Hindawi
Journal of Mathematics
Volume 2022, Article ID 7430272, 15 pages
https://doi.org/10.1155/2022/7430272

Research Article

Characterization of Lipschitz Spaces via Commutators of Maximal Function on the p-Adic Vector Space

Qianjun He [1] **and Xiang Li** [2]

[1] *School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China*
[2] *School of Science, Shandong Jianzhu University, Jinan 250000, China*

Correspondence should be addressed to Xiang Li; lixiang162@mails.ucas.ac.cn

Received 5 August 2022; Revised 8 November 2022; Accepted 10 November 2022; Published 5 December 2022

Academic Editor: Humberto Rafeiro

In this paper, we give characterization of a p-adic version of Lipschitz spaces in terms of the boundedness of commutators of maximal function in the context of the p-adic version of Lebesgue spaces and Morrey spaces, where the symbols of the commutators belong to the Lipschitz spaces. A useful tool is a Lipschitz norm involving the John-Nirenberg-type inequality for homogeneous Lipschitz functions, which is new in the p-adic field context.

1. Introduction and Statement of Main Results

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$. If any nonzero rational number x is represented as $x = p^\gamma (m/n)$, where m and n are integers which are not divisible by p, and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, |x+y|_p \le \max\left\{|x|_p, |y|_p\right\}. \tag{1}$$

It follows from the second property that when $|x|_p \ne |y|_p$, then $|x+y|_p = \max\left\{|x|_p, |y|_p\right\}$. From the standard p-adic analysis [1], we see that any nonzero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$\begin{aligned} x &= p^\gamma \sum_{j=0}^{\infty} a_j p^j, \\ \gamma &= \gamma(x) \in \mathbb{Z}, \end{aligned} \tag{2}$$

where a_j are integers, $0 \le a_j \le p-1$, $a_0 \ne 0$. Series (2) converges in the p-adic norm because $|a_j p^j|_p = p^{-j}$.

The space $\mathbb{Q}_p^d$ consists of points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, d$. The p-adic norm on $\mathbb{Q}_p^d$ is $|\mathbf{x}|_p := \max_{1\le j\le d} |x_j|_p$ for $\mathbf{x} \in \mathbb{Q}_p^d$. Denote by $B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p \le p^\gamma\}$, the ball with center at $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ, and by $S_\gamma(\mathbf{a}) := \left\{\mathbf{x} \in \mathbb{Q}_p^d | \mathbf{x} - \mathbf{a}|_p = p^\gamma\right\}$ the sphere with center at $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(\mathbf{a}) = B_\gamma(\mathbf{a}) / B_{\gamma-1}(\mathbf{a})$ and $B_\gamma(\mathbf{a}) = \cup_{k\le\gamma} S_k(\mathbf{a})$.

Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Harr measure $d\mathbf{x}$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant. We normalize the measure $d\mathbf{x}$ so that

$$\int_{B_0(0)} d\mathbf{x} = \left|B_0(0)\right|_H = 1, \tag{3}$$

where $|E|_H$ denotes the Harr measure of a measurable subset E of $\mathbb{Q}_p^d$. From this integral theory, it is easy to obtain that $|B_\gamma(\mathbf{a})|_H = p^{\gamma d}$ and $|S_\gamma(\mathbf{a})|_H = p^{\gamma d}(1 - p^{-d})$ for any $\mathbf{a} \in \mathbb{Q}_p^d$.

In what follows, we say that a (real-valued) measurable function f defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d)$, $1 \le q \le \infty$, if it satisfies

$$\|f\|_{L^q(\mathbb{Q}_p^d)} := \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})^q \mathrm{d}\mathbf{x}|\right)^{1/q} < \infty, \quad 1 \le q < \infty,$$
$$\|f\|_{L^\infty(\mathbb{Q}_p^d)} := \inf\left\{\alpha\colon \left|\left\{\mathbf{x} \in \mathbb{Q}_p^d\colon |f(\mathbf{x})| > \alpha\right\}\right|_H = 0\right\} < \infty. \tag{4}$$

Here the integral in equation (4) is defined as follows:

$$\begin{aligned}\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \mathrm{d}\mathbf{x} &= \lim_{\gamma \longrightarrow \infty} \int_{B_\gamma(0)} |f(\mathbf{x})|^q \mathrm{d}\mathbf{x} \\ &= \lim_{\gamma \longrightarrow \infty} \sum_{-\infty < k \le \gamma} \int_{S_k(0)} |f(\mathbf{x})|^q \mathrm{d}\mathbf{x}.\end{aligned} \tag{5}$$

If the limit exists. We now mention some of the previous works on harmonic analysis on the p-adic field, see [2–8] and the references therein.

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d)$, we defined the Hardy-Littlewood maximal function of f on $\mathbb{Q}_p^d$ by the following equation:

$$\mathscr{M}^p(f)(\mathbf{x}) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \mathrm{d}\mathbf{y}. \tag{6}$$

In [9, 10], Kim proved L^q boundedness of the version of maximal function $\mathscr{M}^p$ and gave some properties similar to the Euclidean setting.

The maximal commutator of $\mathscr{M}^p$ with a locally integrable function b is defined by the following equation:

$$\mathscr{M}_b^p(f)(\mathbf{x}) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{y}) f(\mathbf{y})| \mathrm{d}\mathbf{y}. \tag{7}$$

The first part of this paper is to study the boundedness of $\mathscr{M}_b^p$ when the symbol belongs to a Lipschitz space (see in Section 2). Some characterizations of the Lipschitz space via such commutator are given. Our first result can be stated as follows:

Theorem 1. *Let b be a locally integrable function and $0 < \beta < 1$, then the following statements are equivalent.*

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^d)$;

(2) $\mathscr{M}_b^p$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$ for all q, r with $1 < q < d/\beta$ and $1/r = 1/q - \beta/d$;

(3) $\mathscr{M}_b^p$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$ for some q, r with $1 < q < d/\beta$ and $1/r = 1/q - \beta/d$;

(4) $\mathscr{M}_b^p$ satisfies the weak-type $(1, d/(d-\beta))$ estimates, namely, there exists a positive constant C such that for any $\lambda > 0$,

$$\left|\left\{\mathbf{x} \in \mathbb{Q}_p^d\colon \mathscr{M}_b^p(f)(\mathbf{x}) > \lambda\right\}\right|_H \le C\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^d)}}{\lambda}\right)^{d/(d-\beta)}, \tag{8}$$

(5) $\mathscr{M}_b^p$ is bounded from $L^{d/\beta}(\mathbb{Q}_p^d)$ to $L^\infty(\mathbb{Q}_p^d)$.

We remark that the boundedness of the commutators of maximal function were pretty much unknown on the p-adic vector space. However, in Euclidean case, the mapping properties of the maximal commutator have been studied intensively by many authors, see [11–21].

It is well-known that the Morrey space introduced by Morrey in 1938, is connected to certain problem in elliptic PDE [22]. Later the Morrey spaces were found to have many important applications to the Navier-Stokes equations, the Schrödinger equations, the elliptic equations with discontinuous coefficients and the potential analysis, see [23–27] and so on. Next, we introduce the p-adic version of Morrey space on p-adic vector space.

For $1 \le q < \infty$ and $0 \le \lambda \le d$. The p-adic version of Morrey space is defined by the following equation:

$$L^{q,\lambda}(\mathbb{Q}_p^d) = \left\{f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^d)\colon \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} < \infty\right\}, \tag{9}$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} := \sup_{\mathbf{x} \in \mathbb{Q}_p^d} \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{x})|^q \mathrm{d}\mathbf{x}\right)^{1/q}. \tag{10}$$

It is well known that if $1 \le q < \infty$, then $L^{q,0}(\mathbb{Q}_p^d) = L^q(\mathbb{Q}_p^d)$ and $L^{q,d}(\mathbb{Q}_p^d) = L^\infty(\mathbb{Q}_p^d)$.

Theorem 2. *Let b be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - q\beta$ and $1/r = 1/p - \beta/(d-\lambda)$. Then, $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ if and only if $\mathscr{M}_b^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$.*

Theorem 3. *Let b be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - q\beta$, $1/r = 1/p - \beta/d$, and $\nu/r = \lambda/q$. Then, $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ if and only if $\mathscr{M}_b^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\nu}(\mathbb{Q}_p^d)$.*

On the other hand, the classical commutator of the p-adic version of maximal function $\mathscr{M}^p$ with a locally integrable function b can be defined by the following equation:

$$[b, \mathscr{M}^p](f)(\mathbf{x}) = b(\mathbf{x})\mathscr{M}^p(f)(\mathbf{x}) - \mathscr{M}^p(bf)(\mathbf{x}). \tag{11}$$

In the Euclidean setting, the commutator of maximal function as in (11) has attracted much more attention. For examples, Milman and Schonbek [28] established a commutator result by real interpolation techniques. As an application, they obtained the L^q-boundedness of the commutator of maximal function $[b, M]$ when $b \in BMO(\mathbb{R}^d)$ and $b \ge 0$. This operator can be used in studying the product of a function in H^1 and a function in BMO (see [29] for instance). Bastero et al. [30] studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on L^q spaces when $1 < q < \infty$. Zhang and Wu [31] extended their results to commutators of the fractional maximal function. The results in [30, 31] were extended to variable Lebesgue spaces in [32, 33]. Recently, Zhang [20] studied the commutator $[b, M]$ when b belongs to Lipschitz

spaces. Some necessary and sufficient conditions for the boundedness of $[b, M]$ on Lebesgue and Morrey spaces are given. Some of the results were extended to variable Lebesgue spaces in [21]. For more information about the characterization of the commutator of maximal operator, see also [19, 20] and the references therein.

