Research Article

On the Approximation of Entire Harmonic Functions in $\mathbb{R}^n$ Having Slow Growth

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1. Introduction

For entire harmonic functions of one variable (identifying $\mathbb{R}^2$ with $\mathbb{C}$), there is a large literature concerning the growth of this topic. For entire harmonic functions (i.e., harmonic functions defined on $\mathbb{R}^n$), elementary characterizations of the growth order and type in terms of Taylor coefficients can be found in the work of Temliakow [1], Fryant [2], Fugard [3], Kapoor and Nautiyal [4], Fryant and Shankar [5], and Veselovska [6].

Several papers about generalized order and type for entire harmonic functions have been published (see, e.g., [3, 7–12]). The historical background of our work is an old result of Bernstein characterization when a continuous real-valued function $f$ on the interval $[-1, 1]$ can be extended to an entire (holomorphic) function in terms of polynomials of degree $\leq k$. In 1968, Varga [13] obtained results giving the order and type of the entire extension.

Analogues of these results for harmonic functions were proved by Kapoor and Nautiyal [4] for $n = 3$. These results were generalized by Veselovska [6] for general dimension. Recently, Veselovska [14] obtained the generalized order and lower order in terms of best approximation errors in the sup norm. The main objective of the present paper is to find an analogue of results of Vakarchuk and Zhir [15] for harmonic functions. We use a new class of functions introduced by Kapoor and Nautiyal [16] to study the slow growth of entire harmonic functions. Our results add new aspects in the results of Veselovska [14], namely, in the case $\rho = 0$.

The harmonic functions have a significant role to study different stationary processes in theoretical mathematics, physics, and mechanics. Therefore, the study of generalized growth parameters of a harmonic function in $n$-dimensional space $\mathbb{R}^n$, $n \geq 3$, has significance.

Let $B_R = \{y \in \mathbb{R}^n: |y| \leq R\}$ be the ball of radius $R$ and $\overline{B_R}$ be its closure. We designate by $H_R$, $0 < R < \infty$, the class of functions harmonic in $B_R^c$ and continuous on $\overline{B_R}$.

Let $u \in H_R \forall r$, $0 < r < R$; then, the Fourier–Laplace series expansion [17] is given as

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x; u)r^k,$$

where $Y^{(k)}$ are the spherical harmonics in the unit sphere $S^n = \{y \in \mathbb{R}^n: |y| = 1\}$ in the space $\mathbb{R}^n$, $n \geq 3$ ([2], pp.157–174, [15]).
\[ Y^{(k)}(x; u) = a_1^{(k)}Y_1^{(k)}(x) + a_2^{(k)}Y_2^{(k)}(x) + \ldots + a_n^{(k)}Y_n^{(k)}(x), \]
\[ a_j^{(k)} = (u, Y_j^{(k)}), \quad j = 1, \ldots, n, \quad x \in S^n, \tag{2} \]

and \((u, Y^{(k)})\) is the scalar product in \(L^2(S^n)\). Here, \(Y_k = ((k^2 + n - 2)(k + n - 3))/((n(2n - 4)))\), and the scalar product in \(L^2(S^n)\) is defined as
\[ (f, g) = \frac{1}{\omega_n} \int_{S^n} f(x)g(x)\,ds, \tag{3} \]

where \(ds\) is an element of area of the sphere \(S^n\) and \(\omega_n = 2(\pi)\frac{n!}{n^{n/2}}(\Gamma(n/2))\) is the area of the surface.

### 2. Preliminaries and Notions

The growth of a continuous function \(u: \mathbb{R}^n \to \mathbb{C}\) can be measured in terms of the order \(\rho\) defined by
\[ \rho = \limsup_{r \to \infty} \frac{\log \log M(r, u)}{\log r}, \tag{4} \]

where \(M(r, u) = \sup_{|x| \leq r} |u(x)|\). If the order is a positive real number, the type \(T\) of the function is defined by
\[ T = \limsup_{r \to \infty} \frac{\log M(r, u)}{r^\rho}. \tag{5} \]

A function has slow growth if the order is equal to 0.

The concept of order and type was generalized in the literature (see, e.g., Seremeta [18]), in particular, to discuss subclasses of the entire function of order \(\rho = 0\) (functions of slow growth). Here, one replaces the log function in the above formulae by more general functions \(a, \beta\) defined on an interval \((r_0, \infty)\), which are assumed to be positive, strictly increasing and tending to infinity as \(r \to \infty\), and satisfying properties of classes \(L^\alpha\) and \(\Lambda\) defined below.

