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Research Article

Existence and Uniqueness for Coupled Systems of Hilfer Type Sequential Fractional Differential Equations Involving Riemann–Stieltjes Integral Multistrip Boundary Conditions

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In this paper, we study a coupled system of Hilfer type sequential fractional differential equations supplemented with Riemann–Stieltjes integral multistrip boundary conditions. The standard tools of the fixed point theory are employed to prove the existence and uniqueness results for the considered problem. Examples are constructed for the illustration of the obtained results.

1. Introduction

Fractional calculus is a generalization of the classical calculus. Fractional differential equations become another necessary tool in solving real-life problems in different research areas such as physics, biology, engineering, and mechanics, see for example the monographs and papers [1–11].

Boundary value problems of fractional differential equations represent an important and interesting branch of applied analysis. Usually, the researchers have given attention in studying fractional differential equations involving Caputo or Riemann–Liouville fractional derivative. But, Caputo or Riemann–Liouville derivative was not considered appropriate in studying some new models in engineering for example. To avoid the difficulties, some new type fractional order derivative operators were introduced in the literature such as Hadamard, Erdeyl-Kober, and Katugampola. Hilfer in [12] introduced a new derivative, which generalizes both Riemann–Liouville and Caputo derivatives. For some applications involving Hilfer fractional derivative, the interested reader is referred to [13–16] and references cited therein.

In [17], Nuchpong et al. investigated a new class of boundary value problems for fractional differential equations for involving sequential Hilfer type fractional derivative and subject to Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha,\beta} + k\,{}^{H}D^{\alpha-1,\beta}\right)u(z) = f(z,u(z)), \quad z \in [c,d], \\ u(c) = 0, \quad u(d) = \lambda \int_{c}^{d} u(s)\mathrm{d}H(s) + \sum_{i-1}^{n}\mu_i \int_{\eta_i}^{\mu_i} u(s)\mathrm{d}s, \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha,\beta}$ denotes the Hilfer fractional derivative operator of order α, $1<\alpha<2$, and parameter β, $0\le\beta\le 1$, f: $[c,d]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function; $\int_c^d x(s)\mathrm{d}H(s)$ is the Riemann–Stieltjes integral with respect to the function H: $[c,d]\longrightarrow\mathbb{R}$, $c\ge 0$, $k,\mu_i\in\mathbb{R}$, $c<\eta_i<\xi_i\le d$, $i=1,2,\ldots,n$. Existence and uniqueness results are established by using basic tools from fixed point theory.

The study of system of Hilfer type was initiated by Wongcharoen et al. [18], by presenting the following system of fractional differential equations:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}u(z)=f(z,u(z),v(z)), & z\in[c,d],\\ {}^{H}D^{\alpha_2,\beta_2}v(z)=g(z,u(z),v(z)), & z\in[c,d],\\ u(c)=0 \quad u(d)=\sum_{i=1}^{m}\overline{\theta}_i I^{\phi_i}v(\xi_i), & \\ v(c)=0 \quad v(d)=\sum_{j=1}^{n}\overline{\zeta}_j I^{\overline{\phi}_j}u(z_i), & \end{cases} \tag{2}$$

in which ${}^{H}D^{\alpha_1,\beta_1}$ and ${}^{H}D^{\alpha_2,\beta_2}$ indicate the Hilfer fractional derivatives of orders α_1 and α_2, $1<\alpha_1,\alpha_2<2$, and parameters β_1,β_2, $0\le\beta_1,\beta_2\le 1$, f,g: $[c,d]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, $c\ge 0$, $\overline{\theta}_i$, $\overline{\zeta}_i\in\mathbb{R}$, and $I^{\phi_i}, I^{\overline{\phi}_j}$ are the Riemann–Liouville fractional integrals of order $\phi_i>0,\overline{\phi}_j>0$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$.

