Research Article

Characterization of Lipschitz Spaces via Commutators of Maximal Function on the $p$-Adic Vector Space

Qianjun He and Xiang Li

1School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China
2School of Science, Shandong Jianzhu University, Jinan 250000, China

Correspondence should be addressed to Xiang Li; lixiang162@mails.ucas.ac.cn

Received 5 August 2022; Revised 8 November 2022; Accepted 10 November 2022; Published 5 December 2022

Academic Editor: Humberto Rafeiro

Copyright © 2022 Qianjun He and Xiang Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we give characterization of a $p$-adic version of Lipschitz spaces in terms of the boundedness of commutators of maximal function in the context of the $p$-adic version of Lebesgue spaces and Morrey spaces, where the symbols of the commutators belong to the Lipschitz spaces. A useful tool is a Lipschitz norm involving the John-Nirenberg-type inequality for homogeneous Lipschitz functions, which is new in the $p$-adic field context.

1. Introduction and Statement of Main Results

For a prime number $p$, let $\mathbb{Q}_p$ be the field of $p$-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic norm $| \cdot |_p$. This norm is defined as follows: $|0|_p = 0$. If any nonzero rational number $x$ is represented as $x = p^m(m/n)$, where $m$ and $n$ are integers which are not divisible by $p$, and $y$ is an integer, then $|x|_p = p^{-y}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

(1)

It follows from the second property that when $|x|_p \neq |y|_p$, then $|x + y|_p = \max\{|x|_p, |y|_p\}$. From the standard $p$-adic analysis [1], we see that any nonzero $p$-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j,$$

(2)

where $a_j$ are integers, $0 \leq a_j \leq p - 1$, $a_0 \neq 0$. Series (2) converges in the $p$-adic norm because $|a_j p^j|_p = p^{-j}$.

The space $\mathbb{Q}_p^d$ consists of points $x = (x_1, x_2, \ldots, x_d)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, d$. The $p$-adic norm on $\mathbb{Q}_p^d$ is $|x|_p := \max_{1 \leq j \leq d}|x_j|_p$ for $x \in \mathbb{Q}_p^d$. Denote by $B_p(a) = \{x \in \mathbb{Q}_p^d, |x - a|_p \leq p^\gamma\}$, the ball with center at $a \in \mathbb{Q}_p^d$ and radius $p^\gamma$, and by $S_p(a) = \{x \in \mathbb{Q}_p^d |x - a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^d$ and radius $p^\gamma$, $\gamma \in \mathbb{Z}$. It is clear that $S_p(a) = B_p(a)/B_{p^{-\gamma}}(a)$ and $B_p(a) = \bigcup_{k \in \mathbb{Z}} S_p(a)$.

Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Harr measure $dx$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant. We normalize the measure $dx$ so that

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

(3)

where $|E|_H$ denotes the Harr measure of a measurable subset $E$ of $\mathbb{Q}_p^d$. From this integral theory, it is easy to obtain that $|B_p(a)|_H = p^{d\gamma}$ and $|S_p(a)|_H = p^{d\gamma} (1 - p^{-d})$ for any $a \in \mathbb{Q}_p^d$. 


In what follows, we say that a (real-valued) measurable function $f$ defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d)$, $1 \leq q \leq \infty$, if it satisfies

$$\|f\|_{L^q(\mathbb{Q}_p^d)} := \left(\int_{\mathbb{Q}_p^d} |f(x)|^q dx\right)^{1/q} < \infty, \quad 1 \leq q < \infty,$$

$$\|f\|_{L^\infty(\mathbb{Q}_p^d)} = \inf\{\alpha: \|f(x) - \alpha\|_H = 0\} < \infty. \quad (4)$$

Here the integral in equation (4) is defined as follows:

$$\int_{\mathbb{Q}_p^d} |f(x)|^q dx = \lim_{y \to \infty} \int_{B_y(0)} |f(x)|^q dx = \lim_{y \to \infty} \sum_{x \in \mathbb{Q}_p^d \cap B_y(0)} |f(x)|^q dy. \quad (5)$$

If the limit exists. We now mention some of the previous works on harmonic analysis on the $p$-adic field, see [2–8] and the references therein.

For a function $f \in L^1_{loc}(\mathbb{Q}_p^d)$, we defined the Hardy-Littlewood maximal function of $f$ on $\mathbb{Q}_p^d$ by the following equation:

$$M^p(f)(x) = \sup_{y \in \mathbb{Z}} \frac{1}{|B_y(x)|_H} \int_{B_y(x)} |f(y)| dy. \quad (6)$$

In [9, 10], Kim proved $L^p$ boundedness of the version of maximal function $M^p$ and gave some properties similar to the Euclidean setting.

The maximal commutator of $M^p$ with a locally integrable function $b$ is defined by the following equation:

$$\mathcal{M}^p_b(f)(x) = \sup_{y \in \mathbb{Z}} \frac{1}{|B_y(x)|_H} \int_{B_y(x)} |b(x) - b(y)f(y)| dy. \quad (7)$$

The first part of this paper is to study the boundedness of $\mathcal{M}^p_b$ when the symbol belongs to a Lipschitz space (see in Section 2). Some characterizations of the Lipschitz space via such commutator are given. Our first result can be stated as follows:

**Theorem 1.** Let $b$ be a locally integrable function and $0 < \beta < 1$, then the following statements are equivalent.

1. $b \in \Lambda^\beta_{\mathbb{Q}_p^d}$;
2. $\mathcal{M}^p_b$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$ for all $q, r$ with $0 < q < d/\beta$ and $1/r = 1/q - \beta/d$;
3. $\mathcal{M}^p_b$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$ for some $q, r$ with $0 < q < d/\beta$ and $1/r = 1/q - \beta/d$;
4. $\mathcal{M}^p_b$ satisfies the weak-type $(1, d/(d - \beta))$ estimates, namely, there exists a positive constant $C$ such that for any $\lambda > 0$,

$$\left\{|x \in \mathbb{Q}_p^d: \mathcal{M}^p_b(f)(x) > \lambda\right\} \leq C \left(\frac{\|f\|_{L^1(\mathbb{Q}_p^d)}}{\lambda}\right)^{d/(d - \beta)}, \quad (8)$$

5. $\mathcal{M}^p_b$ is bounded from $L^{d/\beta}(\mathbb{Q}_p^d)$ to $L^\infty(\mathbb{Q}_p^d)$.

We remark that the boundedness of the commutators of maximal function were pretty much unknown on the $p$-adic vector space. However, in Euclidean case, the mapping properties of the maximal commutator have been studied intensively by many authors, see [11–21].

It is well-known that the Morrey space introduced by Morrey in 1938, is connected to certain problem in elliptic PDE [22]. Later the Morrey spaces were found to have many important applications to the Navier-Stokes equations, the Schrödinger equations, the elliptic equations with discontinuous coefficients and the potential analysis, see [23–27] and so on. Next, we introduce the $p$-adic version of Morrey space on $p$-adic vector space.