We would like to remark that operators $\mathscr{M}_b^p$ and $[b, \mathscr{M}^p]$ essentially differ from each other. For example, $\mathscr{M}_b^p$ is positive and sublinear, but $[b, \mathscr{M}^p]$ is neither positive nor sublinear. Motivated by the papers mentioned above, in this paper, the second part of this paper aims to study the mapping properties of the commutator $[b, \mathscr{M}^p]$ when b belongs to some Lipschitz spaces. More precisely, we will give some new necessary and sufficient conditions for the boundedness of $[b, \mathscr{M}^p]$ on p-adic vector spaces, by which some new characterizations for certain subclasses of Lipschitz spaces are obtained.

Theorem 4. *Let b be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$ and $1/r = 1/p - \beta/d$. Then, the following statements are equivalent.*

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ and $b \geq 0$;

(2) $[b, \mathscr{M}^p]$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$;

(3) There exists a constant $C > 0$ such that

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d} \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H^{\beta/d}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left| b(\mathbf{y}) - \mathscr{M}_{B_\gamma(\mathbf{x})}^p(b)(\mathbf{y}) \right|^r \mathrm{d}\mathbf{y} \right)^{1/r} \leq C. \tag{12}$$

Where $\mathscr{M}_{B_\gamma(\mathbf{x})}^p$ denote the maximal operator with respect to a p-adic ball which is defined by the following equation:

$$\mathscr{M}_{B_\gamma(\mathbf{x})}^p(f)(\mathbf{y}) = \sup_{B_\gamma(\mathbf{x})\supseteq B_{\gamma_0}(\mathbf{y})} \frac{1}{\left|B_{\gamma_0}(\mathbf{y})\right|_H} \int_{B_{\gamma_0}(\mathbf{y})} |f(\mathbf{z})| \mathrm{d}\mathbf{z}. \tag{13}$$

Here, the supremum is taken over all the p-adic $B_{\gamma_0}(\mathbf{y})$ with $B_{\gamma_0}(\mathbf{y}) \subseteq B_\gamma(\mathbf{x})$ for a fixed p-adic ball $B_\gamma(\mathbf{x})$.

Theorem 5. *Let $b \geq 0$ be a locally integrable function, $0 < \beta < 1$ and $b \in \Lambda_\beta(\mathbb{Q}_p^d)$. Then, there exist a positive C such that for any $\lambda > 0$,*

$$\left| \left\{ \mathbf{x} \in \mathbb{Q}_p^d : \left[b, \mathscr{M}^p\right](f)(\mathbf{x}) > \lambda \right\} \right|_H \leq C \left(\frac{\|f\|_{L^1\left(\mathbb{Q}_p^d\right)}}{\lambda} \right)^{d/(d-\beta)}. \tag{14}$$

Theorem 6. *Let b be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - q\beta$, and $1/r = 1/p - \beta/(d - \lambda)$. Then, the following statements are equivalent*

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ and $b \geq 0$;

(2) $[b, \mathscr{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$.

Theorem 7. *Let b be a locally integrable function and $0 < \beta < 1$. Assume that $1 < q < d/\beta$, $0 < \lambda < d - q\beta$, $1/r = 1/p - \beta/d$, and $\nu/r = \lambda/q$. Then, the following statements are equivalent.*

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ and $b \geq 0$;

(2) $[b, \mathscr{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\nu}(\mathbb{Q}_p^d)$.

Remark 1. Theorem 6 is essentially the Adams-type result and Theorem 7 is the Spanne-type result. Also, Theorem 7 can be immediately proved via Theorem 6.

The rest of the present paper is organized as follows: in Section 2, we gave some definitions and lemmas. The Proof of Theorems 1–3 are presented in Section 3. In Section 4, we gave the proof of Theorems 4–7. By $A \lesssim B$ we mean that $A \leq cB$ with some positive constant c independent of appropriate quantities. The positive constants C varies from one occurrence to another. For a real number q, $1 < q < \infty$, q' is the conjugate number of q, that is, $1/q + 1/q' = 1$.

2. Preliminary Definitions and Lemmas

To prove our main results, we need the following definitions and lemmas.

Definition 1. Assume that $0 < \beta < 1$. The p-adic version of homogeneous Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^d)$ (see [34]) is the space of all measurable functions f on $\mathbb{Q}_p^d$ such that

$$\|f\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)} = \sup_{\mathbf{x},\mathbf{y}\in\mathbb{Q}_p^d}, \mathbf{x} \neq \mathbf{y} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|_p^\beta} < \infty. \tag{15}$$

For $1 \leq q < \infty$, the p-adic version of Lipschitz spaces $\mathrm{Lip}_\gamma^q(\mathbb{Q}_p^d)$ is the space of all measurable functions f on $\mathbb{Q}_p^d$ such that

$$\|f\|_{\mathrm{Lip}_\beta^q\left(\mathbb{Q}_p^d\right)} = \sup_{\mathbf{x}\in\mathbb{Q}_p^d} \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left| f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})} \right|^q \mathrm{d}\mathbf{y} \right)^{1/q} < \infty, \tag{16}$$

where $f_{B_\gamma(\mathbf{x})}$ denotes the average of f over $B_\gamma(\mathbf{x})$, i.e., $f_{B_\gamma(\mathbf{x})} = (1/|B_\gamma(\mathbf{x})|_H)\int_{B_\gamma(\mathbf{x})} f(\mathbf{y})\mathrm{d}\mathbf{y}$. When $q = 1$, we use $\mathrm{Lip}_\beta(\mathbb{Q}_p^d)$ as $\mathrm{Lip}_\beta^1(\mathbb{Q}_p^d)$.

We shall give the p-adic version of John-Nirenberg inequality for homogeneous Lipschitz functions in the following lemma.

Lemma 1. *Let $0 < \beta < 1$, then there exist some c_1, $c_2 > 0$ depending on p and d such that*

$$\left|\left\{\mathbf{x} \in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda\right\}\right|_H \le c_1 \exp\left(-\frac{c_2\lambda}{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}\right)|B|_H, \tag{17}$$

for any $f \in Lip_\beta(\mathbb{Q}_p^d)$ and for any p-adic $B \in \mathscr{F}_p$, where $\mathscr{F}_p$ denotes the family of all the p-adic balls which is defined as follows:

$$\mathscr{F}_p = \left\{B_\gamma(\mathbf{x})\colon \gamma \in \mathbb{Z}, \quad \mathbf{x} \in \mathbb{Q}_p^d\right\}. \tag{18}$$

Kim in [9] observed several interesting properties on the family $\mathscr{F}_p$.

Lemma 2. *The family $\mathscr{F}_p$ has the following properties:*

(a) If $\gamma \le \gamma'$, then either $B_\gamma(\mathbf{x}) \cap B_{\gamma'}(\mathbf{y}) = \varnothing$ or $B_\gamma(\mathbf{x}) \subset B_{\gamma'}(\mathbf{y})$.

(b) $B_\gamma(\mathbf{x}) = B_\gamma(\mathbf{y})$ if and only if $\mathbf{y} \in B_\gamma(\mathbf{x})$.