Inspired by the forenamed studies, this article considers the existence and uniqueness of solutions for the following coupled system of Hilfer type fractional differential equations with Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha_1,\beta_1}+\sigma_1^{H}D^{\alpha_1-1,\beta_1}\right)u(z)=f_1(z,u(z),v(z)) & z\in[c,d],\\ \left({}^{H}D^{\alpha_2,\beta_2}+\sigma_2^{H}D^{\alpha_2-1,\beta_2}\right)v(z)=f_2(z,u(z),v(z)) & z\in[c,d],\\ u(c)=0, \quad u(d)=\lambda_1\int_c^d v(s)\mathrm{d}H_1(s)+\sum_{i=1}^{n}\mu_i\int_{\eta_i}^{\xi_i}v(s)\mathrm{d}s, & \\ v(c)=0, \quad v(d)=\lambda_2\int_c^d u(s)\mathrm{d}H_2(s)+\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}u(s)\mathrm{d}s, & \end{cases} \tag{3}$$

in which ${}^{H}D^{\alpha_1,\beta_1}$ and ${}^{H}D^{\alpha_2,\beta_2}$ are the Hilfer fractional derivatives of orders $1<\alpha_1,\alpha_2<2$ and parameters β_1,β_2, $0\le\beta_1,\beta_2\le 1$, f_1,f_2: $[c,d]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, $\int_c^d(\cdot)\mathrm{d}H_1(s)$, $\int_c^d(\cdot)dH_2(s)$ are the Riemann–Stieltjes integrals with respect to the functions H_i: $[c,d]\longrightarrow\mathbb{R}$, $i=1,2$, $c\ge 0$, $\mu_i,\nu_r\in\mathbb{R}$, η_i, ξ_i, ζ_r, $\theta_r\in(c,d)$, $i=1,2,\ldots,n$, $r=1,2,\ldots,p$, λ_1, λ_2, σ_1, $\sigma_2\in\mathbb{R}$.

The remaining of this article has been regulated as follows: In section 2 some concepts, lemmas and theorems are recalled which will be applied throughout this paper. In Section 3, an auxiliary lemma has been proved which concerns a linear variant of system (3) and it is used to convert the coupled system (3) into a fixed point problem. The classical fixed point theorems have been applied in order to obtain the results regarding the existence/uniqueness in Section 4. Thus, the classical Banach fixed point theorem is applied to obtain uniqueness result, while Leray-Schauder alternative and Krasnosel'skiĭ's fixed point theorems are applied to present existence results. Examples are also constructed to illustrate the obtained results.

2. Preliminaries

Now, the following items are reminded which will be applied to fulfil the main results in the next steps.

Throughout the paper, the Banach space of all continuous mappings from $[c,d]$ to $\mathbb{R}$ are denoted by $\mathscr{Y}=C([c,d],\mathbb{R})$ which is equipped with the norm $\|y\|=\sup\{|y(z)|; z\in[c,d]\}$. It is clear that the space

$\mathscr{Y}\times\mathscr{Y}$, equipped with norm defined by $\|(x,y)\| = \|x\| + \|y\|$, is a Banach space.

Also, $AC^n([c,d],\mathbb{R})$ is the n-times absolutely continuous functions defined as

$$AC^n([c,d],\mathbb{R}) = \left\{f\colon [c,d] \longrightarrow \mathbb{R};\, f^{(n-1)} \in AC([c,d],\mathbb{R})\right\}. \tag{4}$$

For a real valued function $g\colon (0,\infty) \longrightarrow \mathbb{R}$, the Riemann–Liouville fractional integral of order $\eta > 0$ is defined by $I^\eta g(t) = \int_0^t ((t-s)^{\eta-1}/\Gamma(\eta))g(s)\mathrm{d}s$, in which the right-hand side is defined point-wise on $(0,\infty)$, see [2]. Besides, for the function g, the Riemann–Liouville fractional derivative of order $\delta > 0$ is defined by $\{^{RL}\}D^\delta g(t) = (1/\Gamma(n-\delta))(d/\mathrm{d}t)^n \int_0^t (g(s)/(t-s)^{s-n+1})\mathrm{d}s$, in which $n = [\delta] + 1$, where $[\delta]$ denotes the integer part of a real number δ, see [2], while the Caputo fractional derivative is defined by $\{^C\}D^s g(t) = (1/\Gamma(n-\delta))\int_0^t (1/ \quad (t-s)^{\delta-n+1})(d/\mathrm{d}s)^n g(s)\mathrm{d}s$, provided that the right-hand side exists.