For $1 \leq q < \infty$ and $0 \leq \lambda \leq d$. The $p$-adic version of Morrey space is defined by the following equation:

$$L^{p, \lambda}(\mathbb{Q}_p^d) = \left\{f \in L^1_{loc}(\mathbb{Q}_p^d): \|f\|_{L^p(\mathbb{Q}_p^d)} < \infty \right\}, \quad (9)$$

where

$$\|f\|_{L^p(\mathbb{Q}_p^d)} = \sup_{y \in \mathbb{Z}} \frac{1}{|B_y(x)|_H} \int_{B_y(x)} |f(x)|^p dx. \quad (10)$$

It is well known that if $1 \leq q < \infty$, then $L^{p,q}(\mathbb{Q}_p^d) = L^q(\mathbb{Q}_p^d)$ and $L^{p,q}(\mathbb{Q}_p^d) = L^\infty(\mathbb{Q}_p^d)$.

**Theorem 2.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - \beta/\lambda$ and $1/r = 1/p - \beta/(d - \lambda)$. Then, $b \in \Lambda^\beta_{\mathbb{Q}_p^d}$ if and only if $\mathcal{M}^p_b$ is bounded from $L^{1, \lambda}(\mathbb{Q}_p^d)$ to $L^{p, \lambda}(\mathbb{Q}_p^d)$.

**Theorem 3.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - \beta/(d - \lambda)$ and $1/r = 1/p - \beta/d$, and $\nu r = \lambda/q$. Then, $b \in \Lambda^\beta_{\mathbb{Q}_p^d}$ if and only if $\mathcal{M}^p_b$ is bounded from $L^{1, \nu r}(\mathbb{Q}_p^d)$ to $L^{p, \lambda}(\mathbb{Q}_p^d)$.

On the other hand, the classical commutator of the $p$-adic version of maximal function $\mathcal{M}^p$ with a locally integrable function $b$ can be defined by the following equation:

$$[b, \mathcal{M}^p](f)(x) = b(x)\mathcal{M}^p(f)(x) - \mathcal{M}^p(bf)(x). \quad (11)$$

In the Euclidean setting, the commutator of maximal function as in (11) has attracted much more attention. For examples, Milman and Schonbek [28] established a commutator result by real interpolation techniques. As an application, they obtained the $L^1$-boundedness of the commutator of maximal function $[b, M]$ when $b \in BMO(R^d)$ and $b \geq 0$. This operator can be used in studying the product of a function in $H^1$ and a function in $BMO$ (see [29] for instance). Bastero et al. [30] studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on $L^q$ spaces when $1 < q < \infty$. Zhang and Wu [31] extended their results to commutators of the fractional maximal function. The results in [30, 31] were extended to variable Lebesgue spaces in [32, 33]. Recently, Zhang [20] studied the commutator $[b, M]$ when $b$ belongs to Lipschitz
spaces. Some necessary and sufficient conditions for the boundedness of \([b, M]\) on Lebesgue and Morrey spaces are given. Some of the results were extended to variable Lebesgue spaces in [21]. For more information about the characterization of the commutator of maximal operator, see also [19, 20] and the references therein.

We would like to remark that operators \(\mathcal{M}_{p}^b\) and \([b, \mathcal{M}]\) essentially differ from each other. For example, \(\mathcal{M}_{p}^b\) is positive and sublinear, but \([b, \mathcal{M}]\) is neither positive nor sublinear. Motivated by the papers mentioned above, in this paper, the second part of this paper aims to study the mapping properties of the commutator \([b, \mathcal{M}]\) when \(b\) belongs to some Lipschitz spaces. More precisely, we will give some new necessary and sufficient conditions for the boundedness of \([b, \mathcal{M}]\) on \(p\)-adic vector spaces, by which some new characterizations for certain subclasses of Lipschitz spaces are obtained.

**Theorem 4.** Let \(b \) be a locally integrable function and \(0 < \beta < 1\). Assume that \(1 < q < d / \beta \) and \(1 / r = 1 / p - \beta / d\). Then, the following statements are equivalent:

1. \(b \in \Lambda_{\rho}(Q_{p}^{d})\) and \(b \geq 0\);
2. \([b, \mathcal{M}]\) is bounded from \(L^{q}(Q_{p}^{d})\) to \(L^{r}(Q_{p}^{d})\);
3. There exists a constant \(C > 0\) such that

\[
\sup_{x \in \Omega_{p}^{d}} \sup_{y \in \mathbb{R}^{d}} \frac{1}{|B_{y}(x)|^{\beta/d}} \int_{B_{r}(x)} |b(y) - \mathcal{M}_{B_{r}(x)}^{p}(b)(y)|^{r/y} dy \leq C. \tag{12}
\]

Where \(\mathcal{M}_{B_{r}(x)}^{p}(f)(y) = \sup_{z \in B_{r}(y)} \frac{1}{|B_{y}(z)|^{\beta/d}} \int_{B_{y}(z)} |f(z)| dz\). (13)

Here, the supremum is taken over all the \(p\)-adic \(B_{r}(y)\) with \(B_{r}(y) \subseteq B_{r}(x)\) for a fixed \(p\)-adic ball \(B_{r}(x)\).

**Theorem 5.** Let \(b \geq 0\) be a locally integrable function, \(0 < \beta < 1\) and \(b \in \Lambda_{\rho}(Q_{p}^{d})\). Then, there exist a positive \(C\) such that for any \(\lambda > 0\),

\[
\left| \left\{ x \in \Omega_{p}^{d} : [b, \mathcal{M}] (f)(x) > \lambda \right\} \right|_{1} \leq C \left( \frac{\|f\|_{L^{1}(Q_{p}^{d})}}{\lambda} \right)^{d/(d - \beta)}. \tag{14}
\]

**Theorem 6.** Let \(b \) be a locally integrable function and \(0 < \beta < 1\). Assume that \(1 < q < d / \beta \), \(0 < \lambda < d - q \beta\), and \(1 / r = 1 / p - \beta / (d - \lambda)\). Then, the following statements are equivalent

1. \(b \in \Lambda_{\rho}(Q_{p}^{d})\) and \(b \geq 0\);
2. \([b, \mathcal{M}]\) is bounded from \(L^{q}(Q_{p}^{d})\) to \(L^{r}(Q_{p}^{d})\).

**Theorem 7.** Let \(b \) be a locally integrable function and \(0 < \beta < 1\). Assume that \(1 < q < d / \beta \), \(0 < \lambda < d - q \beta\), \(1 / r = 1 / p - \beta / d\), and \(v / r = \lambda / q\). Then, the following statements are equivalent.

\[
\|f\|_{\operatorname{Lip}_p(Q_{p}^{d})} = \sup_{x \in \Omega_{p}^{d}} \sup_{y \in \mathbb{R}^{d}} \frac{1}{|B_{y}(x)|^{\beta/d}} \int_{B_{r}(x)} \left| f(y) - f_{B_{r}(x)} \right|^{q/y} dy < \infty. \tag{16}
\]

\[
\|f\|_{L_{\rho}^{q}(Q_{p}^{d})} = \sup_{x \in \Omega_{p}^{d}} \sup_{y \in \mathbb{R}^{d}} \frac{1}{|B_{y}(x)|^{\beta/d}} \int_{B_{r}(x)} \left| f(y) - f_{B_{r}(x)} \right|^{q/y} dy < \infty. \tag{15}
\]

Remark 1. Theorem 6 is essentially the Adams-type result and Theorem 7 is the Spanne-type result. Also, Theorem 7 can be immediately proved via Theorem 6.