Proof of Lemma 1. Let $B \in \mathscr{F}_p$ be a fixed p-adic ball and λ_0 be some positive real number which will be determined later. Applying the p-adic version of the Calderón-Zygmund decomposition (see Corollary 3.4 in [9], or see also Theorem 5.16 in [9]) of B for $|B|_H^{-\beta/d}|f(\mathbf{x}) - f_B|$ relative to λ_0 to obtain a pair-wise disjoint family of p-adic balls $\left\{B_j\right\}_{j=1}^\infty \subset \mathscr{F}_p$ (by Lemma 2) which satisfies the following equation:

$$\left|\left\{\frac{\mathbf{x} \in B}{\cup_{j=1}^\infty B_j}\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda_0\right\}\right|_H = 0, \tag{19}$$

$$\lambda_0 < \frac{1}{\left|B_j\right|_H}\int_{B_j} |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right|\mathrm{d}\mathbf{x} \le p^d\lambda_0, \text{for any } j. \tag{20}$$

and

$$\sum_{j=1}^\infty \left|B_j\right|_H \le \frac{1}{\lambda_0}\int_B |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right|\mathrm{d}\mathbf{x}. \tag{21}$$

Combining $\beta > 0$ and equation (20) gives the following equation:

$$|B|_H^{-\beta/d}\left|f_B - f_{B_j}\right| \le \frac{1}{\left|B_j\right|_H}\int_{B_j} |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right|\mathrm{d}\mathbf{x} + \frac{1}{\left|B_j\right|_H}\int_{B_j}\left|B_j\right|_H^{-\beta/d}\left|f(\mathbf{x}) - f_{B_j}\right|\mathrm{d}\mathbf{x} \le p^d\lambda_0 + \|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}. \tag{22}$$

We denote $t = p^d\lambda_0 + \|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)} > \lambda_0$. For any $x \in B_j$, we have

$$|B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| \le |B|_H^{-\beta/d}\left|f_B - f_{B_j}\right| + |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_{B_j}\right| \le t + \left|B_j\right|_H^{-\beta/d}\left|f(\mathbf{x}) - f_{B_j}\right|. \tag{23}$$

By using equation (19), we can infer that for any $\lambda > 0$

$$\begin{aligned}\left|\left\{\mathbf{x} \in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda + t\right\}\right|_H &\le \left|\left\{\mathbf{x} \in \cup_{j=1}^\infty B_j\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda + t\right\}\right|_H + \left|\left\{\frac{\mathbf{x} \in B}{\cup_{j=1}^\infty B_j}\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda + t\right\}\right|_H \\ &\le \left|\left\{\mathbf{x} \in \cup_{j=1}^\infty B_j\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda + t\right\}\right|_H + \left|\left\{\frac{\mathbf{x} \in B}{\cup_{j=1}^\infty B_j}\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda_0\right\}\right|_H = \sum_j^\infty \left|\left\{\mathbf{x} \in B_j\colon |B|_H^{-\beta/d}\left|f(\mathbf{x}) - f_B\right| > \lambda + t\right\}\right|_H.\end{aligned} \tag{24}$$

Thus, equations (23) and (24) now implies.

$$\left|\left\{\mathbf{x}\in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|>\lambda+t\right\}\right|_H \le \sum_{j=1}^{\infty}\left|\left\{\mathbf{x}\in B_j\colon |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|>\lambda+t\right\}\right|_H \le \sum_{j=1}^{\infty}\left|\left\{\mathbf{x}\in B_j\colon \left|B_j\right|_H^{-\beta/d}\left|f(\mathbf{x})-f_{B_j}\right|>\lambda\right\}\right|_H. \tag{25}$$

For any $\lambda>0$, set

$$G_f(\lambda)=\sup_{B\in\mathscr{F}_p}\frac{1}{|B|_H}\left|\left\{\mathbf{x}\in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|>\lambda\right\}\right|_H. \tag{26}$$

Obviously, $G_f(\lambda)$ is a decreasing function $[0,\infty)$ and $G_f(0)\le 1$. Thus, we have the following equation:

$$\frac{1}{|B|_H}\left|\left\{\mathbf{x}\in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|>\lambda+t\right\}\right|_H \le G_f(\lambda)\frac{1}{|B|_H}\sum_{j=1}^{\infty}\left|B_j\right|_H \le G_f(\lambda)\frac{1}{\lambda_0|B|_H}\int_B |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|d\mathbf{x}. \tag{27}$$

Hence, for any $\lambda>0$, we conclude that

$$G_f(\lambda+t)\le\frac{G_f(\lambda)\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}{\lambda_0}. \tag{28}$$

Taking $\lambda_0=e\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)}$, then t is also a fixed positive number and for any $\lambda\ge 0$,

$$G_f(\lambda+t)\le\frac{G_f(\lambda)}{e}. \tag{29}$$

Using induction argument for any $k\ge 1$, we obtain the following equation:

$$G_f((k+1)t)\le e^{-k}G_f(t). \tag{30}$$

Thus, for $\lambda\in(kt,(k+1)t]$, we have the following equation:

$$G_f(\lambda)\le G_f(kt)\le e^{-k}G_f(0)\le ee^{-\lambda/t}. \tag{31}$$

Notice that this inequality is also true for $\lambda\in[0,t]$, due to $G_f(\lambda)\le G_f(0)=1\le ee^{-\lambda/t}$. Therefore, for any $\lambda\ge 0$, we conclude that

$$\frac{1}{|B|_H}\left|\left\{\mathbf{x}\in B\colon |B|_H^{-\beta/d}\left|f(\mathbf{x})-f_B\right|>\lambda\right\}\right|_H\le ee^{-\lambda/t}. \tag{32}$$

This finishes the proof of the lemma.

Kim in [9] gave the property of $L^q(\mathbb{Q}_p^d)$ as in the Euclidean case. □

Lemma 3. *For $\alpha>0$, the distribution function $\omega_H(\alpha)$ of $|f|$ on $\mathbb{Q}_p^d$ defined by the following equation:*

$$\omega_H(\alpha)=\left|\left\{\mathbf{x}\in\mathbb{Q}_p^d\colon |f(\mathbf{x})|>(\alpha\}\right\}\right|_H. \tag{33}$$

If $f\in L^q(\mathbb{Q}_p^d)$ for $q>0$, then we have the following equation:

$$\int_{\mathbb{Q}_p^d}|f(\mathbf{x})|^q\mathrm{d}\mathbf{x}=-\int_0^\infty\alpha^q\mathrm{d}\omega_H(\alpha)=q\int_0^\infty\alpha^{q-1}\omega_H(\alpha)\mathrm{d}\alpha. \tag{34}$$

We also need the p-adic version of Lebesgue differential theorem, which is due to Kim [10].

Lemma 4. *If $f\in L^1(\mathbb{Q}_p^d)$, then we have that for a.e. $\mathbf{x}\in\mathbb{Q}_p^d$*

$$\lim_{\gamma\longrightarrow-\infty}\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}f(\mathbf{y})\mathrm{d}\mathbf{y}=f(\mathbf{x}). \tag{35}$$

The following lemma is about some properties of Lipschitz space $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^d)$.

Lemma 5. *If $f\in Lip_\beta(\mathbb{Q}_p^d)$ is given, then we have the following properties:*

(a) For $q>1$, then there exists a constant $c_{pq}>0$ such that $\|f\|_{Lip_\beta^q(\mathbb{Q}_p^d)}\le c_{pq}\|f\|_{Lip_\beta(\mathbb{Q}_p^d)}$.

(b) The norm $\|f\|_{Lip_\beta^q(\mathbb{Q}_p^d)}$ is equivalent to the norm $\|f\|_{Lip_\beta(\mathbb{Q}_p^d)}$ on Lipschitz space.

(c) For any λ with $(0<\lambda<c_2/\|f\|_{Lip_\beta(\mathbb{Q}_p^d)})$, where c_2 is the constant given in Lemma 1,

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}}\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\exp\left(\lambda\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\right)\mathrm{d}\mathbf{y}<\infty. \tag{36}$$

Proof.

(a) For $q > 1$, using Lemmas 1 and 3 we give the following equation:

$$\begin{aligned}\int_{B_\gamma(\mathbf{x})}\left(\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\right)^q \mathrm{d}\mathbf{y} &= \int_0^\infty q\lambda^{q-1}\left|\left\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ \left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|>\lambda\right\}\right|_H \mathrm{d}\lambda \\ &\le c_1\int_0^\infty q\lambda^{q-1}\exp\left(-\frac{c_2\lambda}{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}\right)d\lambda\ \cdot\left|B_\gamma(\mathbf{x})\right|_H = c_1 q\left(\frac{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}{c_2}\right)^q\int_0^\infty \lambda^{q-1}e^{-\lambda}\mathrm{d}\lambda\cdot\left|B_\gamma(\mathbf{x})\right|_H \\ &= c_1 q\Gamma(q)c_2^{-q}\|f\|^q_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}\cdot\left|B_\gamma(\mathbf{x})\right|_H = c_{pq}^q\|f\|^q_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}\cdot\left|B_\gamma(\mathbf{x})\right|_H,\end{aligned} \tag{37}$$

where $c_{pq} = (c_1 q\Gamma(q)c_2^{-q})^{1/q}$. Hence, we conclude that $\|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^d)} \le c_{pq}\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)}$.

(b) For $q > 1$, by Hölder's inequality it is obvious that

$$\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|^q\mathrm{d}\mathbf{y} \le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|^q\mathrm{d}\mathbf{y}\right)^{1/q}. \tag{38}$$

for any $(\gamma, \mathbf{x}) \in \mathbb{Z}\times\mathbb{Q}_p^d$. Thus, we have that $\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)} \le \|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^d)}$. Hence, by (a) we can obtain the equivalence of those two norms.