Also, the Hilfer fractional derivative of order α and parameter β of a function is defined by

$$^H D^{\alpha,\beta} u(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} u(t), \tag{5}$$

where $n-1 < \alpha < n$, $0 \le \beta \le 1$, $t > a$, $D = (d/\mathrm{d}t)$, see [12]. Note that if $\beta = 0$ and $\beta = 1$, the Hilfer derivative is reduced to the Riemann–Liouville and Caputo fractional derivatives, respectively.

The following lemma will be applied to prove a lemma in the next section which presents a pattern of existence of solutions for system (1.3).

Lemma 1 (see [13]). *Let $h \in L(c,d)$, $n_1 - 1 < \overline{\alpha} \le n_1$, $n_1 \in \mathbb{N}$, $0 \le \overline{\beta} \le 1$, $I^{(n_1-\overline{\alpha})(1-\overline{\beta})}h \in AC^k[c,d]$. Then, we have the following relation:*

$$\begin{aligned}\left(I^{\overline{\alpha}\,H}D^{\overline{\alpha},\overline{\beta}}h\right)(z) &= h(z) - \sum_{k=0}^{n_1-1} \frac{(z-c)^{k-(n_1-\overline{\alpha})(1-\overline{\beta})}}{\Gamma\left(k-(n_1-\overline{\alpha})(1-\overline{\beta})+1\right)} \\ &\cdot \lim_{z\longrightarrow c^+} \frac{d^k}{dz^k}\left(I^{(1-\beta)(n_1-\overline{\alpha})}h\right)(z).\end{aligned} \tag{6}$$

Finally, we collect the fixed point theorems applied to prove the main results in this paper.

Lemma 2 (Banach fixed point theorem, [19]). *Let D be a closed set in X and $T\colon D \longrightarrow D$ satisfies*

$$\begin{aligned}&|Tu - Tv| \le \lambda|u - v|,\\ &\text{for some}\,\lambda \in (0,1),\\ &\text{for all u},\, v \in D.\end{aligned} \tag{7}$$

Then T admits a unique fixed point in D.

Lemma 3 (Leray–Schauder alternative [20]). *Let the set ω be closed bounded convex in X and O an open set contained in ω with $0 \in O$. Then, for the continuous and compact $T\colon \overline{U} \longrightarrow \omega$, either*

(1) $(\mathfrak{a})T$ admits a fixed point in $\overline{U}$, or

(2) $(\mathfrak{a}\mathfrak{a})$ there exists $\mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.

Lemma 4 (Krasnosel'skiĭ fixed point theorem, [21]). *Let N indicate a closed, bounded, convex, and nonempty subset of a Banach space Y and C, D be operators such that (i) $Cx + Dy \in N$ where $x, y \in N$, (ii) C is compact and continuous, and (iii) D is a contraction mapping. Then, there exists $z \in N$ such that $z = Cz + Dz$.*

3. An Auxiliary Result

Lemma 5. *Let $c \ge 0$, $1 < \alpha_1, \alpha_2 < 2$, $0 \le \beta_1, \beta_2 \le 1$, $\gamma_1 = \alpha_1 + 2\beta_1 - \alpha_1\beta_1$, $\gamma_2 = \alpha_2 + 2\beta_2 - \alpha_2\beta_2$, $h_1, h_2 \in C([c,d],\mathbb{R})$ and $\Theta \ne 0$. Then, the solution of the system*

