{\theta}^T \dot{\tilde{\theta}}. \end{aligned} \quad (17)$$

Substituting (14) and (15) into $\dot{V}(t)$ yields

$$\begin{aligned} \dot{V}(t) &= s(t) \left(\Lambda(t)\tilde{\theta}^T w(x) + \Lambda(t)\varepsilon_f(x) - c_2 s(t) \right) \\ &\quad - \tilde{\theta}^T (\Lambda(t)w(x)s(t) - \lambda_f \hat{\theta}) \\ &= \Lambda(t)s(t)\varepsilon_f(x) - c_2 s^2(t) + \lambda_f \tilde{\theta}^T \hat{\theta}. \end{aligned} \quad (18)$$

Because the inequality described in the following holds,

$$\begin{aligned} \Lambda(t)s(t)\varepsilon_f(x) &\leq \frac{\Lambda^2(t)}{4} s^2(t) + \bar{\varepsilon}_f^2, \\ \lambda_f \tilde{\theta}^T \hat{\theta} &\leq -\frac{\lambda_f}{2} \tilde{\theta}^T \tilde{\theta} + \frac{\lambda_f}{2} \theta^{*T} \theta^*. \end{aligned} \quad (19)$$

Substituting the above inequalities into (18), one gets

$$\dot{V}(t) \leq -\left(c_2 - \frac{\Lambda^2(t)}{4}\right) s^2(t) - \frac{\lambda_f}{2} \tilde{\theta}^T \tilde{\theta} + \tau^*, \quad (20)$$

where $\tau^* = \bar{\varepsilon}_f^2 + (\lambda_f/2)\theta^{*T}\theta^*$, and select positive constant c_3 such that $c_3 < \min\{2c_2 - (\Lambda^2(t)/2), \gamma_f \lambda_f\}$. So, one has

$$\dot{V}(t) \leq -c_3 V(t) + \tau^*, \quad (21)$$

which implies that

$$V(t) \leq \frac{\tau^*}{c_3} + \left(V(0) - \frac{\tau^*}{c_3}\right) e^{-c_3 t} \leq 2 \left(V(0) + \frac{\tau^*}{c_3}\right). \quad (22)$$

Obviously, all signals in (16) are ultimately uniformly bounded. Assume that the bounded value of $s(t)$ is $\bar{s}$. Let $\bar{V}(t) = (1/2)\bar{s}^2(t)$; according to (8), one has

$$\begin{aligned} \dot{\bar{V}}(t) &= z(t)\dot{z}(t) \\ &= z(t)(s(t) - c_1 z(t)) \\ &= z(t)s(t) - c_1 z^2(t) \\ &\leq -\left(c_1 - \frac{1}{4}\right) z^2(t) + \bar{s}^2. \end{aligned} \quad (23)$$

Select parameter c_1 so that $c_1 > (1/4)$, which means that $z(t)$ is bounded. Therefore, we can conclude that $e(t)$ is limited within $(-\rho(t), \rho(t))$. This completes the proof. $\square$

4. Numerical Simulations

In this part, an uncertain gyroscope system [22] is used to show the availability of the proposed method in this paper. The gyroscope system is described as

$$\ddot{x} = f(t, x) + g(t, x)u(t), \quad (24)$$

FIGURE 1: Time response trajectory of tracking error e by using method (26).

FIGURE 3: Time response trajectory of controller u by using method (26).

FIGURE 2: Time response trajectories of x and x_d by using method (26).

where $x = [x, \dot{x}]^T$, $f(t, x) = -A_1^2((1 - \cos x)^2 / \sin^3 x) + A_2 \sin x - A_3 \dot{x} - A_4 \dot{x}^3 + A_5 \sin(wt) \sin x$, and $g(t, x) = 3 + \sin x \sin \dot{x}$. Parameters A_1, A_2, A_3, A_4, A_5 , and w are selected as $A_1 = 10, A_2 = 1, A_3 = 0.5, A_4 = 0.05, A_5 = 35.7$, and $w = 2$. The initial values of system (24) are $x(0) = -1, \dot{x}(0) = 1$, and $x_d(t) = \sin(t)$. The fuzzy sets are defined over $[-5, 5]$ for x and $\dot{x}$. The corresponding fuzzy membership functions are chosen as follows:

$$w(\xi) = \exp\left[-\frac{1}{2}\left(\frac{\xi + 7.5 - 2.5l}{1.2}\right)^2\right], \quad (25)$$

where $\xi = x, \dot{x}$ and $l = 1, 2, 3, 4, 5$. In order to compare with the proposed method in this paper, the traditional feedback method is designed as follows:

FIGURE 4: Time response trajectory of tracking error e by using method (14) under $\rho_1(t)$.

$$\begin{cases} e(t) = x(t) - x_d(t), \\ \bar{s}(t) = \dot{e}(t) + c_1 e(t), \\ u(t) = \frac{1}{g(t, x)} \left[-\hat{\theta}^T w(x) + \ddot{x}_d(t) - c_1 \dot{e}(t) - c_2 \bar{s}(t) \right], \\ \dot{\hat{\theta}} = \gamma_f (w(x) \bar{s}(t) - \lambda_f \hat{\theta}), \end{cases} \quad (26)$$

where parameters c_1, c_2, γ_f , and λ_f are designed as $c_1 = c_2 = 5, \gamma_f = 3$, and $\lambda_f = 0.05$. The control effect of traditional method (26) is shown in Figures 1–3.

FIGURE 5: Time response trajectories of x and x_d by using method (14) under $\rho_1(t)$.

FIGURE 7: Time response trajectory of tracking error e by using method (14) under $\rho_2(t)$.

FIGURE 6: Time response trajectory of controller u by using method (14) under $\rho_1(t)$.

FIGURE 8: Time response trajectories of x and x_d by using method (14) under $\rho_2(t)$.

Obviously, the tracking error $e(t)$ is not effectively controlled by using traditional method (26). With the above same parameters, the prescribed function $\rho(t)$ is designed as

$$\rho_1(t) = \begin{cases} 1.97 \sin^3\left(\frac{(5-t)\pi}{10}\right) + 0.05, & t \leq 5, \\ 0.05, & t > 5. \end{cases} \quad (27)$$

The control effect of controller (14) is shown in Figures 4–6. It can be seen that the control effect of $e(t)$ has been improved, and the tracking error $e(t)$ has been limited within $[-0.05, 0.05]$ after 5 seconds. One strong point of the

proposed method (14) is that it can change the control time and control interval, such as setting the prescribed performance function as

$$\rho_2(t) = \begin{cases} 1.97 \sin^3\left(\frac{(3-t)\pi}{6}\right) + 0.02, & t \leq 3, \\ 0.02, & t > 3. \end{cases} \quad (28)$$

The control effect is shown in Figures 7–9. The above simulation results show that the proposed method (14) has better transient performance in practical applications.

FIGURE 9: Time response trajectory of controller u by using method (14) under $\rho_2(t)$.

5. Conclusion

We studied the finite time control problem of uncertain second-order nonlinear systems in this article. For the sake of the tracking error reaching the preset zone at the preset time, a finite time performance function was introduced. Meanwhile, the fuzzy logic system (FLS) was used to evaluate the uncertain function of the system. Through theoretical analysis, this paper proves that the proposed control method achieves the expected control effect. At the same time, this conclusion is also verified by simulation.

Data Availability

The data used in this paper are reflected in the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This research was supported by the NSFC Grant no. 11872210 and Grant no. MCMS-I-0120G01 and the Key Research Projects of Natural Science in Colleges and Universities of Anhui Province (KJ2019A0695, KJ2020A0644, and KJ2021A0965).

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