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Research Article

Asymptotics of m-Cliques in a Sparse Inhomogeneous Random Graph

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One of the classical questions in random graph theory is to understand the asymptotics of subgraph counts. In inhomogeneous random graph, this question has not been well studied. In this study, we investigate the asymptotic distribution of m-cliques in a sparse inhomogeneous random graph. Under mild conditions, we prove that the number of m-cliques converges in law to the standard normal distribution.

1. Introduction

The random graph theory was founded by Erdös–Rényi [1]. The well-known Erdös–Rényi random graph $\mathscr{G}(n, p)$ is an undirected graph on vertices' (nodes) set $[n] = \{1, 2, \ldots, n\}$, where any two nodes form an edge independently with probability p. In random graph theory, one of the classical questions concerns the asymptotics of the number of subgraphs [2–7]. Much attention has been paid to the limiting distribution of subgraph counts. In a dense Erdös–Rényi graph, the subgraph count converges in law to the standard normal distribution under some conditions [3]. This result was proven to be true for sparse random graph [2]. Analogue results exist for the number of strictly balanced subgraph or short cycles in random regular graphs [4–6, 8].

In practice, a lot of real networks display the inhomogeneity property, that is, the vertex degrees vary a lot. In many cases, the degree follows a power law [9]. To accommodate the inhomogenity, the inhomogeneous random graph has recently been introduced [10]. It is natural to study the asymptotics of subgraphs in an inhomogeneous random graph. Some of the first results concern the asymptotic clique or cycle number in special inhomogeneous random graph [9, 11, 12, 23, 24]. For example, the authors gave the upper bound and lower bound of the average number of cycles in [12]. The authors of [9, 13] obtained the asymptotic order of large cliques in scale-free random graph, and Janson [14] studied the order of the largest component. In [12], the order of expected number of cycles and cliques in a random graph were given. Hu and Dong [15] derived the limiting distribution of the number of edges in a generalized random graph, and Liu and Dong [16] derived the asymptotic distribution of the number of triangles. However, to our knowledge, the limiting distribution of the number of cliques is unknown. In this study, we study this problem in sparse inhomogeneous random graph and derive its asymptotic distribution.

2. The Model and Main Result

In this section, we introduce the model and present the main result. The inhomogeneous random graph $\mathscr{G}(n, p, W)$ is defined as follows: given a positive array $W = \{W_{ij}\}_{1 \le i < j \le n}$, every pair of nodes i, j in $\mathscr{G}(n, p, W)$ are joined as an edge with probability pW_{ij} independently. The adjacency matrix A of a graph is a symmetric $(0, 1)$-matrix with zeros on its diagonal and $A_{ij} = 1$ if (i, j) is an edge, $A_{ij} = 0$ otherwise. For $\mathscr{G}(n, p, W)$, the adjacency matrix A is a symmetric random matrix, with elements following independent Bernoulli distributions, that is,

$$p_{ij} = \mathbb{P}(A_{ij} = 1) = pW_{ij}, \quad 1 \le i < j \le n, \tag{1}$$

and A_{ij} is independent of A_{kl} if $\{i, j\} \neq \{k, l\}$. By symmetry, $p_{ji} = p_{ij}$ for $i \neq j$ and $p_{ii} = 0$ for $i = 1, 2, \ldots, n$. This model was introduced in [10], and it contains the models in [9, 11, 18] as a special case. If $W_{ij} \equiv 1 (1 \leq i < j \leq n)$, the random graph is the homogeneous Erdös–Rényi model, where nodes i, j form an edge with probability p [1].

In graph $\mathcal{G}$, an m-clique is a subgraph of m vertices such that any two distinct vertices are adjacent. Given the adjacency matrix A and vertices $i_1 < i_2 < \cdots < i_m$, the node set $\{i_1, i_2, \ldots, i_m\}$ forms an m-clique if and only if

$$\prod_{1 \leq k < l \leq m} A_{i_k i_l} = 1. \tag{2}$$

Then, the total number N_n of m-clique in $\mathcal{G}$ is

$$N_n = \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \prod_{1 \leq k < l \leq m} A_{i_k i_l}. \tag{3}$$

For the random graph $\mathcal{G}(n, p, W)$, N_n is a sum of dependent random variables. The asymptotic distribution of N_n is given in the following theorem.