(c) We first observe that

$$\int_0^{\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\right)\mathrm{d}t = \exp\left(\lambda\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\right)-1,\quad \mathbf{y}\in B_\gamma(\mathbf{x}). \tag{39}$$

For any $(\gamma, \mathbf{x}) \in \mathbb{Z}\times\mathbb{Q}_p^d$. Then, it follows from Lemma 1 and the p-adic version [1] of changing the order of integration that

$$\begin{aligned}\int_{B_\gamma(\mathbf{x})}\exp\left(\lambda\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\right)\mathrm{d}\mathbf{y} &= \int_{B_\gamma(\mathbf{x})}\left(\int_0^{\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|}\lambda e^{\lambda t}\mathrm{d}t+1\right)\mathrm{d}\mathbf{y} \\ &= \int_0^\infty \lambda e^{\lambda t}\left|\left\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ \left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|>t\right\}\right|_H\mathrm{d}t+\left|B_\gamma(\mathbf{x})\right|_H \\ &\le c_1\lambda\int_0^\infty\exp\left(\lambda t-\frac{c_2\lambda}{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}\right)\mathrm{d}t\cdot\left|B_\gamma(\mathbf{x})\right|_H+\left|B_\gamma(\mathbf{x})\right|_H \\ &= \left[c_1\lambda\left(\frac{c_2}{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}-\lambda\right)^{-1}+1\right]\cdot\left|B_\gamma(\mathbf{x})\right|_H.\end{aligned} \tag{40}$$

for any $(\gamma, \mathbf{x}) \in \mathbb{Z}\times\mathbb{Q}_p^d$, provided that $0 < \lambda < c_2/\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)}$. Therefore, we have the following equation:ecmath;

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}}\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\exp\left(\lambda\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\right)\mathrm{d}\mathbf{y}\le c_1\lambda\left(\frac{c_2}{\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}}-\lambda\right)^{-1}+1<\infty. \tag{41}$$

This completes the proof of the lemma.

To prove the theorems, we need the following key lemma. In Euclidean case, the Lipschitz space $\Lambda_\beta(\mathbb{R}^d)$ coincides with some Morrey-Compana to space [35] and can be characterized by mean oscillation which is due to DeVore and Sharpley [36], Janson et al. [35] and Harboure et al. [37] (see also Paluszyński [38]). □

Lemma 6. *For $1\le q<\infty$ and $0<\beta<1$, then the homogeneous Lipschitz space $\Lambda_\beta(\mathbb{Q}_p^d)$ coincides with the space $Lip_\beta^q(\mathbb{Q}_p^d)$.*

Proof. Using the property (b) of Lemma 5, we only need to prove that for some constant $C>0$,

$$\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}\le\|f\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}\le C\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}. \tag{42}$$

Now, we give the proof of the right side of inequality (42). Assume that $f\in\mathrm{Lip}_\beta(\mathbb{Q}_p^d)$. Given $\mathbf{x},\mathbf{y}\in\mathbb{Q}_p^d$ and $\mathbf{x}\ne\mathbf{y}$, set $B=B(\mathbf{x},|\mathbf{x}-\mathbf{y}|_p)$ and $B\prime=(\mathbf{y},|\mathbf{x}-\mathbf{y}|_p)$. Then we have the following equation:

$$|f(\mathbf{x})-f(\mathbf{y})|\le\left|f(\mathbf{x})-f_B\right|+\left|f(\mathbf{y})-f_{B\prime}\right|+\left|f_B-f_{B'}\right|. \tag{43}$$

We estimate only the first term of the right side, since the others follow similar lines. Letting $B_i=B(\mathbf{x},p^{-i}|\mathbf{x}-\mathbf{y}|_p)$ for $i\ge1$ and $B_0=B$, then by Lemma 4, we show that

$$\begin{aligned}\left|f(\mathbf{x})-f_B\right|&\le\lim_{\gamma\longrightarrow\infty}\left(\left|f(\mathbf{x})-f_{B_\gamma}\right|+\sum_{i=0}^{\gamma-1}\left|f_{B_{i+1}}-f_{B_i}\right|\right)\le C\sum_{i=1}^{\infty}\frac{1}{|B_i|_H}\int_{B_i}\left|f(\mathbf{z})-f_{B_i}\right|\mathrm{d}\mathbf{z}\\&\le C\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}\sum_{i=1}^{\infty}\left|B_i\right|_H^{\beta/d}\le C\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}\sum_{i=1}^{\infty}p^{-\beta i+\log_p|\mathbf{x}-\mathbf{y}|_p^\beta}\le C|\mathbf{x}-\mathbf{y}|_p^\beta\|f\|_{\mathrm{Lip}_\beta\left(\mathbb{Q}_p^d\right)}.\end{aligned} \tag{44}$$

Thus, we obtain $\|f\|_{\Lambda_\beta(\mathbb{Q}_p^d)}\le C\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)}$.

On the other hand, for $\mathbf{y},\quad\mathbf{z}\in B_\gamma(\mathbf{x})$ and $\mathbf{y}\ne\mathbf{z}$, we have the following equation:

$$\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\mathrm{d}\mathbf{y}\le\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{2+\beta/d}}\int_{B_\gamma(\mathbf{x})}\int_{B_\gamma(\mathbf{x})}|f(\mathbf{y})-f(\mathbf{z})|\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}. \tag{45}$$

Since $\mathbf{y},\mathbf{z}\in B_\gamma(\mathbf{x})$ and (b) of Lemma 2 implies that $|\mathbf{y}-\mathbf{z}|_p\le|B_\gamma(\mathbf{x})|_H^{1/d}$. Hence, by using above inequality, we can infer that

$$\begin{aligned}\frac{1}{\left|B_\gamma(\mathbf{x})|\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|f(\mathbf{y})-f_{B_\gamma(\mathbf{x})}\right|\mathrm{d}\mathbf{y}&\le\frac{\|f\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}}{\left|B_\gamma(\mathbf{x})\right|_H^{2+\beta/d}}\int_{B_\gamma(\mathbf{x})}\int_{B_\gamma(\mathbf{x})}|\mathbf{y}-\mathbf{z}|_p^\beta\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\\&\le\frac{\|f\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}}{\left|B_\gamma(\mathbf{x})\right|_H^{2+\beta/d}}\int_{B_\gamma(\mathbf{x})}\int_{B_\gamma(\mathbf{x})}\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\Big]=\|f\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}.\end{aligned} \tag{46}$$

for any $(\gamma,\mathbf{x})\in\mathbb{Z}\times\mathbb{Q}_p^d$. Thus, we have $\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^d)}\le\|f\|_{\Lambda_\beta(\mathbb{Q}_p^d)}$. This concludes the proof of the lemma.

In [34], Taibleson introduced and studied the Riesz potentials on local fields. Let us recall the definition of Riesz potentials on p-adic vector space as follows:

$$I_\alpha^p(f)(\mathbf{x}) = \frac{1}{\Gamma_d(\alpha)} \int_{\mathbb{Q}_p^d} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\alpha}} \mathrm{d}\mathbf{y}, \tag{47}$$

where $\Gamma_d(\alpha) = (1 - p^{\alpha-n}/1 - p^{-\alpha})$ with $\alpha \in \mathbb{C}/\{0\}$.

Let us recall the Hardy-Littlewood-Sobolev inequality for the Riesz potentials I_α^p on p-adic vector space. More details, the interested reader may refer to the book [34]. □

Lemma 7. *Let α be a complex number with $0 < Re\alpha < d$ and $1 \le q < \infty$ such that $1/r = 1/q - Re\alpha/d$. Then, the following statements are held.*

(a) If $q > 1$, then I_α^p is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$.

(b) If $q = 1$, then I_α^p is bounded from $L^1(\mathbb{Q}_p^d)$ to $L^{r,\infty}(\mathbb{Q}_p^d)$.

Let α be a complex number with $0 < \mathrm{Re}\alpha < d$ and f be locally integrable function on $\mathbb{Q}_p^d$, the p-adic version of fractional maximal function of f is given by the following equation:

$$\mathscr{M}_\alpha^p(f)(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1-\alpha/d}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \mathrm{d}\mathbf{y}. \tag{48}$$

Using the definition of the p-adic version of Riesz potential I_α^p in equation (47), we give the following inequality:

$$\left|I_\alpha^p(|f|)(\mathbf{x})\right| = \left|\frac{1}{\Gamma_d(\alpha)} \int_{\mathbb{Q}_p^d} \frac{|f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\alpha}} \mathrm{d}\mathbf{y}\right| \ge \frac{1}{\Gamma_d(\alpha)} \int_{|\mathbf{x}-\mathbf{y}|_p \le p^\gamma} \frac{|f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\mathrm{Re}\alpha}} \mathrm{d}\mathbf{y} \ge \frac{1}{\Gamma_d(\alpha)} \frac{1}{p^{\gamma(n-\mathrm{Re}\alpha)}} \int_{|\mathbf{x}-\mathbf{y}|_p \le p^\gamma} |f(\mathbf{y})| \mathrm{d}\mathbf{y}. \tag{49}$$

So inequality (49) gives the following equation:

$$\left|\mathscr{M}_\alpha^p(f)(\mathbf{x})\right| \le C \left|I_\alpha^p(|f|)(\mathbf{x})\right|. \tag{50}$$

Using inequality (50) and Lemma 7, we give the L^q boundedness of p-adic version of fractional maximal function $\mathscr{M}_\alpha^p$ on the p-adic vector space.