The rest of the present paper is organized as follows: in Section 2, we gave some definitions and lemmas. The Proof of Theorems 1–3 are presented in Section 3. In Section 4, we gave the proof of Theorems 4–7. By \(A \preceq B\) we mean that \(A \leq cB\) with some positive constant \(c\) independent of appropriate quantities. The positive constants \(C\) varies from one occurrence to another. For a real number \(q, 1 < q < \infty, q'\) is the conjugate number of \(q\), that is, \(1/q + 1/q' = 1\).

**2. Preliminary Definitions and Lemmas**

To prove our main results, we need the following definitions and lemmas.

**Definition 1.** Assume that \(0 < \beta < 1\). The \(p\)-adic version of homogeneous Lipschitz spaces \(\Lambda_{\rho}(Q_{p}^{d})\) (see [34]) is the space of all measurable functions \(f \) on \(Q_{p}^{d}\) such that

\[
\|f\|_{\Lambda_{\rho}(Q_{p}^{d})} = \sup_{x \in \Omega_{p}^{d}, y \neq x} \frac{|f(x) - f(y)|}{|x - y|_{p}} < \infty. \tag{15}
\]

For \(1 \leq q < \infty\), the \(p\)-adic version of Lipschitz spaces \(\operatorname{Lip}_p(Q_{p}^{d})\) is the space of all measurable functions \(f \) on \(Q_{p}^{d}\) such that

\[
\|f\|_{\operatorname{Lip}_p(Q_{p}^{d})} = \sup_{x \in \Omega_{p}^{d}} \sup_{y \in \mathbb{R}^{d}} \frac{1}{|B_{y}(x)|^{\beta/d}} \int_{B_{r}(x)} \left| f(y) - f_{B_{r}(x)} \right|^{q/y} dy < \infty. \tag{16}
\]
where $f_{B_i}(x)$ denotes the average of $f$ over $B_i(x)$, i.e., $f_{B_i}(x) = \frac{1}{|B_i(x)|} \int_{B_i} f(y) dy$. When $q = 1$, we use $\text{Lip}_p(Q^d_p)$ as $\text{Lip}_1(Q^d_p)$.

We shall give the $p$-adic version of John-Nirenberg inequality for homogeneous Lipschitz functions in the following lemma.

**Lemma 1.** Let $0 < \beta < 1$, then there exist some $c_1, \ c_2 > 0$ depending on $p$ and $d$ such that

$$\left( x \in B : |B|^d_{H} |f(x) - f_B| > \lambda \right) \leq c_1 \exp \left( \frac{c_2 \lambda}{\|f\|_{\text{Lip}_p(Q^d_p)}} \right) |B|^H_{H},$$

for any $f \in \text{Lip}_p(Q^d_p)$ and for any $p$-adic $B \in \mathcal{F}_p$, where $\mathcal{F}_p$ denotes the family of all the $p$-adic balls which is defined as follows:

$$\mathcal{F}_p = \{ B_p(x) : y \in \mathbb{Z}, \ x \in Q^d_p \}.$$

Kim in [9] observed several interesting properties on the family $\mathcal{F}_p$.

**Lemma 2.** The family $\mathcal{F}_p$ has the following properties:

$$|B|^{-\beta d} |f_B - f_{B_j}| \leq \frac{1}{|B|^H_{H}} \int_{B_j} |B|^{-\beta d} |f(x) - f_B| dx + \frac{1}{|B|^H_{H}} \int_{B_j} |B|^{-\beta d} |f(x) - f_{B_j}| dx \leq p^d \lambda_0 + \|f\|_{\text{Lip}_p(Q^d_p)}.$$  

(a) If $y \leq y'$, then either $B_\gamma(x) \cap B_{\gamma'}(y) = \emptyset$ or $B_\gamma(x) \subset B_{\gamma'}(y)$.

(b) $B_\gamma(x) = B_\gamma(y)$ if and only if $y \in B_\gamma(x)$.

**Proof of Lemma 1.** Let $B \in \mathcal{F}_p$ be a fixed $p$-adic ball and $\lambda_0$ be some positive real number which will be determined later. Applying the $p$-adic version of the Calderón-Zygmund decomposition (see Corollary 3.4 in [9], or see also Theorem 5.16 in [9]) of $B$ for $|B|^{-\beta d} |f(x) - f_B|$ relative to $\lambda_0$ to obtain a pair-wise disjoint family of $p$-adic balls $\{ B_j \}_{j=1}^\infty \subset \mathcal{F}_p$ (by Lemma 2) which satisfies the following equation:

$$\left\{ \frac{x \in B}{\cup_{j=1}^\infty B_j}, |B|^H_{H} |f(x) - f_B| > \lambda \right\} = 0,$$

for $\lambda_0 < \frac{1}{|B|^H_{H}} \int_{B_j} |B|^{-\beta d} |f(x) - f_B| dx \leq p^d \lambda_0$, for any $j$.  

and

$$\sum_{j=1}^\infty |B_j|^H_{H} \leq \frac{1}{\lambda_0} \int_{B_j} |B|^{-\beta d} |f(x) - f_B| dx.$$

Combining $\beta > 0$ and equation (20) gives the following equation:

$$|B|^{-\beta d} |f(x) - f_B| \leq |B|^{-\beta d} |f_B - f_{B_j}| + |B|^{-\beta d} |f(x) - f_{B_j}| \leq t + |B|^{-\beta d} |f(x) - f_{B_j}|.$$

By using equation (19), we can infer that for any $\lambda > 0$

$$\left( x \in B : |B|^d_{H} |f(x) - f_B| > \lambda + t \right) \leq \left( x \in \cup_{j=1}^\infty B_j : |B|^{-\beta d} |f(x) - f_B| > \lambda + t \right) + \left( x \in B_{\beta d} : |B|^d_{H} |f(x) - f_B| > \lambda + t \right).$$

Thus, equations (23) and (24) now implies.
\[ \left\{ x \in B: |B|^{-\beta_d} \| f(x) - f_B \| > \lambda + t \right\} \leq \sum_{j=1}^{\infty} \frac{1}{|B_j|} \left\{ x \in B_j: |B_j|^{-\beta_d} \| f(x) - f_B \| > \lambda + t \right\} \| B_j \|_H \leq \sum_{j=1}^{\infty} \left\{ x \in B_j: |B_j|^{-\beta_d} \| f(x) - f_B \| > \lambda \right\} \| B_j \|_H. \] (25)