Theorem 1. *Let N_n be the number of m-cliques in the random inhomogeneous graph $\mathcal{G}(n, p, W)$ with expectation $\mathbb{E}(N_n)$ and variance $\mathbb{V}(N_n)$. Suppose $m \geq 3$ is a fixed integer, for some $0 < \epsilon < 1$,*

$$\begin{aligned} &\lim_{n \longrightarrow \infty} n^{m-2} p^{(m+1)(m-2)/2} = 0, \\ &\lim_{n \longrightarrow \infty} n^{m(1-\epsilon)} p^{m(m-1)/2} = \infty, \end{aligned} \tag{4}$$

for some positive constants c_1, c_2, $0 \leq W_{ij} \leq c_2$, $1 \leq i < j \leq n$, and

$$c_1 \binom{n}{m} \leq \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq n} \prod_{1 \leq k < l \leq m} W_{i_k i_l}. \tag{5}$$

Then,

$$\frac{N_n - \mathbb{E}(N_n)}{\sqrt{\mathbb{V}(N_n)}} \Rightarrow N(0, 1), \text{as } n \longrightarrow \infty, \tag{6}$$

where '⇒' represents convergence in distribution.

According to Theorem 1, the scaled and centered number of m-cycle in $\mathcal{G}(n, p, W)$ converges in law to the standard normal distribution. Note that Theorem 1 only holds for sparse random graph. The first equation in (4) implies $p = o(n^{-2/m+1})$ and the second one requires the average degree tends to infinity. Our proof relies on the martingale central limit theorem.

The conditions on W seem to be very restrictive. In fact, there are many models which satisfy these conditions. Here, we provide some of the examples that satisfy the conditions of Theorem 1. Consider the rank one model with

$$p_{ij} = pW_{ij}, W_{ij} = \frac{ij}{n^2}, \quad i \neq j, \tag{7}$$

$W_{ii} = 0, 1 \leq i \leq n$. Suppose p satisfies (4). Straightforward computation yields

$$\sum_{1 \leq i_1 < i_2 \ldots < i_m \leq n} \prod_{1 \leq k < l \leq m} W_{i_k i_l} = \sum_{1 \leq i_1 < i_2 \ldots < i_m \leq n} \prod_{1 \leq k \leq m} \left(\frac{i_k}{n}\right)^{m-1} \asymp \frac{n^{2m}}{n^{m(m-1)}} = n^m. \tag{8}$$

Hence, (5) holds and Theorem 1 applies to this model. The average degree d_i of each vertex i is given by

$$d_i = \sum_j p_{ij} = \frac{ip}{2} + \frac{ip}{2n} - \frac{i^2 p}{n^2}, \quad i = 1, 2, \ldots, n. \tag{9}$$

The degree d_n is approximately n times of d_1, that is, $d_n \asymp n d_1$. Hence, this model is essentially different from the homogeneous Erdös–Rényi model. Furthermore, suppose $p = \log n/n$, satisfying (4) for $m = 3$. Then, for some constant $\kappa > 0$,

$$\begin{aligned} &0 < \kappa \leq \max_i\{d_i\} = \frac{\log n}{2} + o(1) \leq \eta \log n, \max_{i \neq j} p_{ij} \leq n^{-1+\eta}, \\ &\eta \in (0.5, 1). \end{aligned} \tag{10}$$

This model is a special case in [17], where the authors studied asymptotic behavior of the extreme eigenvalues.