$$\begin{cases}\left(^H D^{\alpha_1,\beta_1} + \sigma_1\,^H D^{\alpha_1-1,\beta_1}\right)u(z) = h_1(z), & z \in [c,d],\\ \left(^H D^{\alpha_2,\beta_2} + \sigma_2\,^H D^{\alpha_2-1,\beta_2}\right)v(z) = h_2(z), & z \in [c,d],\\ u(c) = 0, \quad u(d) = \lambda_1 \displaystyle\int_c^d v(s)\mathrm{d}H_1(s) + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} v(s)\mathrm{d}s, & \\ v(c) = 0, \quad v(d) = \lambda_2 \displaystyle\int_c^d u(s)\mathrm{d}H_2(s) + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} u(s)\mathrm{d}s, & \end{cases} \tag{8}$$

is given by

$$
\begin{aligned}
u(z) = &-\sigma_1 \int_c^z u(s)\mathrm{d}s + I^{\alpha_1} h_1(z) \\
&+ \frac{(z-c)^{\gamma_1 - 1}}{\Theta\Gamma(\gamma_1)} \Bigg[G_3 \Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) \\
&+ \lambda_1 \int_c^d I^{\alpha_2} h_2(s)\mathrm{d}H_1(s) - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s \\
&+ \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} h_2(s)\mathrm{d}s \Bigg) + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} h_1(d) \Bigg) \\
&+ G_2 \Bigg(-\sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2 \int_c^d I^{\alpha_1} h_1(s)\mathrm{d}H_2(s) \\
&- \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} h_1(s)\mathrm{d}s + \sigma_2 \int_c^d v(s)\mathrm{d}s \\
&- I^{\alpha_2} h_2(d) \Bigg) \Bigg],
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
v(z) = &-\sigma_2 \int_c^z v(s)\mathrm{d}s + I^{\alpha_2} h_2(z) \\
&+ \frac{(z-c)^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \Bigg[G_1 \Bigg(-\sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) \\
&+ \lambda_2 \int_c^d I^{\alpha_1} h_1(s)\mathrm{d}H_2(s) - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s \\
&+ \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} h_1(s)\mathrm{d}s \Bigg) + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} h_2(d) \Bigg) \\
&+ G_4 \Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1 \int_c^d I^{\alpha_2} h_2(s)\mathrm{d}H_1(s) \\
&- \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} h_2(s)\mathrm{d}s \Bigg) + \sigma_1 \int_c^d u(s)\mathrm{d}s \\
&- I^{\alpha_1} h_1(d) \Bigg) \Bigg],
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
G_1 &= \frac{(d-c)^{\gamma_1 - 1}}{\Gamma(\gamma_1)}, \\
G_2 &= \lambda_1 \int_c^d \frac{(z-c)^{\gamma_2 - 1}}{\Gamma(\gamma_2)} \mathrm{d}H_1(z) + \sum_{i=1}^n \mu_i \frac{(\xi_i - c)^{\gamma_2} - (\eta_i - c)^{\gamma_2}}{\Gamma(\gamma_2 + 1)}, \\
G_3 &= \frac{(d-c)^{\gamma_2 - 1}}{\Gamma(\gamma_2)}, \\
G_4 &= \lambda_2 \int_c^d \frac{(z-c)^{\gamma_1 - 1}}{\Gamma(\gamma_1)} \mathrm{d}H_2(z) + \sum_{r=1}^p \nu_r \frac{(\theta_r - c)^{\gamma_1} - (\zeta_r - c)^{\gamma_1}}{\Gamma(\gamma_1 + 1)},
\end{aligned} \tag{11}
$$

$$
\Theta = G_1 G_3 - G_2 G_4. \tag{12}
$$

Proof. Let (u, v) be a solution of system (5). By Lemma 2, we have

$$u(z) = k_1\frac{(z-c)^{\gamma_1-2}}{\Gamma(\gamma_1-1)} + k_2\frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1-)} - \sigma_1\int_c^z u(s)\mathrm{d}s + I^{\alpha_1}h_1(z),$$
$$v(z) = d_1\frac{(z-c)^{\gamma_2-2}}{\Gamma(\gamma_2-1)} + d_2\frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2))} - \sigma_2\int_c^z v(s)\mathrm{d}s + I^{\alpha_2}h_2(z), \tag{13}$$

where k_1, k_2, d_1, d_2 are the arbitrary constants, since $(1-\beta_1)(2-\alpha_1) = \gamma_1$ and $(1-\beta_1)(2-\alpha_2) = \gamma_2$. Applying $u(c) = 0$ and $v(c) = 0$, we deduce that $k_1 = d_1 = 0$. Thus, the previous equations become

$$u(z) = k_2\frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} - \sigma_1\int_c^z u(s)\mathrm{d}s + I^{\alpha_1}h_1(z), \tag{14}$$

$$v(z) = d_2\frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} - \sigma_2\int_c^z v(s)\mathrm{d}s + I^{\alpha_2}h_2(z). \tag{15}$$