For any \( \lambda > 0 \), set
\[ G_f(\lambda) = \sup_{B \in F_p} \frac{1}{|B|_H} \left\{ x \in B: |B|^{-\beta_d} \| f(x) - f_B \| > \lambda \right\} \| B \|_H. \] (26)

Hence, for any \( \lambda > 0 \), we conclude that
\[ G_f(\lambda + t) \leq \frac{G_f(\lambda)}{e}. \] (28)

Taking \( \lambda_0 = e\| f \|_{\text{Lip}_p(Q_p^d)} \), then \( t \) is also a fixed positive number and for any \( \lambda_0 \geq 0 \),
\[ G_f(\lambda + t) \leq \frac{G_f(\lambda)}{e}. \] (29)

Using induction argument for any \( k \geq 1 \), we obtain the following equation:
\[ G_f((k + 1)t) \leq e^{-k}G_f(t). \] (30)

Thus, for \( \lambda \in (kt,(k + 1)t) \), we have the following equation:
\[ G_f(\lambda) \leq G_f(kt) \leq e^{-k}G_f(0) \leq e^{-\lambda t}. \] (31)

Notice that this inequality is also true for \( \lambda \in [0, t] \), due to \( G_f(\lambda) \leq G_f(0) = 1 \leq e^{-\lambda t} \). Therefore, for any \( \lambda \geq 0 \), we conclude that
\[ \frac{1}{|B|_H} \left\{ x \in B: |B|^{-\beta_d} \| f(x) - f_B \| > \lambda \right\} \| B \|_H \leq e^{-\lambda t}. \] (32)

This finishes the proof of the lemma.

Kim in [9] gave the property of \( L^q(Q_p^d) \) as in the Euclidean case.

**Lemma 3.** For \( \alpha > 0 \), the distribution function \( \omega_H(\alpha) \) of \( |f| \) on \( Q_p^d \) defined by the following equation:

\[ \sup_{x \in Q_p^d} \sup_{y \in Z} \frac{1}{|B_\gamma(x)|_H} \int_{B_\gamma(x)} \exp \left( \gamma |B_\gamma(x)|^{-\beta_d} \| f(y) - f_{B_\gamma(x)} \|_H \right) dy < \infty. \] (36)

Obviously, \( G_f(\lambda) \) is a decreasing function \([0, \infty)\) and \( G_f(0) \leq 1 \). Thus, we have the following equation:

\[ \omega_H(\alpha) = \frac{\left\{ x \in Q_p^d: |f(x)| > (\alpha) \right\} \| H \} \leq G_f(\lambda) \frac{1}{\lambda_0 |B|_H} \int_B |B|^{-\beta_d} \| f(x) - f_B \| dx. \] (33)

If \( f \in L_q(Q_p^d) \) for \( q > 0 \), then we have the following equation:

\[ \int_{Q_p^d} |f(x)|^q dx = -\int_0^\infty \alpha^q d\omega_H(\alpha) = q \int_0^\infty \alpha^{q-1} \omega_H(\alpha) d\alpha. \] (34)

We also need the \( p \)-adic version of Lebesgue differential theorem, which is due to Kim [10].

**Lemma 4.** If \( f \in L^1(Q_p^d) \), then we have that for a.e. \( x \in Q_p^d \)
\[ \lim_{y \to x} \frac{1}{|B_\gamma(y)|_H} \int_{B_\gamma(y)} f(y) dy = f(x). \] (35)

The following lemma is about some properties of Lipschitz space \( \text{Lip}_p^d(Q_p^d) \).

**Lemma 5.** If \( f \in \text{Lip}_p^d(Q_p^d) \) is given, then we have the following properties:

(a) For \( q > 1 \), then there exists a constant \( c_q > 0 \) such that
\[ \| f \|_{\text{Lip}_p^d(Q_p^d)} \leq c_q \| f \|_{\text{Lip}_p^d(Q_p^d)}. \]

(b) The norm \( \| f \|_{\text{Lip}_p^d(Q_p^d)} \) is equivalent to the norm \( \| f \|_{\text{Lip}_p^d(Q_p^d)} \) on Lipschitz space.

(c) For any \( \lambda > 0 \), with \( 0 < \lambda \leq c_1 \| f \|_{\text{Lip}_p^d(Q_p^d)} \), where \( c_1 \) is the constant given in Lemma 1.
Proof.

(a) For $q > 1$, using Lemmas 1 and 3 we give the following equation:

\[ \int_{B_y(x)} \left( \left| B_y(x) \right|^{\beta/d} |f(y) - f_{B_y(x)}| \right)^q \, dy = \int_0^\infty q \lambda^{q-1} \left[ \left| y \in B_y(x) \right| B_y(x) \left| f(y) - f_{B_y(x)} \right| > \lambda \right] \, d\lambda \]

\[ \leq c_1 \int_0^\infty q \lambda^{q-1} \exp \left( -\frac{c_2 \lambda}{\|f\|_{\text{Lip}}(Q_x^r)} \right) d\lambda \cdot \left| B_y(x) \right|_{H} = c_1 q \left( \frac{\|f\|_{\text{Lip}}(Q_x^r)}{c_2} \right)^q \int_0^\infty \lambda^{q-1} e^{-\lambda} \, d\lambda \cdot \left| B_y(x) \right|_{H} \]

\[ = c_1 q^r (\|f\|_{\text{Lip}}(Q_x^r))^q \cdot \left| B_y(x) \right|_{H} = c_p q \|f\|_{\text{Lip}}(Q_x^r) \cdot \left| B_y(x) \right|_{H}, \]

where $c_p = (c_2 q^r (\|f\|_{\text{Lip}}(Q_x^r)))^{1/q}$. Hence, we conclude that $\|f\|_{\text{Lip}}(Q_x^r) \leq c_p q \|f\|_{\text{Lip}}(Q_x^r)$.

(b) For $q > 1$, by Hölder’s inequality it is obvious that

\[ \frac{1}{\left| B_y(x) \right|_{H}^{1+\beta/d}} \int_{B_y(x)} \left| f(y) - f_{B_y(x)} \right|^q \, dy \leq \frac{1}{\left| B_y(x) \right|_{H}^{1+\beta/d}} \left( \int_{B_y(x)} \left| f(y) - f_{B_y(x)} \right|^q \, dy \right)^{1/q} \]

for any $(y, x) \in \mathbb{Z} \times \mathbb{Q}_p^d$. Thus, we have that $\|f\|_{\text{Lip}}(Q_x^r) \leq \|f\|_{\text{Lip}}(Q_x^r)$. Hence, by (a) we can obtain the equivalence of those two norms.