Let \(h(\xi)\) be a monotonically increasing function defined on \([a, \infty)\) for some \(a \geq 0\) and \(h(\xi) \to \infty\) as \(\xi \to \infty\). From Seremeta [18], \(h(\xi) \in L^\beta\) if, \(\phi(\xi) \to \infty\) as \(\xi \to \infty\), the equality
\[ \lim_{\xi \to \infty} h\left(1 + (1/\phi(\xi))\right) = 1. \tag{6} \]

The function \(h(\xi) \in \Lambda\) if, for all \(c, 0 < c < \infty\),
\[ \lim_{\xi \to \infty} \frac{h(c\xi)}{h(\xi)} = 1. \tag{7} \]

A function \(u\) is said to be an entire harmonic function if it is harmonic in the entire space \(\mathbb{R}^n\). We denote \(M(r, u) = \max_{x \in S^n} |u(rx)|\).

Considering the functions \(a, \beta\) from the classes \(L^\alpha\) and \(\Lambda\), following [18], Veselovska [14] defined the generalized order \(\rho_{a, \beta}(u)\) and lower order \(\lambda_{a, \beta}(u)\) of an entire harmonic function \(u \in \mathbb{R}^n\) by the following formulae:

\[ \rho_{a, \beta}(u) = \limsup_{r \to \infty} \frac{\alpha(\log M(r, u))}{\beta(r)}, \tag{8} \]
\[ \lambda_{a, \beta}(u) = \liminf_{r \to \infty} \frac{\alpha(\log M(r, u))}{\beta(r)}. \tag{9} \]

Kapoor and Nautiyal [16] defined generalized order \(\rho(a, u)\) of slow growth as
\[ \rho(a, u) = \limsup_{r \to \infty} \frac{\alpha(\log M(r, u))}{\alpha(\log r)}, \tag{9} \]

where \(a(\xi)\) either belongs to \(\Omega\) or \(\overline{\Omega}\). Here, \(\Omega\) is the class of functions \(h(\xi)\) defined on \([a, \infty)\), strictly increasing, and differentiable, and \(h(\xi) \to \infty\) as \(\xi \to \infty\) with the condition
\[ 0 < K_1 \leq \frac{d(h(\xi))}{d(\log \xi)} \leq K_2 < \infty, \tag{10} \]

for all \(\xi > \xi_0\) and \(\delta(\xi) \in \Omega\), and \(\overline{\Omega}\) is the class of functions \(h(\xi)\) which satisfies
\[ \lim_{\xi \to \infty} \frac{d(h(\xi))}{d(\log \xi)} = K, \quad 0 < K < \infty. \tag{11} \]

Kapoor and Nautiyal [16] showed that classes \(\Omega\) and \(\overline{\Omega}\) are contained in \(\Lambda\) and \(\Omega \cap \overline{\Omega} = \phi\).

From Vakarchuk and Zhir [15] for \(u \in H_R\), we set
\[ M_q(r, u) = \left(\int_{S^n} |u(re^{i\theta})|^q \,d\theta\right)^{1/q}, \tag{12} \]

\(q \in [1, \infty), \quad 0 < r < R, \quad 0 < R < \infty.\)

We assume the Hardy space of functions \(u(rx)\) denoted by \(H_q^r\) which satisfy the condition
\[ \|u\|_{H_q^r} = \lim_{r \to 0} M_q(r, u) < \infty, \tag{13} \]

and the Bergman space of functions \(u(rx)\) denoted by \(H_q^{**}\) satisfy the following relation:
\[ \|u\|_{H_q^{**}} = \left(\frac{1}{\omega_n} \int_{S^n} |u(y)|^q \,ds(y)\right)^{1/q} < \infty, \quad q = n - \frac{2}{2}. \tag{14} \]

For \(q = \infty\), let \(\|u\|_{H_q^{**}} = \|u\|_{H_q} = \sup_{y \in S^n} |u(y)|\), \(y \in S^n\). Then, \(H_q^{**}\) and \(H_q^{**}\) are Banach spaces for \(q \geq 1\). The spaces \(B(p, q, \lambda), 0 < p < q \leq \infty, \lambda > 0\), of functions \(u \in H_R\) with the norm
\[ \|u\|_{B(p, q, \lambda)} = \left\{ \int_{S^n} \left(\frac{(R-r)}{R}\right)^{(1/p)\lambda} |u(y)|^q \,dy\right\}^{1/q}, \quad \lambda < \infty, \]
\[ \|u\|_{B(p, q, \lambda)} = \sup_{\phi \in B(p, q, \lambda)} \left\{ \left(\frac{(R-r)}{R}\right)^{(1/p)\lambda} M_q(r, u)\right\}, \quad \lambda = \infty, \tag{15} \]