Another interesting model is the power-law graphs [22]:

$$p_{ij} = \frac{1}{n}\left(\frac{i}{n}\right)^{-1/p}\left(\frac{j}{n}\right)^{-1/p}, \quad i \neq j, \ p > 0. \tag{11}$$

Rewrite the probabilities as

$$p_{ij} = \frac{a_n}{n} \frac{n^{2/p}}{a_n} \frac{1}{i^{1/p}} \frac{1}{i^{1/p}} = pW_{ij}, \tag{12}$$

where $p = a_n/n$ and $W_{ij} = n^{2/p}/a_n 1/i^{1/p} 1/i^{1/p}$. If $a_n \asymp n^{2/p}$, then $W_{ij} \leq c_2 < \infty$ for some constant c_2. For some generic positive constant c_3,

$$\sum_{1 \leq i_1 < i_2 \ldots < i_m \leq n} \prod_{1 \leq k < l \leq m} W_{i_k i_l} \geq c_3 \sum_{1 \leq i_1 < i_2 \ldots < i_m \leq n} \prod_{1 \leq k \leq m} \left(\frac{1}{i_k^{1/p}}\right)^{m-1} \asymp \frac{n^{2m}}{n^{m(m-1)}} = n^m. \tag{13}$$

3. Proof of Main Result

In this section, we prove Theorem 1. For convenience, denote $\alpha_n \asymp \beta_n$ if $0 < c_1 \le \alpha_n/\beta_n \le c_2 < \infty$ for some constants c_1 and c_2; $\alpha_n \ll \beta_n$ if $\alpha_n = o(\beta_n)$. The proof relies on the following result.

Proposition 1 (see [20]). *Suppose that, for every $n \in \mathbb{N}$ and $k_n \longrightarrow \infty$, the random variables $X_{n,1}, \ldots, X_{n,k_n}$ are a martingale difference sequence relative to an arbitrary filtration $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \cdots \mathcal{F}_{n,k_n}$. If (I) $\sum_{i=1}^{k_n} \mathbb{E}(X_{n,i}^2 | \mathcal{F}_{n,i-1}) \longrightarrow 1$ in probability, (II) $\sum_{i=1}^{k_n} \mathbb{E}(X_{n,i}^2 I[|X_{n,i}| > \epsilon] | \mathcal{F}_{n,i-1}) \longrightarrow 0$ in probability, for every $\epsilon > 0$, then $\sum_{i=1}^{k_n} X_{n,i} \longrightarrow N(0,1)$ in distribution.*

Now, we prove Theorem 1. Let $M = \binom{m}{2} = m(m-1)/2$, $S = \{(i_k, i_l) | 1 \le k < l \le m\}$, and

$$S_t = \{s_t | s_t \text{ is } a \text{ subset consisting of } t \text{ elements in } S\}. \tag{14}$$

Denote s_t^c, the complement set of s_t in S. By Taylor expansion, for generic constants c_i, $1 \le i \le M-1$, it follows

$$\begin{aligned} N_n - \mathbb{E}(N_n) &= \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \left(\prod_{1 \le k < l \le m} A_{i_k i_l} - \prod_{1 \le k < l \le m} pW_{i_k i_l} \right) \\ &= \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \prod_{1 \le k < l \le m} \left(A_{i_k i_l} - pW_{i_k i_l}\right) + \sum_{t=1}^{M-1} R_t, \end{aligned} \tag{15}$$

where

$$R_t = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \sum_{s_t \in S_t} \prod_{(i,j) \in s_t} \left(A_{ij} - pW_{ij}\right) \prod_{(i,j) \in s_t^c} pW_{ij}. \tag{16}$$

Next, we show the first term in (15) is the leading term. Note that if $\{i_k, i_l\} \ne \{j_k, j_l\}$, by the independence of $A_{i_k i_l}$ and $A_{j_k j_l}$, it follows

$$\mathbb{E}\left[\left(A_{i_k i_l} - pW_{i_k i_l}\right)\left(A_{j_k j_l} - pW_{j_k j_l}\right)\right] = 0. \tag{17}$$