(c) We first observe that

\[ \int_0^{\left| B_y(x) \right|_{H}^{1+\beta/d}} \left| f(y) - f_{B_y(x)} \right| \, dy = \exp \left( \lambda \left| B_y(x) \right|_{H}^{\beta/d} \left| f(y) - f_{B_y(x)} \right| \right) - 1, \quad y \in B_y(x). \]

For any $(y, x) \in \mathbb{Z} \times \mathbb{Q}_p^d$, then it follows from Lemma 1 and the $p$-adic version [1] of changing the order of integration that

\[ \int_{B_y(x)} \exp \left( \lambda \left| B_y(x) \right|_{H}^{\beta/d} \left| f(y) - f_{B_y(x)} \right| \right) \, dy = \int_{B_y(x)} \left( \int_0^{\left| B_y(x) \right|_{H}^{1+\beta/d}} \left| f(y) - f_{B_y(x)} \right| \, d\lambda \right) \, dy \]

\[ = \int_0^\infty \exp \left( \lambda \left| B_y(x) \right|_{H}^{\beta/d} \left| f(y) - f_{B_y(x)} \right| \right) \, d\lambda \cdot \left| B_y(x) \right|_{H} \]

\[ \leq c_1 \lambda \int_0^\infty \exp \left( \lambda - \frac{c_2 \lambda}{\|f\|_{\text{Lip}}(Q_x^r)} \right) \, d\lambda \cdot \left| B_y(x) \right|_{H} \]

\[ = c_1 \lambda \left( \frac{c_2}{\|f\|_{\text{Lip}}(Q_x^r)} - \lambda \right)^{-1} \cdot \left| B_y(x) \right|_{H}, \]

for any $(y, x) \in \mathbb{Z} \times \mathbb{Q}_p^d$, provided that $0 < \lambda < c_2/\|f\|_{\text{Lip}}(Q_x^r)$. Therefore, we have the following equation:
Lemma 6. For $1 \leq q < \infty$ and $0 < \beta < 1$, then the homogenous Lipschitz space $\Lambda_p Q^d_p$ coincides with the space $\text{Lip}^\beta_p Q^d_p$.

Proof. Using the property (b) of Lemma 5, we only need to prove that for some constant $C > 0$,

$$|f(x) - f_B| \leq \lim_{y \to x} \left( |f(x) - f_{B_i}| + \sum_{i=0}^{p-1} |f_{B_{i+1}} - f_{B_i}| \right) \leq C \sum_{i=1}^{\infty} \frac{1}{|B_i|^p} \int_{B_i} |f(x) - f| dz$$

Thus, we obtain $\|f\|_{\Lambda_p Q^d_p} \leq C \|f\|_{\text{Lip}^\beta_p Q^d_p}$.

On the other hand, for $y, \ z \in B_r(x)$ and $y \neq z$, we have the following equation:

$$\frac{1}{|B_r(x)|^{1+\beta d}} \int_{B_r(x)} |f(y) - f_{B_r}| dy \leq \frac{1}{|B_r(x)|^{2+\beta d}} \int_{B_r(x)} \int_{B_r(x)} |f(y) - f(z)| dz \ dy.$$

Since $y, z \in B_r(x)$ and (b) of Lemma 2 implies that $|y - z|^\beta \leq |B_r(x)|^{\beta d}$. Hence, by using above inequality, we can infer that

$$\frac{1}{|B_r(x)|^{1+\beta d}} \int_{B_r(x)} |f(y) - f_{B_r}| dy \leq \frac{\|f\|_{\Lambda_p Q^d_p}}{|B_r(x)|^{1+\beta d}} \int_{B_r(x)} \int_{B_r(x)} |y - z|^\beta dz \ dy$$

for any $(y, x) \in \mathbb{Z} \times Q^d_p$. Thus, we have $\|f\|_{\text{Lip}^\beta_p Q^d_p} \leq \|f\|_{\Lambda_p Q^d_p}$. This concludes the proof of the lemma.

In [34], Tableson introduced and studied the Riesz potentials on local fields. Let us recall the definition of Riesz potentials on $p$-adic vector space as follows:
\[ I^p_a(f)(x) = \frac{1}{\Gamma_d(a)} \int_{\mathbb{Q}_p^d} \frac{f(y)}{|x-y|^{d-a}_p} \, dy, \quad (47) \]

where \( \Gamma_d(a) = (1-p^{-d/a}/1-p^{-a}) \) with \( a \in \mathbb{C}/(0). \)

Let us recall the Hardy-Littlewood-Sobolev inequality for the Riesz potentials \( I^p_a \) on \( p \)-adic vector space. More details, the interested reader may refer to the book [34]. \( \square \)

**Lemma 7.** Let \( a \) be a complex number with \( 0 < \Re a < d \) and \( 1 \leq q < \infty \) such that \( 1/r = 1/q - \Re a/d \). Then, the following statements are held.

(a) If \( q > 1 \), then \( I^p_a \) is bounded from \( L^q(\mathbb{Q}_p^d) \) to \( L^r(\mathbb{Q}_p^d) \).

(b) If \( q = 1 \), then \( I^p_a \) is bounded from \( L^q(\mathbb{Q}_p^d) \) to \( L^r(\mathbb{Q}_p^d) \).

Using the definition of the \( p \)-adic version of Riesz potential \( I^p_a \) in equation (47), we give the following inequality:

\[ |I^p_a(f)(x)| = \frac{1}{\Gamma_d(a)} \int_{\mathbb{Q}_p^d} \frac{|f(y)|}{|x-y|^{d-a}_p} \, dy \geq \frac{1}{\Gamma_d(a)} |x-y|^{d-a}_p \int_{|x-y|_p \leq p} |f(y)| \, dy. \quad (49) \]

So inequality (49) gives the following equation:

\[ |\mathcal{M}_a^p(f)(x)| \leq C|I^p_a(f)(x)|. \quad (50) \]

Using inequality (50) and Lemma 7, we give the \( L^q \) boundedness of \( p \)-adic version of fractional maximal function \( \mathcal{M}_a^p \) on the \( p \)-adic vector space.

**Lemma 8.** Let \( a \) be a complex number with \( 0 < \Re a < d \) and \( 1 \leq q < \infty \) such that \( 1/r = 1/q - \Re a/d \). Then, the following statements are held.

(a) If \( q > 1 \), then \( \mathcal{M}_a^p \) is bounded from \( L^q(\mathbb{Q}_p^d) \) to \( L^r(\mathbb{Q}_p^d) \).

(b) If \( q = 1 \), then \( \mathcal{M}_a^p \) is bounded from \( L^q(\mathbb{Q}_p^d) \) to \( L^{r,\infty}(\mathbb{Q}_p^d) \).

We now state the boundedness of \( I^p_a \) on Morrey space which will be useful in proving our results. This lemma is similar to the results of ones from [39–41]. In Euclidean setting, Adams [23] and Spanne [42] studied the corresponding results, and see also [43].

**Lemma 9.** Let \( 0 < \Re a < d \), \( 1 < q < d/\Re a \), and \( 0 < \lambda < d - q\Re a \).

(1) If \( 1/r = 1/q - \Re a/(\lambda - \lambda) \), then \( I^p_a \) is bounded from \( L^{r,\lambda}(\mathbb{Q}_p^d) \) to \( L^{r,\lambda}(\mathbb{Q}_p^d) \).