Lemma 8. *Let α be a complex number with $0 < Re\alpha < d$ and $1 \le q < \infty$ such that $1/r = 1/q - Re\alpha/d$. Then, the following statement are held.*

(a) If $q > 1$, then $\mathscr{M}_\alpha^p$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$.

(b) If $q = 1$, then $\mathscr{M}_\alpha^p$ is bounded from $L^1(\mathbb{Q}_p^d)$ to $L^{r,\infty}(\mathbb{Q}_p^d)$.

We now state the boundedness of I_α^p on Morrey space which will be useful in proving our results. This lemma is similar to the results of ones from [39–41]. In Euclidean setting, Adams [23] and Spanne [42] studied the corresponding results, and see also [43].

Lemma 9. *Let $0 < Re\alpha < d$, $1 < q < d/Re\alpha$, and $0 < \lambda < d - qRe\alpha$.*

(1) If $1/r = 1/q - Re\alpha/(d-\lambda)$, then I_α^p is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$.

(2) If $1/r = 1/q - Re\alpha/d$ and $\lambda/q = \nu/r$, then I_α^p is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\nu}(\mathbb{Q}_p^d)$.

Proof. (1) Let $x \in \mathbb{Q}_p^d$ and $\gamma \in \mathbb{Z}$. Then, we split the following equation:

$$\Gamma_d(\alpha) I_\alpha^p(f)(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|_p \le p^\gamma} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{d-\alpha}} d\mathbf{y} + \int_{|\mathbf{x}-\mathbf{y}|_p > p^\gamma} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{d-\alpha}} d\mathbf{y} =: I_1(\mathbf{x}) + I_2(\mathbf{x}). \tag{51}$$

First, we estimate $I_1(\mathbf{x})$. Using the definition of $\mathscr{M}^p$, we obtain the following equation:

$$\left|I_1(\mathbf{x})\right| \le \sum_{k=-\infty}^{\gamma} \int_{S_\gamma(\mathbf{x})} \frac{|f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{d-\mathrm{Re}\alpha}} \mathrm{d}\mathbf{y}, = \sum_{k=-\infty}^{\gamma} p^{k(\mathrm{Re}\alpha-d)} \int_{S_\gamma(\mathbf{x})} |f(\mathbf{y})| \mathrm{d}\mathbf{y} \le C p^{\gamma \mathrm{Re}\alpha} \mathscr{M}^p(f)(\mathbf{x}). \tag{52}$$

On the other hand, by Hölder's inequality, we have that $\mathrm{Re}\alpha - (d-\lambda)/q < 0$

$$\left|I_2(\mathbf{x})\right| \le \sum_{k=\gamma+1}^{\infty} p^{k(\mathrm{Re}\alpha-d)} \int_{S_\gamma(\mathbf{x})} |f(\mathbf{y})| \mathrm{d}\mathbf{y} \le \sum_{k=\gamma+1}^{\infty} p^{k(\mathrm{Re}\alpha-(d-\lambda)/q)} \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)} \le C p^{\gamma(\mathrm{Re}\alpha-(d-\lambda)/q)} \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}. \tag{53}$$

Combining equations (52) and (53) gives the following equation:

$$\left|I_\alpha^p(f)(\mathbf{x})\right| \lesssim p^{\gamma \mathrm{Re}\alpha} \mathscr{M}^p(f)(\mathbf{x}) + p^{\gamma(\mathrm{Re}\alpha-(d-\lambda)/q)} \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}. \tag{54}$$

Taking

$$p^\gamma = \left(\frac{\|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}}{\mathscr{M}^p(f)(\mathbf{x})}\right)^{(q/d-\lambda)}. \tag{55}$$

then equation (54) becomes as follows:

$$\left|I_\alpha^p(f)(\mathbf{x})\right| \lesssim \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}^{\frac{q\mathrm{Re}\alpha}{d-\lambda}} \left|\mathscr{M}^p(f)(\mathbf{x})\right|^{1-(q\mathrm{Re}\alpha/d-\lambda)}. \tag{56}$$

Since $1/r = 1/q - \mathrm{Re}\alpha/(d-\lambda)$, we have $(d-\lambda-q\mathrm{Re}\alpha)r/(d-\lambda) = q$. Thus, by the L^q boundedness of the p-adic version of maximal operator $\mathscr{M}^p$, we can infer that

$$\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} \left|I_\alpha^p(f)(\mathbf{y})\right|^r d\mathbf{y} \lesssim \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}^{(qr\mathrm{Re}\alpha/d-\lambda)} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} \left|\mathscr{M}^p(f)(\mathbf{y})\right|^q d\mathbf{y} \lesssim \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}^{r-q} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|^q d\mathbf{y} \le \|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}^r, \tag{57}$$

for any $(\gamma, \mathbf{x}) \in \mathbb{Z} \times \mathbb{Q}_p^d$. Hence, we conclude that $\|I_\alpha^p(f)\|_{L^{r,\lambda}(\mathbb{Q}_p^d)} \le C\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)}$.

Now, we start to prove (2). Assume that $r_1, q, \alpha, \lambda, d$ satisfy the condition of (1), then $1/r_1 = 1/q - \mathrm{Re}\alpha/(d-\lambda)$. It follows that

$$\frac{1}{r} = \frac{1}{q} - \frac{\mathrm{Re}\alpha}{d} \ge \frac{1}{q} - \frac{\mathrm{Re}\alpha}{d-\lambda} = \frac{1}{r_1}. \tag{58}$$

that is equivalent to $r \le r_1$. Thus, by using Hölder's inequality, we have the following equation:

$$\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\nu/d}} \int_{B_\gamma(\mathbf{x})} \left|I_\alpha^p(f)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r}, \le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\nu/(dr)-1/r+1/r_1-\lambda/(dr_1)}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} \left|I_\alpha^p(f)(\mathbf{y})\right|^{r_1} d\mathbf{y}\right)^{1/r_1}. \tag{59}$$

Using conditions $1/r_1 = 1/q - \mathrm{Re}\alpha/(d-\lambda)$, $1/r = 1/q - \mathrm{Re}\alpha/d$, and $\lambda/q = \nu/r$, we show that

$$\frac{\nu}{dr} - \frac{1}{r} + \frac{1}{r_1} - \frac{\lambda}{dr_1} = \frac{\lambda}{dq} - \frac{1}{q} + \frac{\mathrm{Re}\alpha}{d} + \frac{n-\lambda}{d} \cdot \left(\frac{1}{q} - \frac{\mathrm{Re}\alpha}{d-\lambda}\right) = 0. \tag{60}$$

Hence, by using the conclusion of equation (1), we obtain the following equation:

$$\left\|I_\alpha^p(f)\right\|_{L^{r,\nu}\left(\mathbb{Q}_p^d\right)} \le C\|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}. \tag{61}$$

This finishes the proof of the lemma.

Using inequality (50) and Lemma 9, we immediately obtain the following lemma: □

Lemma 10. *Let $0 < Re\alpha < d, 1 < q < d/Re\alpha$, and $0 < \lambda < d - qRe\alpha$.*

(1) If $1/r = 1/q - Re\alpha/(d-\lambda)$, then $\mathscr{M}_\alpha^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$.

(2) If $1/r = 1/q - Re\alpha/d$ and $\lambda/q = \nu/r$, then $\mathscr{M}_\alpha^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\nu}(\mathbb{Q}_p^d)$.

Considering the characteristic function $\chi_{B_\gamma(\mathbf{x})}$, we have the following property.

Lemma 11. *Let $1 \le q < \infty$ and $0 < \lambda < d$, then there exist a constant $C > 0$ such that*

$$\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)} = \left|B_\gamma(\mathbf{x})\right|_H^{d-\lambda/dq}. \tag{62}$$

Proof. Using the definition of $L^{q,\lambda}(\mathbb{Q}_p^d)$ and Lemma 2, it is easy to compute that for $0 < \lambda < d$

$$\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)} = p^{\gamma(d-\lambda)/q} = \left|B_\gamma(\mathbf{x})\right|_H^{d-\lambda/dq}, \tag{63}$$

and the lemma is proved. □

3. The Proof of Theorems 1.1–1.3

Proof of Theorem 1. If $b \in \Lambda_\beta(\mathbb{Q}_p^d)$, then

$$\begin{aligned}\mathscr{M}_b^p(f)(\mathbf{x}) &= \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{x})-b(\mathbf{y})f(\mathbf{y})|d\mathbf{y} \le \|b\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)} \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1-\beta/d}}\int_{B_\gamma(\mathbf{x})}|f(\mathbf{y})|d\mathbf{y}\\ &= \|b\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}\mathscr{M}_\beta^p(f)(\mathbf{x}).\end{aligned} \tag{64}$$

Obviously, equations (2)–(5) follow from Lemma 8 and (64).