Then, it s easy to obtain

$$\begin{aligned} &\mathbb{E}\left[\sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \prod_{1 \le k < l \le m} \left(A_{i_k i_l} - pW_{i_k i_l}\right)\right]^2 \\ &= \sum_{\substack{1 \le i_1 < \ldots < i_m \le n \\ 1 \le j_1 < \ldots < j_m \le n}} \mathbb{E}\left[\prod_{1 \le k < l \le m} \left(A_{i_k i_l} - pW_{i_k i_l}\right)\left(A_{j_k j_l} - pW_{j_k j_l}\right)\right] \\ &= \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \prod_{1 \le k < l \le m} \mathbb{E}\left(A_{i_k i_l} - pW_{i_k i_l}\right)^2 \\ &= \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \prod_{1 \le k < l \le m} pW_{i_k i_l}\left(1 - pW_{i_k i_l}\right). \end{aligned} \tag{18}$$

For convenience, let

$$\sigma_n^2 = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \prod_{1 \le k < l \le m} pW_{i_k i_l}\left(1 - pW_{i_k i_l}\right). \tag{19}$$

Note that $\sigma_n^2 \asymp n^{m(1-\epsilon)} p^M$ under the assumption of Theorem 1. To get the order of R_1, let $s_1 = \{(i_1, i_2)\}$ and $s_1^* = \{(j_1, j_2)\}$; then,

$$\begin{aligned} &\mathbb{E}\left[\sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} \left(A_{i_1 i_2} - pW_{i_1 i_2}\right) \prod_{(i,j) \in s_1^c} pW_{ij}\right]^2 \\ &= \sum_{\substack{1 \le i_1 < \ldots < i_m \le n \\ 1 \le j_1 < \ldots < j_m \le n}} \mathbb{E}\left[\left(A_{i_1 i_2} - pW_{i_1 i_2}\right)\left(A_{j_1 j_2} - pW_{j_1 j_2}\right) \prod_{(i,j) \in s_1^c} pW_{ij} \prod_{(i,j) \in s_1^{*c}} pW_{ij}\right] \\ &= \sum_{\substack{1 \le i_1 < \ldots < i_m \le n \\ i_2 < j_3 < \ldots < j_m \le n}} \mathbb{E}\left[\left(A_{i_1 i_2} - pW_{i_1 i_2}\right)^2 \prod_{(i,j) \in s_1^c} pW_{ij} \prod_{(i,j) \in s_1^{*c}} pW_{ij}\right] \\ &= \sum_{\substack{1 \le i_1 < \ldots < i_m \le n \\ i_2 < j_3 < \ldots < j_m \le n}} pW_{i_1 i_2}\left(1 - pW_{i_1 i_2}\right) \prod_{(i,j) \in s_1^c} pW_{ij} \prod_{(i,j) \in s_1^{*c}} pW_{ij} \\ &= O_P\left(n^{2m-2} p^{2M-1}\right), \end{aligned} \tag{20}$$

which implies $R_1^2 = O_P(n^{2m-2}p^{2M-1})$. For R_{M-1}, let $s_{M-1} = \{(i_1,i_2),(i_1,i_3),\ldots,(i_{m-2},i_{m-1})\}$ and $s_{M-1}^* = \{(j_1,j_2),(j_1,j_3),\ldots,(j_{m-2},j_{m-1})\}$. Then,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{1\le i_1<i_2<\ldots<i_m\le n} pW_{i_{m-1}i_m}\prod_{(i,j)\in s_{M-1}}\left(A_{ij}-pW_{ij}\right)\right]^2\\
&=\sum_{\substack{1\le i_1<\ldots<i_m\le n\\ 1\le j_1<\ldots<j_m\le n}}\mathbb{E}\left[pW_{i_{m-1}i_m}pW_{j_{m-1}j_m}\prod_{(i,j)\in s_{M-1}}\left(A_{ij}-pW_{ij}\right)\prod_{(i,j)\in s_{M-1}^*}\left(A_{ij}-pW_{ij}\right)\right]\\
&=\sum_{1\le i_1<\ldots<i_m\le n}\left[p^2W_{i_{m-1}i_m}^2\prod_{(i,j)\in s_{M-1}}\mathbb{E}\left(A_{ij}-pW_{ij}\right)^2\right]\\
&=\sum_{1\le i_1<\ldots<i_m\le n}\left[p^2W_{i_{m-1}i_m}^2\prod_{(i,j)\in s_{M-1}}pW_{ij}\left(1-pW_{ij}\right)\right]\\
&=O_P\left(n^m p^{M+1}\right),
\end{aligned}\tag{21}$$