(2) If \( 1/r = 1/q - \Re a/\lambda \), then \( I^p_a \) is bounded from \( L^{r,\lambda}(\mathbb{Q}_p^d) \) to \( L^{r,\lambda}(\mathbb{Q}_p^d) \).

**Proof.** (1) Let \( x \in \mathbb{Q}_p^d \) and \( y \in \mathbb{Z} \). Then, we split the following equation:

\[ \Gamma_d(a)I^p_a(f)(x) = \int_{|x-y|_p \leq p} \frac{f(y)}{|x-y|^{d-a}_p} \, dy + \int_{|x-y|_p > p} \frac{f(y)}{|x-y|^{d-a}_p} \, dy = : I_1(x) + I_2(x). \quad (51) \]

First, we estimate \( I_1(x) \). Using the definition of \( \mathcal{M}_a^p \), we obtain the following equation:

\[ |I_1(x)| \leq \sum_{k=-\infty}^{\infty} \int_{S_k(x)} |f(y)| \, dy \leq \sum_{k=-\infty}^{\infty} \int_{S_k(x)} p^{k(\Re a - d)} \, dy \leq \int_{\mathbb{Q}_p^d} |f(y)| \, dy \leq C p^{y\Re a} \mathcal{M}_a^p(f)(x). \quad (52) \]

On the other hand, by Hölder’s inequality, we have that \( \Re a - (d - \lambda)/q < 0 \)

\[ |I_2(x)| \leq \sum_{k=1}^{\infty} p^{k(\Re a - d)} \int_{S_k(x)} |f(y)| \, dy \leq \sum_{k=1}^{\infty} p^{k(\Re a - (d - \lambda)/q)} \| L^{r,\lambda}(\mathbb{Q}_p^d) \| f \| L^{r,\lambda}(\mathbb{Q}_p^d) \| f \| L^{r,\lambda}(\mathbb{Q}_p^d). \quad (53) \]
Combining equations (52) and (53) gives the following equation:

\[
\|P^{\operatorname{Re}a}_M(P f)(x)\|_{L^d(Q^d_p)} \leq p^{\operatorname{Re}a} \|P f\|_{L^d(Q^d_p)} dB(x).
\]  

(54)

Taking

\[
p^\alpha = \left( \frac{\|f\|_{L^d(Q^d_p)}(q^{\alpha})}{\|P f\|_{L^d(Q^d_p)}} \right) \quad (q^{\alpha}).
\]  

(55)

then equation (54) becomes as follows:

\[
\|P^{\operatorname{Re}a}M(P f)(x)\|_{L^d(Q^d_p)} \leq p^\alpha \|P f\|_{L^d(Q^d_p)} dB(x).
\]  

Since \(1/r = 1/q - \operatorname{Re}a/(d - \lambda)\), we have \((d - \lambda - q \operatorname{Re}a)/r/(d - \lambda) = q\). Thus, by the \(L^q\) boundedness of the \(p\)-adic version of maximal operator \(M\), we can infer that

\[
\|P^{\operatorname{Re}a}M(P f)(x)\|_{L^d(Q^d_p)} \leq \|P f\|_{L^d(Q^d_p)} dB(x).
\]  

(56)

for any \((y, \alpha) \in \mathbb{Z} \times Q^d_p\). Hence, we conclude that

\[
\|P^\alpha f\|_{L^d(Q^d_p)} \leq C \|P f\|_{L^d(Q^d_p)}.
\]

(57)

Now, we start to prove (2). Assume that \(r_1, q, \alpha, \lambda, d\) satisfy the condition of (1), then \(1/r_1 = 1/q - \operatorname{Re}a/(d - \lambda)\). It follows that

\[
\left( \frac{1}{B_{r_1}(x)} \int_{B_{r_1}(x)} \|P^\alpha f(y)\|_{L^d(Q^d_p)} dy \right)^{1/r_1} \leq \frac{1}{B_{r_1}(x)} \int_{B_{r_1}(x)} \|P f(y)\|_{L^d(Q^d_p)} dy \leq \frac{1}{B_{r_1}(x)} \int_{B_{r_1}(x)} \|P f(y)\|_{L^d(Q^d_p)} dy.
\]  

(58)

Considering the characteristic function \(\chi_{B_{r_1}(x)}\), we have the following property.

**Lemma 11.** Let \(1 \leq q < \infty\) and \(0 < \lambda < d\), then there exists a constant \(C > 0\) such that

\[
\|\chi_{B_{r_1}(x)}\|_{L^{d/q}(Q^d_p)} = \|B_{r_1}(x)\|_{L^{d/q}(Q^d_p)}.
\]  

(62)

**Proof.** Using the definition of \(L^{d/q}(Q^d_p)\) and Lemma 2, it is easy to compute that for \(0 < \lambda < d\)

\[
\|\chi_{B_{r_1}(x)}\|_{L^{d/q}(Q^d_p)} = p^{\lambda/(d - \lambda)q} = \|B_{r_1}(x)\|_{L^{d/q}(Q^d_p)},
\]  

(63)

and the lemma is proved.

**3. The Proof of Theorems 1.1–1.3**

**Proof of Theorem 1.** If \(b \in A_{\beta}(Q^d_p)\), then
\[ \mathcal{M}_p^p(f)(x) = \frac{1}{|B_p(x)|_H} \int_{B_p(x)} |b(x) - b(y) - f(y)| dy \leq \|b\|_{\Lambda_p} \frac{1}{|B_p(x)|_H} \int_{B_p(x)} |f(y)| dy \]

\[ = \|b\|_{\Lambda_p} \mathcal{M}_p^p(f)(x). \]  

(64)

Obviously, equations (2)–(5) follow from Lemma 8 and (64).

(3) ⇒ (1): Assume that \( \mathcal{M}_p^p \) is bounded from \( L^q(Q^d_p) \) to \( L^q(Q^d_p) \) for some \( q, r \) with \( 1 < q < d/\beta \) and \( 1/r = 1/q - \beta/n \).

\[
\frac{1}{|B_p(x)|_H} \int_{B_p(x)} |b(y) - b_{B_p(x)}| dy \leq \frac{1}{|B_p(x)|_H} \int_{B_p(x)} \left( \frac{1}{|B_p(x)|_H} \int_{B_p(x)} |b(y) - b(z)| dz \right) dy
\]

\[ = \frac{1}{|B_p(x)|_H} \int_{B_p(x)} \left( \frac{1}{|B_p(x)|_H} \int_{B_p(x)} |b(y) - b(z)| \chi_{B_p(x)}(z) dz \right) dy \]

\[ \leq \frac{1}{|B_p(x)|_H} \int_{B_p(x)} M_p^p(\chi_{B_p(x)}(y)) dy \leq \frac{1}{|B_p(x)|_H} \int_{B_p(x)} \left( \int_{B_p(x)} M_p^p(\chi_{B_p(x)}(y)) \right) dy^{1/r} \]

\[ = \left( \int_{B_p(x)} \chi_{B_p(x)}(y) dy \right)^{1/r} \leq \frac{C}{|B_p(x)|_H} \|M_p^p\|_{L^q(Q^d_p)} \|X_{B_p(x)}\|_{L^q(Q^d_p)} \leq C|B_p(x)|_H \rightarrow L^r(Q^d_p) \]  

(65)

This together with Lemma 6 implies that \( b \in \Lambda_p(Q^d_p) \).