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for \( \min (q, \lambda) \geq 1, B(p, q, \lambda) \) are Banach spaces. It is known from [19] that \( B(p, q, \lambda) \) is a Banach space for \( p > 0 \) and \( q, \lambda \geq 1 \); otherwise, it is a Frechet space. From [20], we have

\[
H^*_q \subseteq H^{**}_q = B\left(\frac{q}{2}, q, q\right), \quad 1 \leq q < \infty. \tag{16}
\]

Let \( X \) denote one of the Banach spaces defined above, and the approximation error of functions \( u \in H_k \) by harmonic polynomials \( P \in \pi_k \) is defined as

\[
E_k(u, X) = \inf\{\|u(y) - P(y)\|_X, y \in B_k\}, \tag{17}
\]

where \( \pi_k \) denotes a set of harmonic polynomials of maximum degree \( k \).

Veselovska [14] studied generalized \((\alpha, \beta)\)-orders and generalized \((\alpha, \beta)\)-types of \( u \in H_k \) in terms of approximation errors \( E_k(u, X) \) and also the \((\alpha, \beta)\)-lower order in terms of ratios of approximation errors. The generalized type \( T(\alpha, u) \) of function \( u \in H_k \) with \( 0 < \rho(\alpha, u) < \infty \) is defined as

\[
T(\alpha, u) = \limsup_{r \to \infty} \frac{\alpha(\log M(r, u))}{[\alpha(\log r)]^\rho}, \tag{18}
\]

where \( \alpha(\xi) \) either belongs to \( \Omega \) or \( \Omega \).

This paper is organized as follows. First, the generalized type has been characterized in terms of coefficients occurring in the Fourier-Laplace series expansion of \( u(rx) \). Finally, the necessary and sufficient conditions of \( T(\alpha, u) \) for the function of order zero in certain Banach spaces have been investigated. Our results are different from those of Veselovska [14].

Notations are as follows:

1. \( F[\xi; T, \rho] = \alpha^{-1}\left\{ T[\alpha(\xi)]^{1/\rho}\right\} \) where \( \rho \) is nonzero finite and \( T = T + \epsilon \)
2. \( E[F[\xi; T, \rho]] \) denotes an integer part of \( F[\xi; T, \rho] \)

3. Main Results

**Theorem 1.** Let \( \alpha(\xi) \in \Omega \); then, the entire harmonic function \( u \in \mathbb{R}^n, n \geq 3 \), with nonzero finite \( \rho \), is of generalized type \( T \) if and only if

\[
\frac{\alpha(k/\rho)}{dF(\xi; T, \rho)/d \log \xi = O(1)} \text{ as } \xi \to \infty \forall \text{ nonzero finite } T.
\]

**Proof.** Let

\[
\limsup_{r \to \infty} \frac{\alpha(\log M(r, u))}{[\alpha(\log r)]^\rho} = T. \tag{20}
\]

Consider \( T < \infty \). Then, for every \( \epsilon > 0 \), there exists \( r(\epsilon) \) such that

\[
\frac{\alpha(\log M(r, u))}{[\alpha(\log r)]^\rho} \leq T + \epsilon = T_\forall r \geq r(\epsilon)
\]

or

\[
\log M(r, u) \leq \left\{ \alpha^{-1}\left[ T[\alpha(\log r)]^\rho\right]\right\}. \tag{21}
\]

Let \( r = r(k) \) satisfy the equation

\[
\frac{\alpha(k/\rho)}{dF(\xi; T, \rho)/d \log \xi = O(1)} \text{ as } \xi \to \infty \forall \text{ nonzero finite } T.
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Let \( r = r(k) \) satisfy the equation

\[
\frac{\alpha(k/\rho)}{dF(\xi; T, \rho)/d \log \xi = O(1)} \text{ as } \xi \to \infty \forall \text{ nonzero finite } T.
\]
\[
\max_{x \in S^p} |Y^{(k)}(x; u)| \leq \exp \left\{ -kF + \frac{k\rho}{\rho - 1} \right\} + \frac{\sqrt{2} (k + 2\nu)^{2\nu}}{\sqrt{(2\nu)!}} \\
\text{or } \frac{\rho}{\rho - 1} \log \left( \max_{x \in S^p} |Y^{(k)}(x; u)| \right)^{-1/k} \\
\geq \alpha^{-1} \left\{ \left[ \left( \frac{\alpha}{\rho} \right)^{1/(\rho - 1)} \right] \right\} + O(1), \\
\text{Inequality (26) obviously holds when } T = \infty. \\
\text{In order to prove the reverse inequality, let }
\]
(25)

\[
T \geq \limsup_{k \to \infty} \left\{ \alpha \left[ \frac{\rho}{(\rho - 1)} \log \left( \max_{x \in S^p} |Y^{(k)}(x; u)| \right)^{-1/(1/k)} \right] \right\}^{p - \eta} = \eta. \\
(26)
\]