from which it yields $R_{M-1}^2 = O_P(n^m p^{M+1})$. For $t \ge m$, any s_t involves all the vertices $i_1, i_2, \ldots, i_m$. Hence, by a similar proof of (20) and (21), it follows that $R_t^2 = O_P(n^m p^{2M-t})$. For $3 \le t \le m-1$, any s_t involves at least t distinct vertices; then, $R_t^2 = O_P(n^{2m-t}p^{2M-t})$. When $t=2$, s_t contains at least 3 distinct vertices; then, $R_2^2 = O_P(n^{2m-3}p^{2M-2})$.

We claim the following hold:

$$\frac{n^{2m-2}p^{2M-1}}{n^m p^M} = n^{m-2}p^{M-1} \longrightarrow 0, \quad t=1, \tag{22}$$

$$\frac{n^{2m-3}p^{2M-2}}{n^m p^M} = n^{m-3}p^{M-2} \longrightarrow 0, \quad t=2, \tag{23}$$

$$\frac{n^{m}p^{2M-t}}{n^m p^M} = p^{M-t} \longrightarrow 0, \quad m \le t \le M-1, \tag{24}$$

$$\frac{n^{2m-t}p^{2M-t}}{n^m p^M} = n^{m-t}p^{M-t} \longrightarrow 0, \quad 3 \le t \le m-1. \tag{25}$$

Firstly, according to the first equation in (4), (22) and (24) hold trivially. Let $p = a_n/n$; then, $a_n \ll n^{M-m+1/M-1}$ by (22). Equations (23) and (25) hold if

$$a_n \ll n^{M-m+1/M-2}, a_n \ll n^{M-m/M-t}, \quad (3 \le t \le m-1). \tag{26}$$

Simple algebra yields

$$\frac{M-m+1}{M-1} - \frac{M-m}{M-3} = -\frac{(m-2)(m-3)}{2(M-3)(M-1)} \le 0, \quad (m \ge 3). \tag{27}$$

Hence, $M-m+1/M-1 \le M-m/M-t$ $(3 \le t \le m-1)$. Then, the first equation in (4) implies (22)–(25).

As a result, by (15), one has

$$N_n - \mathbb{E}(N_n) = \sum_{1\le i_1<i_2<\ldots<i_m\le n}\prod_{1\le k<l\le m}\left(A_{i_k i_l} - pW_{i_k i_l}\right) + o_P\left(\sqrt{n^m p^M}\right), \tag{28}$$

and $\mathbb{V}(N_n) = \sigma_n^2 + o(n^m p^M)$. Therefore, to prove Theorem 1, it suffices to prove

$$X_n = \frac{\sum_{1\le i_1<i_2<\ldots<i_m\le n}\prod_{1\le k<l\le m}\left(A_{i_k i_l} - pW_{i_k i_l}\right)}{\sigma_n} \Rightarrow N(0,1). \tag{29}$$

In the following, we use Proposition 1 to prove (29). To this end, define $Z_t = X_t - X_{t-1}$ for $4 \le t \le n$, where $X_3 = 0$ and

$$X_t = \frac{\sum_{1\le i_1<i_2<\ldots<i_m\le t}\prod_{1\le k<l\le m}\left(A_{i_k i_l} - pW_{i_k i_l}\right)}{\sigma_n}. \tag{30}$$