(3) ⇒ (1): Assume that $\mathscr{M}_b^p$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^q(\mathbb{Q}_r^d)$ for some q, r with $1<q<d/\beta$ and $1/r = 1/q-\beta/n$. For any p-adic ball $B_\gamma(\mathbf{x})\subset\mathbb{Q}_p^d$, by Hölder's inequality and noting that $1/q+1/r' = 1+\beta/d$, one gives that

$$\begin{aligned}\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})-b_{B_\gamma(\mathbf{x})}\right|\mathrm{d}\mathbf{y} &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/\mathrm{d}}}\int_{B_\gamma(\mathbf{x})}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\mathrm{d}\mathbf{z}\right)\mathrm{d}\mathbf{y}\\ &= \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_\gamma(\mathbf{x})}(\mathbf{z})\mathrm{d}\mathbf{z}\right)d\mathbf{y}\\ &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}M_b^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})d\mathbf{y} \le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\left(\int_{B_\gamma(\mathbf{x})}\left|M_B^P\left(\chi_{B_\gamma(\mathbf{x})}(\mathbf{y})\right)\right|^r\mathrm{d}\mathbf{y}\right)^{1/r}\\ &\quad\left(\int_{B_\gamma(\mathbf{x})}\chi_{B_\gamma(\mathbf{x})}(\mathbf{y})\mathrm{d}\mathbf{y}\right)^{1/r'}\\ &\le \frac{C}{\left|B_\gamma\right|}(\mathbf{x})_H^{1+\beta/d}\left\|M_b^p\right\|_{L^q\left(\mathbb{Q}_p^d\right)\longrightarrow L^r\left(\mathbb{Q}_p^d\right)}X_{B_\gamma(\mathbf{x})}\left\|L^q\left(Q_P^d\right)\right\|\left\|X_{B_\gamma(\mathbf{x})}\right\|L^{r'}\left(Q_p^d\right) \le C\left\|M_b^p\right\|_{L^q\left(\mathbb{Q}_P^D\right)\longrightarrow L^r\left(\mathbb{Q}_P^d\right)}.\end{aligned} \tag{65}$$

This together with Lemma 6 implies that $b\in\Lambda_\beta(\mathbb{Q}_p^d)$.

(4) ⇒ (1): We assume that (8) is true and will verify $b\in\Lambda_\beta(\mathbb{Q}_p^d)$. For any p-adic $B_{\gamma_0}(\mathbf{x})\subset\mathbb{Q}_p^d$, since for any $\mathbf{y}\in B_{\gamma_0}(\mathbf{x})$

$$\left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right| \le \frac{1}{\left|B_{\gamma_0}(\mathbf{x})\right|_H}\int_{B_{\gamma_0}(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|d\mathbf{z}, \tag{66}$$

then, for all $\mathbf{y}\in B_{\gamma_0}(\mathbf{x})$

$$\begin{aligned}\mathscr{M}_b^p\left(\chi_{B_{\gamma_0}(\mathbf{x})}\right)(\mathbf{y}) &= \sup_{\gamma\in\mathbb{Z}}\frac{1}{\left|B_\gamma(\mathbf{y})\right|_H}\int_{B_\gamma(\mathbf{y})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_{\gamma_0}(\mathbf{x})}(\mathbf{z})d\mathbf{z}, \ge \frac{1}{\left|B_{\gamma_0}(\mathbf{x})\right|_H}\int_{B_{\gamma_0}(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_{\gamma_0}(\mathbf{x})}(\mathbf{z})d\mathbf{z}\\ &= \frac{1}{\left|B_{\gamma_0}(\mathbf{x})\right|_H}\int_{B_{\gamma_0}(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|d\mathbf{z}, \ge \left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|.\end{aligned} \tag{67}$$

This together with equation (8) gives the following equation:

$$\begin{aligned}\left|\left\{\mathbf{y}\in B_{\gamma_0}(\mathbf{x}): \left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|>\lambda\right\}\right|_H &\le \left|\left\{\mathbf{y}\in B_{\gamma_0}(\mathbf{x}): \mathscr{M}_b^p\left(\chi_{B_{\gamma_0}(\mathbf{x})}\right)(\mathbf{y})>\lambda\right\}\right|_H\\ &\le C\left(\lambda^{-1}\left\|\chi_{B_{\gamma_0}(\mathbf{x})}\right\|_{L^1\left(\mathbb{Q}_p^d\right)}\right)^{d/(d-\beta)} = C\left(\lambda^{-1}\left|B_{\gamma_0}(\mathbf{x})\right|_H\right)^{d/(d-\beta)}.\end{aligned} \tag{68}$$

Let $t>0$ be a constant to be determined later, then by using Lemma 3, we have the following equation:

$$\begin{aligned}\int_{B_{\gamma_0}(\mathbf{x})}\left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|d\mathbf{y} &= \int_0^\infty\left|\left\{\mathbf{y}\in B_{\gamma_0}(\mathbf{x}):\ \left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|>\lambda\right\}\right|_H d\lambda\\ &= \int_0^t\left|\left\{\mathbf{y}\in B_{\gamma_0}(\mathbf{x}):\ \left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|>\lambda\right\}\right|_H d\lambda+\int_t^\infty\left|\left\{\mathbf{y}\in B_{\gamma_0}(\mathbf{x}):\ \left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|>\lambda\right\}\right|_H d\lambda\\ &\le t\left|B_{\gamma_0}(\mathbf{x})\right|_H+C\int_t^\infty\left(\lambda^{-1}\left|B_{\gamma_0}(\mathbf{x})\right|_H\right)^{d/(d-\beta)}d\lambda\ \le t\left|B_{\gamma_0}(\mathbf{x})\right|_H+C\left|B_{\gamma_0}(\mathbf{x})\right|_H^{d/(d-\beta)}\int_t^\infty\lambda^{-d/(d-\beta)}d\lambda\\ &\le C\left(t\left|B_{\gamma_0}(\mathbf{x})\right|_H+t^{1-d/(d-\beta)}\left|B_{\gamma_0}(\mathbf{x})\right|_H^{d/(d-\beta)}\right).\end{aligned}\tag{69}$$

Taking $t=|B_{\gamma_0}(\mathbf{x})|_H^{\beta/d}$ in the above estimate, we have the following equation:

$$\int_{B_{\gamma_0}(\mathbf{x})}\left|b(\mathbf{y})-b_{B_{\gamma_0}(\mathbf{x})}\right|d\mathbf{y}\le C\left|B_{\gamma_0}(\mathbf{x})\right|_H^{1+\beta/d}.\tag{70}$$

It follows from Lemma 6 that $b\in\Lambda_\beta(\mathbb{Q}_p^d)$ since $B_{\gamma_0}(\mathbf{x})$ is an arbitrary p-adic ball in $\mathbb{Q}_p^d$.

(5) $\Rightarrow$ (1): If $\mathscr{M}_b^p$ is bounded from $L^{d/\beta}(\mathbb{Q}_p^d)$ to $L^\infty(\mathbb{Q}_p^d)$, then for any p-adic ball $B_\gamma(\mathbf{x})\subset\mathbb{Q}_p^d$

$$\begin{aligned}\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})-b_{B_\gamma(\mathbf{x})}\right|d\mathbf{y} &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_\gamma(\mathbf{x})}(\mathbf{z})d\mathbf{z}\right)d\mathbf{y}\\ &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}}\int_{B_\gamma(\mathbf{x})}\mathscr{M}_b^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})d\mathbf{y}\le\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left\|\mathscr{M}_b^p\left(\chi_{B_\gamma(\mathbf{x})}\right)\right\|_{L^\infty\left(\mathbb{Q}_p^d\right)}\\ &\le\frac{C}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left\|\mathscr{M}_b^p\right\|_{L^{d/\beta}\left(\mathbb{Q}_p^d\right)\longrightarrow L^\infty\left(\mathbb{Q}_p^d\right)}\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^{d/\beta}\left(\mathbb{Q}_p^d\right)}\le C\left\|\mathscr{M}_b^p\right\|_{L^{d/\beta}\left(\mathbb{Q}_p^d\right)\longrightarrow L^\infty\left(\mathbb{Q}_p^d\right)}.\end{aligned}\tag{71}$$

This together with Lemma 6 implies that $b\in\Lambda_\beta(\mathbb{Q}_p^d)$.