Suppose \( \eta < \infty \). Then, for every \( \varepsilon > 0 \) and \( \forall k \geq N(\varepsilon) \), we have

\[
\alpha(k/\rho) \left\{ \frac{\rho}{(\rho - 1)} \log \left( \max_{x \in S^p} |Y^{(k)}(x; u)| \right)^{-1/(1/k)} \right\}^{p - \eta} = \eta + \varepsilon = \bar{\eta}. \\
(27)
\]

The inequality
\[
\left( \max_{x \in S^p} |Y^{(k)}(x; u)| \right)^{-1/(1/k)} \leq \exp \left( (\rho - 1)/(\rho - 1)F \cdot (k/\rho); (1/\rho - 1) \right) \leq \frac{1}{2} \\
(29)
\]
is fulfilled beginning with some \( k = k(r) \). Then,
\[
\sum_{k=k(r)+1}^{\infty} \max_{x \in S^p} |Y^{(k)}(x; u)|^{r/k} \leq \sum_{k=k(r)+1}^{\infty} \frac{1}{2^k} \leq 1. \\
(30)
\]

From (29), we have
\[
2r \leq e^{\left( (\rho - 1)/(\rho - 1)F \cdot (k/\rho); (1/\rho - 1) \right)}, \\
(31)
\]
and choose \( k(r) = E \left[ \rho \alpha^{-1} \left\{ T(\alpha/(\log r + \log 2))^{\rho - 1} \right\} \right] \). Consider the function \( \phi(\xi) = r^2 e^{-\left( (\rho - 1)/(\rho - 1)F \cdot (\xi/\rho); (1/\rho - 1) \right)} \). We have

\[
\frac{\phi'(\xi)}{\phi(\xi)} = \log r - \left( \frac{\rho - 1}{\rho} \right) F \left[ \frac{\xi}{\rho} - \frac{1}{\rho} \right] - \frac{dF[\xi/(\rho); (1/\rho - 1)]}{d\log \frac{\xi}{\rho}} = 0. \\
(32)
\]
As $ξ → ∞$, in view of the theorem, for $η(0 < η < ∞)$, $dF[(ξ/ρ): (1/η), ρ − 1]/d log ξ$ is bounded. So, there is $b > 0$ such that, for $ξ ≥ ξ$, we have

$$\left| dF[(ξ/ρ): (1/η), ρ − 1]/d log ξ \right| ≤ b.$$  \hspace{1cm} (33)

Choose $b > log 2$; then, inequalities (29) and (30) hold for $k > k_1(ξ) = E[ρα − 1\{η(α(log r + A))^{(ρ−1)}\}] + 1$. Let $k_0 = max(N(ε), E[ξ] + 1)$. For $r > r_1(k_0)$, we have $(ϕ(ξ)/φ(k_0)) > 0$. From (32) and (33), it follows that

$$\max_{k_0 < k < k_1(ξ)}\left(\max_{x \in S^r}Y^{(k)}(x; u)\right)^k ≤ \max_{k_0 < ξ < k_1(ξ)} ϕ(ξ) = \frac{ρα−1\{η(α(log r + A))^{(ρ−1)}\}}{exp[α(α(log r + A))^{(ρ−1)}]}$$  \hspace{1cm} (35)

Furthermore,

$$M(r, u) ≤ \sum_{k=0}^{∞} \max_{x \in S^r}Y^{(k)}(x; u)^k \leq O(r^{b_1}) + k_1(ξ) \max_{k_0 < k < k_1(ξ)}\left(\max_{x \in S^r}Y^{(k)}(x; u)^k\right) + 1$$  \hspace{1cm} (36)

$$= \exp\{α(α(log r + A))^{(ρ−1)}\},$$

$$\alpha(α(log M(r, u)) ≤ \bar{η}[α(α(log r + b))]^{ρ−1} ≤ \bar{η}[α(α(log r + b))].$$

Now, we have

$$\frac{α\left( (bp + o(1))^{-1}\log M(r, u) \right)}{[α(α(log r + b))]} ≤ \bar{η} = η + ε.$$  \hspace{1cm} (37)

Since $α(ξ) ∈ Ω < Λ$, proceeding to limits, we get

$$\frac{α(α(log M(r, u))}{[α(α(log r))]} ≤ η.$$  \hspace{1cm} (38)

Combining inequalities (26) and (38), we get the required result.