(4) ⇒ (1): We assume that (8) is true and will verify \( b \in \Lambda_p(Q^d_p) \). For any \( p \)-adic \( B_{y_0}(x) \subset Q^d_p \), since for any \( y \in B_{y_0}(x) \)

\[ \mathcal{M}_p^p(\chi_{B_{y_0}(x)})(y) = \sup_{y \in Z} \frac{1}{|B_p(y)|_H} \int_{B_p(y)} |b(y) - b(z)| \chi_{B_{y_0}(x)}(z) dz \geq \frac{1}{|B_p(y)|_H} \int_{B_p(y)} |b(y) - b(z)| \chi_{B_{y_0}(x)}(z) dz \]

\[ = \frac{1}{|B_p(y)|_H} \int_{B_p(y)} |b(y) - b(z)| dz \geq |b(y) - b_{B_p(x)}| \]  

(67)

This together with equation (8) gives the following equation:

\[ \left| \left\{ y \in B_{y_0}(x): |b(y) - b_{B_{y_0}(x)}| > \lambda \right\} \right|_H \leq \left\{ y \in B_{y_0}(x): \mathcal{M}_p^p(\chi_{B_{y_0}(x)})(y) > \lambda \right\} \right|_H \]

\[ \leq C \left( \lambda^{-1} \|X_{B_{y_0}(x)}\|_{L^q(Q^d_p)} \right)^{d/(d-\beta)} \]

(68)

Let \( t > 0 \) be a constant to be determined later, then by using Lemma 3, we have the following equation:
\[
\int_{B_{r_0}(x)} |b(y) - b_{B_{r_0}}(x)| \, dy = \int_0^\infty \left\{ y \in B_{r_0}(x) : |b(y) - b_{B_{r_0}}(x)| > \lambda \right\} \, d\lambda \\
= \int_0^\infty \left\{ y \in B_{r_0}(x) : |b(y) - b_{B_{r_0}}(x)| > \lambda \right\} \, d\lambda + \int^\infty_t \left\{ y \in B_{r_0}(x) : |b(y) - b_{B_{r_0}}(x)| > \lambda \right\} \, d\lambda \\
\leq t |B_{r_0}(x)|_H + C \int^\infty_t \left( \lambda^{-1} |B_{r_0}(x)|_H \right)^{d/d(\beta - 1)} \, d\lambda \leq t |B_{r_0}(x)|_H + C |B_{r_0}(x)|_H^{d(\beta - 1)/d} \int^\infty_t \lambda^{-d(\beta - 1)} \, d\lambda \\
\leq C \left( t |B_{r_0}(x)|_H + t^{1-d(\beta - 1)} |B_{r_0}(x)|_H^{d(\beta - 1)/d} \right). \\
\] (69)

Taking \( t = |B_{r_0}(x)|_H^{1/d} \) in the above estimate, we have the following equation:

\[
\int_{B_{r_0}(x)} |b(y) - b_{B_{r_0}}(x)| \, dy \leq C |B_{r_0}(x)|_H^{1+1/d}. \\
\] (70)

\[
\frac{1}{|B_p(x)|_H^{1+1/d}} \int_{B_p(x)} |b(y) - b_{B_p(x)}| \, dy \leq \frac{1}{|B_p(x)|_H^{1+1/d}} \int_{B_p(x)} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \int_{B_p(x)} |b(y) - b(z)| \chi_{B_p(x)}(z) \, dz \, dy \\
\leq \frac{1}{|B_p(x)|_H^{1+1/d}} \int_{B_p(x)} \mathcal{M}_p^p \left( \chi_{B_p(x)} \right)(y) \, dy \leq \frac{1}{|B_p(x)|_H^{1+1/d}} \left\| \mathcal{M}_p^p \left( \chi_{B_p(x)} \right) \right\|_{L^\infty(O^p)} \leq C \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)} \\
\leq C \frac{1}{|B_p(x)|_H^{1+1/d}} \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)} \leq C \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)} \\
\leq C \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)} \leq C \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)} \leq C \left\| \mathcal{M}_p^p \right\|_{L^1(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^\infty(O^p)}. \\
\] (71)

This together with Lemma 6 implies that \( b \in \Lambda_p(Q_p^d) \).

The above together with (2) \( \Rightarrow \) (1) follows from (3) \( \Rightarrow \) (1).

\[ \square \]

**Proof of Theorem 2.** Assume that \( b \in \Lambda_p(Q_p^d) \). By using equations (64) and (1) of Lemma 9, we have the following equation:

\[
\frac{1}{|B_p(x)|_H^{1+1/d}} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \int_{B_p(x)} |b(y) - b_{B_p(x)}| \, dy \\
\leq \frac{1}{|B_p(x)|_H^{1+1/d}} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \int_{B_p(x)} |b(y) - b(z)| \chi_{B_p(x)}(z) \, dz \, dy \\
\leq \frac{1}{|B_p(x)|_H^{1+1/d}} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \mathcal{M}_p^{1/q} \left( \chi_{B_p(x)} \right)(y) \, dy \\
= \frac{1}{|B_p(x)|_H^{1+1/d}} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \mathcal{M}_p^{1/q} \left( \chi_{B_p(x)} \right)(y) \, dy \\
\leq \frac{1}{|B_p(x)|_H^{1+1/d}} \left( \frac{1}{|B_p(x)|_H^{1+1/d}} \right) \mathcal{M}_p^{1/q} \left( \chi_{B_p(x)} \right)(y) \, dy \\
\leq |B_p(x)|_H^{1+1/d} \left\| \mathcal{M}_p \right\|_{L^q(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^q(O^p)} \leq C \left\| \mathcal{M}_p \right\|_{L^q(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^q(O^p)} \leq C \left\| \mathcal{M}_p \right\|_{L^q(O^p)} \left\| \chi_{B_p(x)} \right\|_{L^q(O^p)}. \\
\] (73)
where in the last step we have used $1/r = 1/q - \beta/(d - \lambda)$ and Lemma 10.