Now, applying the boundary conditions $u(d) = \lambda_1\int_c^d v(s)\mathrm{d}H_1(s) + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} v(s)\mathrm{d}s$ and $v(d) = \lambda_2\int_c^d u(s)dH_2(s) + \sum_{r=1}^p \int_{\zeta_r}^{\theta_r} u(s)ds$, we get

$$\begin{aligned}
&k_2\frac{(d-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} - d_2\left[\lambda_1\int_c^d \frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)}\mathrm{d}H_1(z) + \sum_{i=1}^n \mu_i\frac{(\xi_i-c)^{\gamma_2}-(\eta_i-c)^{\gamma_2}}{\Gamma(\gamma_2+1)}\right] \\
&= -\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2}h_2(s)\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2}h_2(s)\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}h_1(d), \\
&d_2\frac{(d-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} - k_2\left[\lambda_2\int_c^d \frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathrm{d}H_2(z) + \sum_{r=1}^p \nu_r\frac{(\theta_r-c)^{\gamma_1}-(\zeta_r-c)^{\gamma_1}}{\Gamma(\gamma_1+1)}\right] \\
&= -\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1}h_1(s)\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1}h_1(s)\mathrm{d}s + \sigma_2\int_c^d v(s)ds - I^{\alpha_2}h_2(d).
\end{aligned} \tag{16}$$

Consequently, we have the system

$$k_2G_1 - d_2G_2 = \Omega_1,\ d_2G_3 - k_2G_4 = \Omega_2, \tag{17}$$

where

$$\begin{aligned}
\Omega_1 &= -\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2}h_2(s)\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2}h_2(s)\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}h_1(d), \\
\Omega_2 &= -\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1}h_1(s)\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1}h_1(s)\mathrm{d}s + \sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2}h_2\mathrm{d}.
\end{aligned} \tag{18}$$

By solving the above system, we have

$$k_2 = \frac{\Omega_1 G_3 + \Omega_2 G_2}{\Theta}, \quad d_2 = \frac{G_1 \Omega_2 + G_4 \Omega_1}{\Theta}. \tag{19}$$

Substituting the values of k_2 and d_2 into (10) and (14), respectively, we obtain the solutions (9) and (10). We can obtain the converse by direct computation. The proof is finished. □

4. Existence and Uniqueness Result

Due to Lemma 5, we define an operator $\mathcal{Q}$: $\mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y} \times \mathcal{Y}$ by

$$\mathcal{Q}(u, v)(z) := (\mathcal{Q}_1(u, v)(z), \mathcal{Q}_2(u, v)(z)), \tag{20}$$