**Theorem 2.** Suppose $α(ξ) ∈ Ω$; then, a necessary and sufficient condition for $u ∈ B(p, q, λ)$ to be of generalized type $T$ with nonzero finite order $ρ$ is

$$T = \limsup_{k → ∞} \frac{α(k/ρ)}{[α(α(log r))]} ≤ η.$$  \hspace{1cm} (39)

**Proof.** In order to prove our theorem, first, we take the space $B(p, q, λ)$, $q = 2$, $0 < p < 2$, and $λ ≥ 1$. Assume that $u ∈ B(p, q, λ)$ having $T$ with order $ρ$. Using Theorem 1, it gives

$$\limsup_{k → ∞} \frac{α(k/ρ)}{[α(α(log r))]} ≤ η.$$  \hspace{1cm} (40)

For given $ε > 0$ and all $k > k_0 = k_0(ε)$, we have

$$\max_{x \in S^r}Y^{(k)}(x; u) ≤ \frac{1}{exp\{(ρ−1)(k/ρ)pF([k/ρ]; (1/T), ρ−1)]}.$$  \hspace{1cm} (41)

Let us consider the function $Q_κ(ξ) = \sum_{j=0}^{∞} Y^{(i)}(ξ; u)r^j$, $r > 0$ and $ξ ∈ S^r$. It is $κ$th partial sum of the Fourier–Laplace series of the harmonic function in $R^n$, $n ≥ 3$. Following pp.1396 of [21], we have
Here, $B(a,b)$ $(a,b > 0)$ has been used as the beta function. Using (41), we have

\[
E_k^\lambda (B(p,q,\lambda); u) \leq B^{1/\lambda} \left[ (k+1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right] \left\{ \sum_{j=1}^\infty \left( \max_{\xi < 0} Y^{(j)}(\xi; u)R^j \right)^2 \right\}^{1/2}.
\]  

(42)

where

\[
\psi_j(\alpha) = \frac{\exp\left\{ \left( (k+1)/\rho \right) (\rho - 1) \left[ \alpha^{-1}\left( \frac{(1/\rho)}{T + \varepsilon} \right)^{(p-1)} \right] \right\}}{\exp\left\{ \left( j/\rho \right) (\rho - 1) \left[ \alpha^{-1}\left( \frac{(1/\rho)}{T + \varepsilon} \right)^{(p-1)} \right] \right\}} R^{(k+1-j)}.
\]

(43)

Set

\[
\psi(\alpha) = \exp\left\{ -\frac{1}{\rho} (\rho - 1) \left[ \alpha^{-1}\left( \frac{(1/\rho)}{T + \varepsilon} \right)^{(p-1)} \right] \right\}.
\]

(45)

Taking \( j \geq k + 1 \) and since \( \alpha(\xi) \) is an increasing function, it gives

\[
\psi_j(\alpha) \leq \psi^{-(k+1)}(\alpha).
\]

(46)

Since \( \psi(\alpha) < 1 \), we get the following from equations (43) and (46):

\[
R^{-k} E_k^\lambda (B(p,q,\lambda); u) \leq \frac{B^{1/\lambda} \left[ (k+1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right]}{(1 - \psi^2(\alpha)))^{1/2} \left\{ \sum_{j=1}^\infty \left( \max_{\xi < 0} Y^{(j)}(\xi; u)R^j \right)^2 \right\}^{1/2}.\]

(47)

For \( k > k_0 \), equation (47) gives

\[
T + \varepsilon \geq \frac{\alpha(\frac{1}{p})(1+1/n)\log\left( R \left\{ E_k^\lambda \right\}^{1/\lambda} \right) + \log \left( \left( B^{1/\lambda} \left( k+1 \lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right) \right) \left( \left( 1 - \psi^2(\alpha) \right)^{1/2} \right)^{1/\lambda} \right)^{1/k} \right]{\varepsilon}^{p-1}.
\]

(48)