Let $F_t = \{A_{ij} | 1 \le i < j \le t\}$ and $C_t = \{(i_k, i_m) | 1 \le k \le m-1, i_m = t\}$. Then,

$$\sigma_n \mathbb{E}[Z_t | F_{t-1}] = \sum_{1\le i_1<i_2<\ldots<i_m=t}\prod_{(i,j)\in C_t^c}\left(A_{ij} - pW_{ij}\right)\prod_{(i,j)\in C_t}\mathbb{E}\left(A_{ij} - pW_{ij}\right) = 0, \tag{31}$$

which implies Z_t is a Martingale difference.

Check condition (I) in Proposition 1. It suffices to show the following:

$$\mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2|F_{t-1}\right)\right]\longrightarrow 1,$$
$$\mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2|F_{t-1}\right)\right]^2\longrightarrow 1. \tag{32}$$

For (30), direct computation yields

$$\begin{aligned}\mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2|F_{t-1}\right)\right] &= \mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(X_t^2-2X_tX_{t-1}^2+X_{t-1}^2|F_{t-1}\right)\right]\\ &=\sum_{t=m}^{n}\left(\mathbb{E}X_t^2-\mathbb{E}X_{t-1}^2\right)=\mathbb{E}X_n^2=1.\end{aligned} \tag{33}$$

For (32), let $C_t^*=\{(j_k,j_m)|1\le k\le m-1, j_m=t\}$. Then,

$$\begin{aligned}\sigma_n^2\mathbb{E}\left(Z_t^2|F_{t-1}\right) &= \sum_{\substack{1\le i_1<\ldots<i_m=t\\ 1\le j_1<\ldots<j_m=t}}\prod_{(i,j)\in C_t^c}\left(A_{ij}-pW_{ij}\right)\prod_{(i,j)\in C_t^{*c}}\left(A_{ij}-pW_{ij}\right)\\ &\quad\times\mathbb{E}\left[\prod_{(i,j)\in C_t}\left(A_{ij}-pW_{ij}\right)\prod_{(i,j)\in C_t^*}\left(A_{ij}-pW_{ij}\right)\right]\\ &=\sum_{1\le i_1<\ldots<i_m=t}\prod_{(i,j)\in C_t^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in C_t}\mathbb{E}\left[\left(A_{ij}-pW_{ij}\right)^2\right]\\ &=\sum_{1\le i_1<\ldots<i_m=t}\prod_{(i,j)\in C_t^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in C_t}pW_{ij}\left(1-pW_{ij}\right).\end{aligned} \tag{34}$$

Let $D=\{(i_k,i_m)|1\le k\le m-1\}$ and $D^*=\{(j_k,j_m)|1\le k\le m-1\}$. Then,

$$\begin{aligned}\sigma_n^2\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2|F_{t-1}\right) &= \sum_{t=m}^{n}\sum_{1\le i_1<\ldots<i_m=t}\prod_{(i,j)\in C_t^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in C_t}pW_{ij}\left(1-pW_{ij}\right)\\ &=\sum_{1\le i_1<\ldots<i_m\le n}\prod_{(i,j)\in D^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in D}pW_{ij}\left(1-pW_{ij}\right).\end{aligned} \tag{35}$$

Now, (32) follows from the following calculation:

$$\begin{aligned}\sigma_n^4\mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2|F_{t-1}\right)\right]^2 &= \sum_{1\le i_1<\ldots<i_m\le n 1\le j_1<\ldots<j_n\le n}\mathbb{E}\left[\prod_{(i,j)\in D^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in D}p\right.\\ &\quad\left.\times\prod_{(i,j)\in D^{*c}}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in D^*}pW_{ij}\left(1-pW_{ij}\right)\right]\\ &=\sum_{1\le i_1<\ldots<i_m\le n}\mathbb{E}\left[\prod_{(i,j)\in D^c}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in D}pW_{ij}\left(1-pW_{ij}\right)\right]\\ &\quad\times\sum_{1\le j_1<\ldots<j_n\le n}\mathbb{E}\left[\prod_{(i,j)\in D^{*c}}\left(A_{ij}-pW_{ij}\right)^2\prod_{(i,j)\in D^*}pW_{ij}\left(1-pW_{ij}\right)\right]+o\left(\sigma_n^4\right)=\sigma_n^4+o\left(\sigma_n^4\right).\end{aligned} \tag{36}$$