where

$$\begin{aligned}
&\mathcal{Q}_1(u,v)(z) \\
&= -\sigma_1 \int_c^z u(s)\mathrm{d}s + I^{\alpha_1} f_1(z, u(z), v(z)) \\
&\quad + \frac{(z-c)^{\gamma_1 - 1}}{\Theta\Gamma(\gamma_1)} \Bigg[G_3 \Bigg(-\sigma_2 \lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) \\
&\quad + \lambda_1 \int_c^d I^{\alpha_2} f_2(s, u(s), v(s))\mathrm{d}H_1(s) - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s, u(s), v(s))\mathrm{d}s + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d, u(d), v(d)) \\
&\quad + G_2 \Bigg(-\sigma_1 \lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2 \int_c^d I^{\alpha_1} f_1(s, u(s), v(s))\mathrm{d}H_2(s) \\
&\quad - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s, u(s), v(s))\mathrm{d}s \\
&\quad + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d, u(d), v(d)),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathcal{Q}_2(u,v)(z) \\
&= -\sigma_2 \int_c^z v(s)\mathrm{d}s + I^{\alpha_2} f_2(z, u(z), v(z)) \\
&\quad + \frac{(z-c)^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} G_1 - \sigma_1 \lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) \\
&\quad + \lambda_2 \int_c^d I^{\alpha_1} f_1(s, (u(s), v(s))\mathrm{d}H_2(s) - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s \\
&\quad + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s, u(s), v(s))\mathrm{d}s + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d, u(d), v(d)) \\
&\quad + G_4 \Bigg(-\sigma_2 \lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1 \int_c^d I^{\alpha_2} f_2(s, u(s), v(s))\mathrm{d}H_1(s) \\
&\quad - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s, u(s), v(s))\mathrm{d}s \\
&\quad + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d, u(d), v(d)) \Bigg) \Bigg].
\end{aligned} \tag{22}$$

For convenience, we set

$$\begin{aligned}
\mathscr{C}_1 &= \frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\frac{1}{\Gamma(\alpha_1+1)} \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\lambda_2\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s) \\
&\quad + \frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}|\nu_r|\left[(\theta_r-c)^{\alpha_1+1}-(\zeta_r-c)^{\alpha_1+1}\right]\Bigg), \\
\mathscr{C}_2 &= \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\Bigg(\frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}|\mu_i|\left[(\xi_i-c)^{\alpha_2+1}-(\eta_i-c)^{\alpha_2+1}\right] \\
&\quad + \frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg) + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}, \\
\mathscr{C}_3 &= |\sigma_1|(d-c) + |G_3|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_1| + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_1||\lambda_2|\int_c^d (s-c)\mathrm{d}H_2(s) \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|\sigma_1|\sum_{r=1}^{p}|\nu_r|\frac{1}{2}\left[(\theta_r-c)^2-(\zeta_r-c)^2\right], \\
\mathscr{C}_4 &= \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3||\sigma_2||\lambda_1|\int_c^d (s-c)\mathrm{d}H_1(s) + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3||\sigma_2|\sum_{i=1}^{n}|\mu_i|\frac{1}{2}\left[(\xi_i-c)^2-(\eta_i-c)^2\right] \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_2|,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathscr{D}_1 &= \frac{1}{\Gamma(\alpha_2+1)} + \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1|\frac{1}{\Gamma(\alpha_2+1)} \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}\Bigg(|\lambda_1|\frac{1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s) + \frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}|\mu_i|\left[(\xi_i-c)^{\alpha_2+1}-(\eta_i-c)^{\alpha_2+1}\right]\Bigg), \\
\mathscr{D}_2 &= \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1|\Bigg(\frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}|\nu_r|\left[(\theta_r-c)^{\alpha_1+1}-(\zeta_r-c)^{\alpha_1+1}\right] + \frac{\lambda_2}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\Bigg) \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}\frac{1}{\Gamma(\alpha_1+1)}, \\
\mathscr{D}_3 &= |\sigma_2|(d-c) + |G_1|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_2| + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_2||\lambda_1|\int_c^d (s-c)\mathrm{d}H_1(s) \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|\sigma_2|\sum_{i=1}^{n}\mu_i\frac{1}{2}\left[(\xi_i-c)^2-(\eta_i-c)^2\right], \\
\mathscr{D}_4 &= \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1||\sigma_1||\lambda_2|\int_c^d (s-c)\mathrm{d}H_2(s) + \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1||\sigma_1|\sum_{r=1}^{p}|\nu_r|\frac{1}{2}\left[(\theta_r-c)^2-(\zeta_r-c)^2\right] \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_1|.
\end{aligned} \tag{24}$$

Now, Banach's fixed point theorem is applied to present the following uniqueness result.