We have
Now, taking equation (40) into account, we get
\begin{equation}
B \left[ (k + 1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right] = \frac{\Gamma ((k + 1)\lambda + 1)\Gamma ((1/p) - (1/2))}{\Gamma ((k + (1/2) + (1/p))\lambda + 1)}. \tag{49}
\end{equation}

Hence,
\begin{equation}
B \left[ (k + 1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right] = \frac{e^{-\left[(k+1)\lambda]\right]}}{e^{[\lambda(1/p)\lambda + (1/2)]}} \frac{\Gamma ((1/p) - (1/2))}{\Gamma ((k + (1/2) + (1/p))\lambda + 1)}. \tag{50}
\end{equation}

Thus,
\begin{equation}
\left\{ B \left[ (k + 1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right] \right\}^{1/(k+1)} = 1. \tag{51}
\end{equation}

In view of equation (51), we have
\begin{equation}
E^k_B (B(p, 2, \lambda); u) \geq \max_{\xi \in S^n} Y^{(k+1)} (\xi; u) R^{(k+1)} B^{(1/2)} \left[ (k + 1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{2} \right) \right]. \tag{53}
\end{equation}

Then, for sufficiently large \( k \), we obtain
\begin{equation}
\alpha (k/\rho)
\left[ \alpha \left( \rho/(\rho - 1) \right) \log \left( R^{(1/2)} (B(p, q, \lambda); u) \right) \right] = \left[ \alpha \left( \rho/(\rho - 1) \right) \log \left( \max_{\xi \in S^n} Y^{(1/2)} (\xi; u) \right) \right] \geq \alpha (k/\rho)
\end{equation}

where \( S^* = \log (B^{-\rho/(\rho k)} (k + 1)\lambda + 1; \lambda (1/p - (1/2))) \).

Now, taking equation (40) into account, we get
\begin{equation}
\limsup_{k \to \infty} \frac{\alpha (k/\rho)}{\left( \alpha \left( \rho/(\rho - 1) \right) \log \left( R^{(1/2)} (B(p, q, \lambda); u) \right) \right)} = T. \tag{55}
\end{equation}

Combining equations (52) and (55), we obtain the result
\begin{equation}
T = \limsup_{k \to \infty} \frac{\alpha (k/\rho)}{\left( \alpha \left( \rho/(\rho - 1) \right) \log \left( R^{(1/2)} (B(p, q, \lambda); u) \right) \right)} \geq T. \tag{56}
\end{equation}

Now, we discuss the spaces \( B(p, q, \lambda) \) for \( 0 < p < q, q \neq 2, \) and \( q, \lambda \geq 1. \) For \( p \geq p_1, q \leq q_1, \) and \( \lambda \leq \lambda_1, \) Gvaradze [19] observed that if at least one of the inequalities is strict, then \( B(p, q, \lambda) \subset B(p_1, q_1, \lambda_1) \) holds, and it gives
\[ \|u\|_{p,q,\lambda} \leq 2^{(1/q)-(1/p)} \left[ \lambda \left( \frac{1}{p} - \frac{1}{q} \right) \right]^{(1/\lambda)-(1/p)} \|u\|_{p,q}. \]  

(57)

For any harmonic function \( u(rx) \in B(p,q,\lambda) \), the last relation yields

\[ E^k_R(B(p,q,\lambda);u) \leq 2^{(1/2)-(1/q)} \left[ \lambda \left( \frac{1}{p} - \frac{1}{q} \right) \right]^{(1/\lambda)-(1/p)} E^k_R(B(p,q,\lambda);u). \]  

(58)

We will prove the necessity of condition (39) for \( B(p,q,\lambda), \, q \neq 2 \).

\[ E^k_R(B(p,q,\lambda);u) = \|u(\tau \zeta) - Q(\tau \zeta)\|_{p,q,\lambda} \leq \left( \int_{s^n} \left( \frac{R-r}{R} \right)^{((1/p)-(1/q)) - 1} M^\lambda_\zeta(r,u)dr \right)^{1/\lambda}. \]  

(59)

Hence,

\[ |u(rx)|^q \leq \left( \sum_{k=0}^{\infty} Y^{(k)}(x;u) r^k \right)^q \leq \left( \sum_{j=k+1}^{\infty} Y^{(j)}(x;u) \right)^q. \]  