It follows from Lemma 6 that $b \in \Lambda_p(Q_p^d)$. This finishes the Proof of Theorem 2. □

Proof of Theorem 3. By using similar proof to the one of Theorem 2, we can obtain Theorem 3. Hence, we omit the details. □

$$\left| \left[ b, \mathcal{M}^p \right] (\mathbb{R}^d) \right| = \left| \left[ b, \mathcal{M}^p \right] (\mathbb{R}^d) - \mathcal{M}^p (b \mathbb{R}^d) \right|$$

$$\leq \sup_{y \in \mathbb{Z}} \frac{1}{|B_y(x)|} \int_{B_y(x)} b(y) f(y) dy - \sup_{y \in \mathbb{Z}} \frac{1}{|B_y(x)|} \int_{B_y(x)} b(y) f(y) dy$$

$$= \mathcal{M}^p (x).$$

It follows from Theorem 1 that $[b, \mathcal{M}^p]$ is bounded from $L^q(Q_p^d)$ to $L^r(Q_p^d)$ since $b \in \Lambda_p(Q_p^d)$. (2) ⇒ (1): For any fixed $p$-adic ball $B_y(x) \subset Q_p^d$ and all $y \in B_y(x)$, we claim that

$$\mathcal{M}^p (\chi_{B_y(x)}) (y) = \chi_{B_y(x)} (x),$$

$$\mathcal{M}^p (\chi_{\mathbb{R}^d} (y) = \mathcal{M}^p (y).$$

(75)

$$\frac{1}{|B_y(x)|^{d/\alpha}} \left( \frac{1}{|B_y(x)|} \int_{B_y(x)} b(y) - \mathcal{M}^p (y) dy \right)^{1/r} = \frac{1}{|B_y(x)|^{d/\alpha}} \left( \frac{1}{|B_y(x)|} \int_{B_y(x)} b(y) \mathcal{M}^p (\chi_{B_y(x)} (y) - \mathcal{M}^p (\chi_{B_y(x)} (y) dy \right)^{1/r}$$

$$= \frac{1}{|B_y(x)|^{d/\alpha}} \left( \frac{1}{|B_y(x)|} \int_{B_y(x)} \left| b(y) \mathcal{M}^p (\chi_{B_y(x)} (y) dy \right|^{1/r}$$

$$\leq \frac{1}{|B_y(x)|^{d/\alpha}} \left\| b \mathcal{M}^p \right\|_{L^r} \leq \frac{C}{|B_y(x)|^{d/\alpha}} \leq C.$$

(76)

4. The Proof of Theorems 4–7

Proof of Theorem 4. (2) ⇒ (1): For any fixed $x \in Q_p^d$ such that $\mathcal{M}^p (f) (x) < \infty$, since $b \geq 0$ then we have the following equation:

In fact, similar to the proof of Lemma 2.12, it is easy to obtain the claim. Then, by using the above claim, we have the following equation:

$$\frac{1}{|B_y(x)|^{d/\alpha}} \left( \frac{1}{|B_y(x)|} \int_{B_y(x)} b(y) - \mathcal{M}^p (y) dy \right)^{1/r} \leq \frac{1}{|B_y(x)|^{d/\alpha}} \left\| b \mathcal{M}^p \right\|_{L^r} \leq \frac{C}{|B_y(x)|^{d/\alpha}} \leq C.$$

(77)

which gives that (3) since the $p$-adic ball $B_y(x) \subset Q_p^d$ is arbitrary.

(3) ⇒ (1): To prove $b \in \Lambda_p(Q_p^d)$, by using Lemma 6, it suffices to verify that there is a constant $C > 0$ such that for any $p$-adic ball $B_y(x) \subset Q_p^d$,

$$\frac{1}{|B_y(x)|} \int_{B_y(x)} \left| b(y) - \mathcal{M}^p (y) \right| dy \leq C.$$

(78)
Since for any \( y \in E \) we have \( b(y) \leq b_{B_y(x)} \leq M_{B_y(x)}^p (b)(y) \), then for any \( y \in E \),

\[
|b(y) - b_{B_y(x)}| \leq |b(y) - M_{B_y(x)}^p (b)(y)|. \tag{79}
\]

Thus, we can conclude that

\[
\frac{1}{|B_y(x)|^\frac{\beta_d}{H}} \int_{B_y(x)} |b(y) - b_{B_y(x)}| \, dy = \frac{1}{|B_y(x)|^\frac{\beta_d}{H}} \int_{E} |b(y) - b_{B_y(x)}| \, dy
\]

\[
= \frac{2}{|B_y(x)|^\frac{\beta_d}{H}} \int_{B_y(x)} |b(y) - b_{B_y(x)}| \, dy
\]

\[
= \frac{2}{|B_y(x)|^\frac{\beta_d}{H}} \int_{E} |b(y) - M_{B_y(x)}^p (b)(y)| \, dy = \frac{2}{|B_y(x)|^\frac{\beta_d}{H}} \int_{E} |b(y) - M_{B_y(x)}^p (b)(y)| \, dy
\]

\[
\leq \frac{2}{|B_y(x)|^\frac{\beta_d}{H}} \int_{E} |b(y) - M_{B_y(x)}^p (b)(y)| \, dy \tag{80}
\]

On the other hand, it follows from Hölder’s inequality and equation (12) that

\[
|b_{B_y(x)}| \leq \frac{1}{|B_y(x)|^\frac{\beta_d}{H}} \left( \int_{B_y(x)} |b(y) - M_{B_y(x)}^p (b)(y)| \, dy \right)^{\frac{1}{r}} \leq C.
\]

Combining equations (76) with (80) it follows that \( b \in A_{\beta_d}^p (Q^d_\kappa) \).

In order to prove \( b \geq 0 \), it suffices to show \( b^\ast = 0 \), where \( b^\ast = \min \{ b, 0 \} \). Let \( b^\ast = |b| - b^\ast \), then \( b = b^\ast + b^\ast \). For any fixed \( p \)-adic ball \( B_y(x) \), observe that

\[
0 \leq b^\ast(y) \leq |b(x)| \leq M_{B_y(x)}^p (b)(y), \tag{82}
\]

for \( y \in B_y(x) \) and therefore we have that for \( y \in B_y(x) \),

\[
0 \leq b^\ast(y) \leq M_{B_y(x)}^p (b)(y) - b^\ast(y) + b^\ast(y) = M_{B_y(x)}^p (b)(y) - b(y). \tag{83}
\]

Then, it follows from equation (12) that for any \( p \)-adic ball \( y \in B_y(x) \),

\[
\frac{1}{|B_y(x)|^\frac{\beta_d}{H}} \left( \frac{1}{|B_y(x)|^\frac{\beta_d}{H}} \int_{B_y(x)} |M_{B_y(x)}^p (b)(y) - b(y)| \, dy \right)^{\frac{1}{r}} \leq C|B_y(x)|^\frac{\beta_d}{H}. \tag{84}
\]
Thus, $b^{-} = 0$ follows from Lemma 4. Hence, the Proof of Theorems 4 is completed.

**Proof of Theorem 5.** Obviously, Theorem 5 follows from equation (74) and Theorem 1. Hence, we omit the details. □

**Data Availability**

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

**Ethical Approval**

All procedures were in accordance with the ethical standards of the institutional research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This work was supported by National Natural Science Foundation of China (Grant nos. 11871452 and 12071473) and Shandong Jianzhu University Foundation (Grant no. X20075Z0101).

**References**