The Proof of Theorem 1 is completed since (2) $\Rightarrow$ (1) follows from (3) $\Rightarrow$ (1). □

Proof of Theorem 2. Assume that $b\in\Lambda_\beta(\mathbb{Q}_p^d)$. By using equations (64) and (1) of Lemma 9, we have the following equation:

$$\left\|\mathscr{M}_b^p(f)\right\|_{L^{r,\lambda}\left(\mathbb{Q}_p^d\right)}\le\|b\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}\left\|\mathscr{M}_\beta^p\right\|_{L^{r,\lambda}\left(\mathbb{Q}_p^d\right)}\le C\|b\|_{\Lambda_\beta\left(\mathbb{Q}_p^d\right)}\|f\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}.\tag{72}$$

Conversely, if $\mathscr{M}_b^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$, then for any p-adic ball $B_\gamma(\mathbf{x})\subset\mathbb{Q}_p^d$

$$\begin{aligned}&\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})-b_{B_\gamma(\mathbf{x})}\right|^q d\mathbf{y}\right)^{1/q}\\ &\le\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left[\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_\gamma(\mathbf{x})}(\mathbf{z})d\mathbf{z}\right]^q d\mathbf{y}\right)^{1/q}\\ &\le\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|\mathscr{M}_b^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^q d\mathbf{y}\right)^{1/q}\\ &=\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}}{\left|B_\gamma(\mathbf{x})\right|_H}\right)^{1/q}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/d}}\int_{B_\gamma(\mathbf{x})}\left|\mathscr{M}_b^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^q d\mathbf{y}\right)^{1/q}\\ &\le\left|B_\gamma(\mathbf{x})\right|_H^{-\beta/d-1/q+\lambda/(dq)}\left\|\mathscr{M}_b^p\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)\longrightarrow L^{r,\lambda}\left(\mathbb{Q}_p^d\right)}\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}\le C\left\|\mathscr{M}_b^p\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)\longrightarrow L^{r,\lambda}\left(\mathbb{Q}_p^d\right)},\end{aligned}\tag{73}$$

where in the last step we have used $1/r = 1/q - \beta/(d-\lambda)$ and Lemma 10.

It follows from Lemma 6 that $b \in \Lambda_\beta(\mathbb{Q}_p^d)$. This finishes the Proof of Theorem 2. □

Proof of Theorem 3. By using similar proof to the one of Theorem 2, we can obtain Theorem 3. Hence, we omit the details. □ □

4. The Proof of Theorems 4–7

Proof of Theorem 4. (2) ⇒ (1): For any fixed $\mathbf{x} \in \mathbb{Q}_p^d$ such that $\mathcal{M}^p(f)(\mathbf{x}) < \infty$, since $b \geq 0$ then we have the following equation:

$$\begin{aligned}
\left|[b,\mathcal{M}^p](f)(\mathbf{x})\right| &= \left|b(\mathbf{x})\mathcal{M}^p(f)(\mathbf{x}) - \mathcal{M}^p(bf)(\mathbf{x})\right| \\
&= \left|\sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})} b(\mathbf{x})|f(\mathbf{y})|d\mathbf{y} - \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})} b(\mathbf{y})|f(\mathbf{y})|d\mathbf{y}\right| \leq \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{y})f(\mathbf{y})|d\mathbf{y}| \\
&= \mathcal{M}_b^p(\mathbf{x}).
\end{aligned} \tag{74}$$

It follows from Theorem 1 that $[b,\mathcal{M}^p]$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^r(\mathbb{Q}_p^d)$ since $b \in \Lambda_\beta(\mathbb{Q}_p^d)$.

(2) ⇒ (3): For any fixed p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$ and all $y \in B_\gamma(\mathbf{x})$, we claim that

$$\begin{aligned}
\mathcal{M}^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y}) &= \chi_{B_\gamma(\mathbf{x})}(\mathbf{x}), \\
\mathcal{M}^p\left(b\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y}) &= \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}).
\end{aligned} \tag{75}$$

In fact, similar to the proof of Lemma 2.12, it is easy to obtain the claim. Then, by using the above claim, we have the following equation:

$$\begin{aligned}
\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r} &= \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})\mathcal{M}^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y}) - \mathcal{M}^p\left(b\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r} \\
&= \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|[b,\mathcal{M}^p]\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r} \\
&\leq \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d+1/r}}\left\|[b,\mathcal{M}^p]\left(\chi_{B_\gamma(\mathbf{x})}\right)\right\|_{L^r\left(\mathbb{Q}_p^d\right)} \leq \frac{C}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d+1/r}}\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^q\left(\mathbb{Q}_p^d\right)} \leq C.
\end{aligned} \tag{76}$$

which gives that (3) since the p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$ is arbitrary.

(3) ⇒ (1): To prove $b \in \Lambda_\beta(\mathbb{Q}_p^d)$, by using Lemma 6, it suffices to verify that there is a constant $C > 0$ such that for any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$,

$$\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right|d\mathbf{y} \leq C. \tag{77}$$

For any fixed p-adic ball $B_\gamma(\mathbf{x})$, let $E = \left\{\mathbf{y} \in B_\gamma(\mathbf{x}): b(\mathbf{y}) \leq b_{B_\gamma(\mathbf{x})}\right\}$ and $B_\gamma(\mathbf{x})$, let $E = \left\{\mathbf{y} \in B_\gamma(\mathbf{x}): b(\mathbf{y}) < b_{B_\gamma(\mathbf{x})}\right\}$. The following equality is trivially true.

$$\int_E \left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right|d\mathbf{y} = \int_F \left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right|d\mathbf{y}. \tag{78}$$

Since for any $\mathbf{y} \in E$ we have $b(\mathbf{y}) \le b_{B_\gamma(\mathbf{x})} \le \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})$, then for any $\mathbf{y} \in E$,

$$\left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right| \le \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right|. \tag{79}$$

Thus, we can conclude that

$$\begin{aligned}
&\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{B_\gamma(\mathbf{x})} \left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right| d\mathbf{y} \\
&= \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{E\cup F} \left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right| d\mathbf{y} \\
&= \frac{2}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{E} \left|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}\right| d\mathbf{y} \\
&= \frac{2}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{E} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right| d\mathbf{y} \\
&= \frac{2}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{E} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right| d\mathbf{y} \\
&= \frac{2}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{B_\gamma(\mathbf{x})} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right| d\mathbf{y}.
\end{aligned} \tag{80}$$

On the other hand, it follows from Hölder's inequality and equation (12) that

$$\begin{aligned}
\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \int_{B_\gamma(\mathbf{x})} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right| d\mathbf{y} &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{1+\beta/d}} \left(\int_{B_\gamma(\mathbf{x})} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right|^r d\mathbf{y} \right)^{1/r} \\
\left|B_\gamma(\mathbf{x})\right|_H^{1/r'} &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left|b(\mathbf{y}) - \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right|^r d\mathbf{y} \right)^{1/r} \le C.
\end{aligned} \tag{81}$$

Combining equations (76) with (80) it follows that $b \in \Lambda_\beta(\mathbb{Q}_p^d)$.

In order to prove $b \ge 0$, it suffices to show $b^- = 0$, where $b^- = -\min\{b, 0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed p-adic ball $B_\gamma(\mathbf{x})$, observe that

$$0 \le b^+(\mathbf{y}) \le |b(\mathbf{x})| \le \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}), \tag{82}$$

for $\mathbf{y} \in B_\gamma(\mathbf{x})$ and therefore we have that for $\mathbf{y} \in B_\gamma(\mathbf{x})$,

$$0 \le b^-(\mathbf{y}) \le \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b^+(\mathbf{y}) + b^-(\mathbf{y}) = \mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y}). \tag{83}$$

Then, it follows from equation (12) that for any p-adic ball $\mathbf{y} \in B_\gamma(\mathbf{x})$,

$$\begin{aligned}
\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} b^-(\mathbf{y}) d\mathbf{y} &\le \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left|\mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y})\right| d\mathbf{y} \le \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left|\mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y})\right|^r d\mathbf{y} \right)^{1/r} \\
&= \left|B_\gamma(\mathbf{x})\right|_H^{\beta/d} \left\{ \frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}} \left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H} \int_{B_\gamma(\mathbf{x})} \left|\mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y})\right|^r d\mathbf{y} \right)^{1/r} \right\} \le C \left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}.
\end{aligned} \tag{84}$$

Thus, $b^- = 0$ follows from Lemma 4. Hence, the Proof of Theorems 4 is completed. □

Proof of Theorem 5. Obviously, Theorem 5 follows from equation (74) and Theorem 1. Hence, we omit the details. □

Proof of Theorem 6. (1) ⇒ (2): Assume that $b \geq 0$ and $b \in \Lambda_\beta(\mathbb{Q}_p^d)$, then by using equation (74) and Theorem 2, we show that $[b, \mathscr{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$.