(60)

\[ E^k_R(B(p,q,\lambda);u) \leq B^{1/\lambda} \left[ (k+1)\lambda + 1; \lambda \left( \frac{1}{p} - \frac{1}{q} \right) \right] \sum_{j=k+1}^{\infty} Y^{(j)}(x;u) \]

\[ \leq \frac{B^{1/\lambda} [(k+1)\lambda + 1; \lambda ((1/p)-(1/q))] \exp \left( \frac{1}{(1-\psi(\alpha))} \left[ \alpha^{-1} \left( \left( \alpha ((k+1)/\rho)\right)(\rho - 1) \left( (\alpha ((k+1)/\rho) (T + \varepsilon) - 1) \right) \right] \right] \right)}{(1 - \psi(\alpha))}. \]  

(61)

For \( k > k_0 \), from (61), we have

\[ T + \varepsilon \geq \alpha ((k+1)/\rho) \left\{ \alpha \left( \rho/(1+1/n) \right) \log \left( R \right)^{1/\lambda} \left( \log \left( R \right)^{1/\lambda} \right) + \log \left( \left( B^{1/\lambda} [(k+1)\lambda + 1; \lambda ((1/p)-(1/2))] \right)/\left( (1 - \psi(\alpha)) \right) \right) \right\}^{\rho - 1}. \]  

(62)

As \( \psi(\alpha) < 1 \) and \( \alpha \in \bar{\Omega} \), in view of equation (51), we obtain
where \( \alpha \) is a positive constant. Using equation (69) and following on the lines of the previous case where \( S_0 \), obtain

\[
\limsup_{k \to \infty} k \rho \tau_0 \left\{ \alpha \left( \frac{\rho}{(\rho - 1) \log \left( \frac{E_k^B(B(p,q,\lambda);u)}{E_{k-1}^B(B(p,q,\lambda);u)} \right)^{(1/k)}} \right) \right\}^{p-1} = T. 
\]

(63)

Now, consider \( 0 < p \leq q < 2 \), and we know that \( M_2(r,u) \leq M_q(r,u), \quad 0 < r < R, \) and it gives

\[
E_k^B(B(p,q,\lambda);u) \geq \max_{x \in B^q} Y^{(k+1)}(\zeta;u) R^{(k+1)} B^{\lambda \zeta} \left( k + 1 \lambda + \lambda \left( \frac{1}{p} - \frac{1}{q} \right) \right).
\]

(66)

For a sufficiently large value of \( k \), we have

\[
\limsup_{k \to \infty} k \rho \tau_0 \left\{ \alpha \left( \frac{\rho}{(\rho - 1) \log \left( \frac{E_k^B(X;u)}{E_{k-1}^B(X;u)} \right)^{(1/k)}} \right) \right\}^{p-1} \geq \limsup_{k \to \infty} k \rho \tau_0 \left\{ \alpha \left( \frac{\rho}{(\rho - 1) \log \left( \frac{\max_{x \in B^q} Y^{(k+1)}(x;u) R^{(k+1)}}{\max_{x \in B^q} Y^{(k)}(x;u) R^{(k)}} \right)^{(1/k)}} \right) \right\}^{p-1} \geq \limsup_{k \to \infty} k \rho \tau_0 \left\{ \alpha \left( \frac{\rho}{(\rho - 1) \log \left( \frac{\max_{x \in B^q} Y^{(k+1)}(x;u) R^{(k+1)}}{\max_{x \in B^q} Y^{(k)}(x;u) R^{(k)}} \right)^{(1/k)}} \right) \right\}^{p-1} \]

(67)

where \( S^{**} = \log (B^{-r/(p\lambda)}) [(k + 1)\lambda + 1; (1/p) - (1/q)] \).

Now, proceeding to limits and using equation (40), we have

\[
\limsup_{k \to \infty} k \rho \tau_0 \left\{ \alpha \left( \frac{\rho}{(\rho - 1) \log \left( \frac{Y^{(k)}(x;u)}{Y^{(k+1)}(x;u)} \right)^{(1/k)}} \right) \right\}^{p-1} = T. 
\]

(68)

Now, assume that \( 2 \leq p < q \). Set \( q_1 = q, \lambda_1 = \lambda \), and \( 0 < p_1 < 2 \) in equation (58), and putting \( p_1 \) for \( p \) in (66), we obtain

\[
E_k^B(B(p,q,\lambda);u) \geq \left| Y^{(k+1)}(x;u) R^{(k+1)} B^{\lambda \zeta} \left( k + 1 \lambda + \lambda \left( \frac{1}{p_1} - \frac{1}{q} \right) \right) \right|
\]

(69)

Using equation (69) and following on the lines of the previous case \( 0 < p \leq q < 2 \), for sufficiently large \( k \), we obtain
\[
\limsup_{k \to \infty} \left[ a\left(\frac{\alpha}{\rho}\right) \right]^{p-1} \leq T.
\]

(70)

Combining inequalities (52) and (55) with (70), we get the required result.