Hence, $\sum_{t=m}^{n}\mathbb{E}(Z_t^2|F_{t-1})\longrightarrow 1$ in probability.

Check condition (II) in Proposition 1. By Cauchy–Schwarz inequality and Markov inequality, for any $\epsilon>0$ and generic constant $C>0$, it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2 I\left[\left|Z_t\right|>\epsilon\right]\middle|F_{t-1}\right)\right]\\
&\leq \mathbb{E}\left[\sum_{t=m}^{n}\sqrt{\mathbb{E}\left(Z_t^4|F_{t-1}\right)\mathbb{E}\left(I\left[\left|Z_t\right|>\epsilon\right]|F_{t-1}\right)}\right]\\
&\leq \frac{1}{\epsilon^2}\mathbb{E}\left[\sum_{t=m}^{n}\sqrt{\mathbb{E}\left(Z_t^4|F_{t-1}\right)}\sqrt{\mathbb{E}\left(Z_t^4|F_{t-1}\right)}\right]\\
&= \frac{1}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}\mathbb{E}\left(Z_t^4\right)\\
&= \frac{1}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}\sum_{\substack{1\leq i_1<\ldots<i_m=t\,1\leq j_1<\ldots<j_m=t\\ 1\leq k_1<\ldots<k_m=t\,1\leq l_1<\ldots<l_m=t}}\mathbb{E}\left[\prod_{1\leq r<s\leq m}\left(A_{i_ri_s}-pW_{i_ri_s}\right)\prod_{1\leq r<s\leq m}\left(A_{j_rj_s}-pW_{j_rj_s}\right)\right.\\
&\quad\left.\times\prod_{1\leq r<s\leq m}\left(A_{k_rk_s}-pW_{k_rk_s}\right)\prod_{1\leq r<s\leq m}\left(A_{l_rl_s}-pW_{l_rl_s}\right)\right]\\
&= \frac{C}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}\sum_{\substack{1\leq i_1<\ldots<i_m=t\\ 1\leq j_1<\ldots<j_m=t}}\mathbb{E}\left[\prod_{1\leq r<s\leq m}\left(A_{i_ri_s}-pW_{i_ri_s}\right)^2\prod_{1\leq r<s\leq m}\left(A_{j_rj_s}-pW_{j_rj_s}\right)^2\right]\\
&\quad+\frac{C}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}\sum_{1\leq i_1<\ldots<i_m=t}\mathbb{E}\left[\prod_{1\leq r<s\leq m}\left(A_{i_ri_s}-pW_{i_ri_s}\right)^4\right]\\
&= \frac{C}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}t^{2m-2}p^{2M}+\frac{C}{\epsilon^2\sigma_n^4}\sum_{t=m}^{n}t^{m-1}p^{M}\\
&= O\left(\frac{n^{2m-1}p^{2M}}{\sigma_n^4}+\frac{n^mp^M}{\sigma_n^4}\right)=O\left(\frac{1}{n}+\frac{1}{n^mp^M}\right)\longrightarrow 0,
\end{aligned}
\tag{37}
$$

if $n^m p^M \longrightarrow \infty$. Hence, $\sum_{t=m}^{n}\mathbb{E}\left(Z_t^2 I\left[\left|Z_t\right|>\epsilon\right]|F_{t-1}\right)=o_P(1)$. By Proposition 1, the proof is complete.[19, 21]

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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