Theorem 1. *Let* $\Theta \neq 0$ *and* $f_1, f_2\colon [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be two continuous functions such that for all* $z \in [c,d]$ *and* $\overline{u}_i, \overline{v}_i \in \mathbb{R}$, $i = 1,2$, *we have*

$$\begin{aligned}&\left|f_1(z,\overline{u}_1,\overline{v}_1) - f_1(z,\overline{u}_2,\overline{v}_2)\right| \leq \ell_1\left(\left|\overline{u}_1 - \overline{u}_2\right| + \left|\overline{v}_1 - \overline{v}_2\right|\right),\\&\left|f_2(z,\overline{u}_1,\overline{v}_1) - f_2(z,\overline{u}_2,\overline{v}_2)\right| \leq \ell_2\left(\left|\overline{u}_1 - \overline{u}_2\right| + \left|\overline{v}_1 - \overline{v}_2\right|\right).\end{aligned} \tag{25}$$

where ℓ_1, ℓ_2 *are positive constants and* $\overline{u}_i, \overline{v}_i \in \mathbb{R}$, $i = 1,2$. *Then, there exists a unique solution of system (3) on* $[c,d]$ *provided that*

$$\ell_1\left(\mathscr{C}_1 + \mathscr{D}_1\right) + \ell_2\left(\mathscr{C}_2 + \mathscr{D}_2\right) + \mathscr{C}_3 + \mathscr{C}_4 + \mathscr{D}_3 + \mathscr{D}_4 < 1. \tag{26}$$

Proof. It suffices to display that the operator Q has a unique fixed point. For this aim, Banach's theorem will be applied. Put $\sup_{z\in[c,d]}|f_1(z,0,0)| \coloneqq M < \infty$ and $\sup_{z\in[c,d]}|f_2(z,0,0)| \coloneqq N < \infty$. Now, we locate $B_r = \{(u,v) \in \mathscr{Y} \times \mathscr{Y};\ \|(u,v)\| \leq r\}$, in which

$$r \geq \frac{M\left(\mathscr{C}_1 + \mathscr{D}_1\right) + N\left(\mathscr{C}_2 + \mathscr{D}_2\right)}{1 - \left[\ell_1\left(\mathscr{C}_1 + \mathscr{D}_1\right) + \ell_2\left(\mathscr{C}_2 + \mathscr{D}_2\right) + \mathscr{C}_3 + \mathscr{C}_4 + \mathscr{D}_3 + \mathscr{D}_4\right]}. \tag{27}$$

First, we indicate that $\mathcal{Q}(B_r) \subseteq B_r$. Assume that $(u,v) \in B_r$ and $z \in [c,d]$. Due to (), we have

$$\left|f_1(z,u(z),v(z))\right| \leq \left|f_1(z,u(z),v(z)) - f_1(z,0,0)\right| + \left|f_1(z,0,0)\right| \leq \ell_1(|u(z)| + |v(z)|) + M = \ell_1 r + M. \tag{28}$$

Similarly, we have

$$\left|f_2(z,u(z),v(z))\right| \leq \ell_2 r + N. \tag{29}$$

Hence, we infer that

$$\begin{aligned}&\left|\mathcal{Q}_1(u,v)(z)\right|\\&\quad\leq r\Bigg[\left|\sigma_1\right|(d-c) + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d (s-c)\mathrm{d}H_1(s)\\&\quad+\left|\sigma_2\right|\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s + \left|\sigma_1\right|(d-c)\Bigg)\\&\quad+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d(s-c)\mathrm{d}H_2(s) + \left|\sigma_1\right|\sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\\&\quad+\left|\sigma_2\right|(d-c)\Bigg)\Bigg] + (\ell_1 r + M)\Bigg[\frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}\\&\quad+\left|G_2\right|\frac{(d-c)^{\gamma-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\lambda_2\right|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d(s-c)^{\alpha_1}\mathrm{d}H_2(s)\\&\quad+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^p\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\Bigg)\Bigg]\\&\quad+(\ell_2 r + N)\Bigg[\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s\\&\quad+\frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d(s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg) + \left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\Bigg]\\&\quad= (\ell_1 r + M)\mathscr{C}_1 + (\ell_2 r + N)\mathscr{C}_2 + r\left(\mathscr{C}_3 + \mathscr{C}_4\right).\end{aligned} \tag{30}$$