**Theorem 3.** Let \( \alpha \in \overline{\mathbb{H}} \). A necessary and sufficient condition for \( u(x) \in \mathcal{H}_k^n \) to be an entire harmonic function in space \( \mathbb{R}^n \), \( n \geq 3 \), of generalized type \( W(\alpha) > 0 \) with \( 0 < \rho < \infty \) is

\[
\limsup_{k \to \infty} \left[ a\left(\frac{\rho}{\alpha}\right) \right]^{p-1} = W(\alpha).
\]

(71)

Proof. Let \( u(x) = \sum_{k=0}^{\infty} Y^{(k)}(x; \rho) \) be an entire harmonic function with nonzero finite order \( \rho \) and generalized type \( T \). Since

\[
\lim_{k \to \infty} |Y(\alpha)|^{1/k} = 0,
\]

(72)

\( u(x) \in B(\rho, q, \lambda) \), with \( 0 < \rho < q \leq \infty \) and \( q, \lambda \geq 1 \). In view of [1], it gives

\[
E_k^\alpha(B(\frac{q}{2}, q, q); u) \leq \eta_{\alpha}^{\ast} E_k^\alpha(\mathcal{H}_k^n; u), \quad 1 \leq q < \infty,
\]

(73)

where \( \eta_{\alpha}^{\ast} \) is a constant independent of \( \alpha \) and \( u \).

For \( \mathcal{H}_k^n \), we have

\[
E_k^\alpha(B(\rho, q, \infty, \infty); u) \leq E_k^\alpha(\mathcal{H}_k^n; u), \quad 1 < \rho < \infty.
\]

(74)

Set

\[
W(\alpha, u) = \limsup_{k \to \infty} \left[ a\left(\frac{\rho}{\alpha}\right) \right]^{p-1}
\]

(75)

\[
\geq \limsup_{k \to \infty} \left[ a\left(\frac{\rho}{\alpha}\right) \right]^{p-1}
\]

(76)

In view of equation (74), we can easily prove inequality (75) for \( q = \infty \). To prove the reverse inequality we use relation (41) as follows:

\[
E_k^\alpha(\mathcal{H}_k^n; u) \leq \|u(x) - Q_k(x)\|_{\mathcal{H}_k^n} \leq \sum_{j=k+1}^{\infty} \max_{(j, k)} |Y^{(j)}(\xi); u| R^j
\]

\[
\leq \left[ \exp\left(\frac{(k+1)/\rho)(\rho-1)}{\alpha-1}\left\{((\alpha((k+1)/\rho))(T+\varepsilon))^{(p-1)}\right\}\right]\right] \sum_{j=k+1}^{\infty} R^j \psi_j(\alpha).
\]

(77)

In view of equation (46), we have

\[
E_k^\alpha(\mathcal{H}_k^n; u) \leq \|u(x) - Q_k(x)\|_{\mathcal{H}_k^n}
\]

\[
\leq \frac{1}{(1 - \psi(\alpha)) \left[ \exp\left(\frac{(k+1)/\rho)(\rho-1)}{\alpha-1}\left\{((\alpha((k+1)/\rho))(T+\varepsilon))^{(p-1)}\right\}\right]\right] \sum_{j=k+1}^{\infty} R^j \psi_j(\alpha),
\]

(78)

This gives
\[
T + \varepsilon \geq \frac{\alpha((k + 1)/\rho)}{\log\left(\left[\log\left|\mathcal{R}\mathcal{E}_n^q(H^\ast, u)\right|^{-1/(k+1)}\right]\right)} + \log\left((1 - \psi(\alpha))^{-1/(k+1)}\right)
\]

Proceeding to limits as \( k \to \infty \) in (79), we obtain equation (76). Hence, we have

\[
W(\alpha, u) = T.
\]  

(80)

This completes the proof of Theorem 3.

The case of Bergman spaces for \( 1 \leq q < \infty \) follows from Theorem 3 and for \( 1 \leq q < \infty \) from Theorem 2.

4. Conclusion

Kapoor and Nautiyal [4, 16] studied the growth of functions harmonic in \( \mathbb{R}^3 \). These results were generalized by Veselovska [6, 14] for general dimension. In this paper, we have investigated the slow growth of entire harmonic functions in general dimension and added new aspects in the results of Veselovska.

Data Availability

No data were used to support this study.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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