(2) ⇒ (1): Assume that $[b, \mathscr{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{r,\lambda}(\mathbb{Q}_p^d)$. Similar to estimate for equation (76), we have that for any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$,

$$\begin{aligned}
&\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})-\mathscr{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r}\\
&=\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{y})\mathscr{M}^p\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})-\mathscr{M}^p\left(b\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r}\\
&=\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d}}\left(\frac{1}{\left|B_\gamma(\mathbf{x})\right|_H}\int_{B_\gamma(\mathbf{x})}\left|[b,\mathscr{M}^p]\left(\chi_{B_\gamma(\mathbf{x})}\right)(\mathbf{y})\right|^r d\mathbf{y}\right)^{1/r}\leq\frac{\left|B_\gamma(\mathbf{x})\right|_H^{\lambda/(dq)}}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d+1/r}}\left\|[b,\mathscr{M}^p]\left(\chi_{B_\gamma(\mathbf{x})}\right)\right\|_{L^{r,\lambda}\left(\mathbb{Q}_p^d\right)}\\
&\leq C\frac{\left(B_\gamma(\mathbf{x})\right)_H^{\lambda/(dq)}}{\left|B_\gamma(\mathbf{x})\right|_H^{\beta/d+1/r}}\left\|\chi_{B_\gamma(\mathbf{x})}\right\|_{L^{q,\lambda}\left(\mathbb{Q}_p^d\right)}\leq C,
\end{aligned}\tag{85}$$

where in the last step we have used the condition $1/r = 1/q - \beta/(d-\lambda)$ and Lemma 10. Thus, by using Theorem 4, we give that $b \in \Lambda_\beta(\mathbb{Q}_p^d)$ and $b \geq 0$. This finish the Proof of Theorem 6. □

Proof of Theorem 7. By using the same way of the Proof of Theorem 6, Theorem 2.2 can be proven. Hence, we omit the details. □

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Ethical Approval

All procedures were in accordance with the ethical standards of the institutional research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grant nos. 11871452 and 12071473) and Shandong Jianzhu University Foundation (Grant no. X20075Z0101).

References

[1] V. Vladimirov, I. Volovich, and E. Zelenov, *p-adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1992.

[2] Q. He, M. Wei, and D. Yan, "Characterizations of p-adic central Campanato spaces via commutator of p-adic Hardy type operators," *Journal of the Korean Mathematical Society*, vol. 56, no. 3, pp. 767–787, 2019.

[3] A. Hussain, N. Sarfraz, and F. Gürbüz, "Sharp weak bounds for p-adic Hardy operators on p-adic linear spaces," *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*vol. 71, no. 4, pp. 1–11, 2022.

[4] R. Liu and J. Zhou, "Weighted multilinear p-adic Hardy operators and commutators," *Open Mathematics*, vol. 15, no. 1, pp. 1623–1634, 2017.

[5] K. Rim and J. Lee, "Estimate of weighted Hard-Littlewood averages on the p-adic vector space," *Journal of Mathematical Analysis and Applications*, vol. 324, pp. 1470–1477, 2006.

[6] K. Rogers, "A van der Corput lemma for the p-adic numbers," *Proceedings of the American Mathematical Society*, vol. 133, no. 12, pp. 3525–3534, 2005.

[7] K. Rogers, "Maximal averages along curves over the p-adic numbers," *Bulletin of the Australian Mathematical Society*, vol. 70, no. 3, pp. 357–375, 2004.

[8] N. Sarfraz and F. Gürbüz, "Weak and strong boundedness for p-adic fractional hausdorff operator and its commutator," *International Journal of Nonlinear Sciences and Numerical Stimulation*, pp. 1–12, 2021, https://doi.org/10.1515/ijnsns-2020-0290.

[9] Y. Kim, "Carleson measures and the BMO space on the p-adic vector space," *Mathematische Nachrichten*, vol. 282, no. 9, pp. 1278–1304, 2009.

[10] Y. Kim, "L^q-estimates of maximal operators on the p-adic vector space," *Commun. Korean Math. Soc.*vol. 24, no. 3, pp. 367–379, 2009.

[11] J. García-Cuerva, E. Harboure, C. Segovia, and J. Torrea, "Weighted norm inequalities for commutators of strongly singular integrals," *Indiana University Mathematics Journal*, vol. 40, no. 4, pp. 1397–1420, 1991.

[12] F. Gürbüz, "Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces," *Canadian Mathematical Bulletin*, vol. 60, no. 1, pp. 131–145, 2017.

[13] F. Gürbüz, "Weighted anisotropic Morrey Spaces estimates for anisotropic maximal operators," *Journal of Advances in Applied Mathematics*, vol. 2, no. 3, pp. 143–150, 2017.

[14] G. Hu, H. Lin, and D. Yang, "Commutators of the Hardy-Littlewood maximal operator with BMO symbols on spaces of homogeneous type," *Abstract and Applied Analysis*, vol. 2008, Article ID 237937, 21 pages, 2008.

[15] G. Hu and D. Yang, "Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 249–262, 2009.

[16] G. G. Pradolini and W. A. Ramos, "Characterization of Lipschitz functions via the commutators of singular and fractional integral operators in variable Lebesgue spaces," *Potential Analysis*, vol. 46, no. 3, pp. 499–525, 2017.

[17] C. Segovia and J. Torrea, "Vector-valued commutators and applications," *Indiana University Mathematics Journal*, vol. 38, no. 4, pp. 959–971, 1989.

[18] C. Segovia and J. L. Torrea, "Higher order commutators for vector-valued Calderón-Zygmund operators," *Transactions of the American Mathematical Society*, vol. 336, no. 2, pp. 537–556, 1993.

[19] P. Zhang, "Multiple weighted estimates for commutators of multilinear maximal function," *Acta Mathematica Sinica, English Series*, vol. 31, no. 6, pp. 973–994, 2015.

[20] P. Zhang, "Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function," *Comptes Rendus Mathematique*, vol. 355, no. 3, pp. 336–344, 2017.

[21] P. Zhang, "Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces," *Anal. Math. Phys.* vol. 9, no. 3, pp. 1411–1427, 2019.

[22] C. B. Morrey, "On the solutions of quasi-linear elliptic partial differential equations," *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.

[23] D. Adams, "A note on Riesz potentials," *Duke Mathematical Journal*, vol. 42, pp. 765–778, 1975.

[24] L. Caffarelli, "Elliptic second order equations," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 58, no. 1, pp. 253–284, 1988.

[25] F. Gürbüz, "Weighted Morrey and weighted fractional Sobolev-Morrey spaces estimates for a large class of pseudo-differential operators with smooth symbols," *J. Pseudo-Differ. Oper. Appl.*vol. 7, no. 4, pp. 595–607, 2016.

[26] T. Kato, "Strong solutions of the Navier-Stokes equation in Morrey spaces," *Boletim da Sociedade Brasileira de Matemtica*, vol. 22, no. 2, pp. 127–155, 1992.

[27] Z. Shen, "Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains," *American Journal of Mathematics*, vol. 125, no. 5, pp. 1079–1115, 2003.

[28] M. Milman and T. Schonbek, "Second order estimates in interpolation theory and applications," *Proceedings of the American Mathematical Society*, vol. 110, no. 4, pp. 961–969, 1990.

[29] A. Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister, "On the product of functions in BMO and H," *Annales de l'Institut Fourier*, vol. 57, no. 5, pp. 1405–1439, 2007.

[30] J. Bastero, M. Milman, and F. Ruiz, "Commutators for the maximal and sharp functions," *in Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3329–3334, 2000.

[31] P. Zhang and J. Wu, "Commutators of the fractional maximal functions," *Acta. Math. Sin., Chin. Ser.* vol. 52, no. 6, pp. 1235–1238, 2009.

[32] P. Zhang and J. Wu, "Commutators of the fractional maximal function on variable exponent Lebesgue spaces," *Czechoslovak Mathematical Journal*, vol. 64, no. 1, pp. 183–197, 2014.

[33] P. Zhang and J. Wu, "Commutators for the maximal functions on Lebesgue spaces with variable exponent," *Mathematical Inequalities and Applications*, vol. 17, no. 4, pp. 1375–1386, 2014.

[34] M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, NJ, USA, 1975.

[35] S. Janson, M. Taibleson, and G. Weiss, "Elementary characterization of the Morrey-Campanato spaces," *Lecture Notes in Mathematics*, vol. 992, pp. 101–114, 1983.

[36] R. A. DeVore and R. C. Sharpley, "Maximal functions measuring smoothness," *Memoirs of the American Mathematical Society*, vol. 47, no. 293, pp. 0–115, 1984.

[37] E. Harboure, O. Salinas, and B. Viviani, "Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces," *Transactions of the American Mathematical Society*, vol. 349, no. 1, pp. 235–255, 1997.

[38] M. Paluszyński, "Characterization of the besov spaces via the commutator operator of coifman, rochberg and weiss," *Indiana University Mathematics Journal*, vol. 44, no. 1, pp. 0–17, 1995.

[39] S. S. Volosivets, "Maximal function and Riesz potential on p-adic linear spaces," *P-Adic Numbers, Ultrametric Analysis, and Applications*, vol. 5, no. 3, pp. 226–234, 2013.

[40] S. Volosivets, "Generalized fractional integrals in p-adic Morrey annd Herz spaces," *Adic Numbers Ultrametric Anal. Appl.*vol. 9, no. 1, pp. 53–61, 2017.

[41] Q. Wu and Z. Fu, "Hardy-Littlewood-Sobolev inequalities on p-adic central Morrey spaces," *J. Funct. Space*, vol. 2015, Article ID 419532, 7 pages, 2015.

[42] J. Peetre, "On the theory of $L^{p,\lambda}$ spaces," *Journal of Functional Analysis*, vol. 4, pp. 71–87, 1969.

[43] F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function," *Rend. Math. Appl.* vol. 7, pp. 273–279, 1987.