Consequently,

$$\|\mathcal{Q}_1(uv)\| \leq (\ell_1 r + M)\mathcal{C}_1 + (\ell_2 r + N)\mathcal{C}_2 + r(\mathcal{C}_3 + \mathcal{C}_4). \tag{31}$$

In the same way, we have

$$\|\mathcal{Q}_2(u,v)\| \leq (\ell_1 r + M)\mathcal{D}_1 + (\ell_2 r + N)\mathcal{D}_2 + r(\mathcal{D}_3 + \mathcal{D}_4). \tag{32}$$

Hence,

$$\begin{aligned}\|\mathcal{Q}(u,v)\| &\leq (\ell_1 r + M)(\mathcal{C}_1 + \mathcal{D}_1) + (\ell_2 r + N)(\mathcal{C}_2 + \mathcal{D}_2)\\ &\quad + r(\mathcal{C}_3 + \mathcal{C}_4 + \mathcal{D}_3 + \mathcal{D}_3) \leq r,\end{aligned} \tag{33}$$

which yields that $\mathcal{Q}(B_r)\subseteq B_r$.

Now, it is proved $\mathcal{Q}$: $\mathcal{Y}\times\mathcal{Y}\longrightarrow\mathcal{Y}\times\mathcal{Y}$ is a contraction. Applying condition (25), for any $(u_1,v_1),(u_2,v_2)\in\mathcal{Y}\times\mathcal{Y}$ and for each $z\in[c,d]$, we have

$$\begin{aligned}&\left|\mathcal{Q}_1(u_1,v_1)(z) - \mathcal{Q}_1(u_2,v_2)(z)\right|\\
&\leq \ell_1\left(\|u_1-u_2\| + \|v_1-v_2\|\right)\left\{\frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\frac{1}{\Gamma(\alpha_1+1)}\right.\\
&+|G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left(\lambda_2\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\right.\\
&\left.\left.+\frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}|\nu_r|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\right)\right\}\\
&+\ell_2\left(\|u_1-u_2\| + \|v_1-v_2\|\right)\left\{\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\times\right.\\
&\left(\frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}|\mu_i|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s + \frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\right)\\
&\left.+|G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\right\} + \|u_1-u_2\|\left\{|\sigma_1|(d-c)\right.\\
&+|G_3|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)|\sigma_1| + |G_2|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)\times\\
&\left.|\sigma_1||\lambda_2|\int_c^d (s-c)\mathrm{d}H_2(s) + |G_2|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}|\sigma_1|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\right\}\\
&+\|v_1-v_2\|\left\{\left|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\right||G_3||\sigma_2||\lambda_1|\int_c^d (s-c)\mathrm{d}H_1(s)\right.\\
&\left.+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3||\sigma_2|\sum_{i=1}^{n}|\mu_i|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s + |G_2|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)|\sigma_2|\right\}\\
&= (\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2)\left(\|u_1-u_2\| + \|v_1-v_2\|\right) + \mathcal{C}_3\|u_1-u_2\| + \mathcal{C}_4\|v_1-v_2\|\\
&\leq ((\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2) + \mathcal{C}_3 + \mathcal{C}_4)\left(\|u_1-u_2\| + \|v_1-v_2\|\right),\end{aligned} \tag{34}$$

and hence

$$\|\mathcal{Q}_1(u_1,v_1) - \mathcal{Q}_1(u_2,v_2)\| \leq ((\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2) + \mathcal{C}_3 + \mathcal{C}_4)\left(\|u_1-u_2\| + \|v_1-v_2\|\right). \tag{35}$$

Furthermore, we deduce that

$$\left\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\right\| \le \left(\left(\ell_1\mathcal{D}_1+\ell_2\mathcal{D}_2\right)+\mathcal{D}_3+\mathcal{D}_4\right)\left(\left\|u_1-u_2\right\|+\left\|v_1-v_2\right\|\right). \tag{36}$$

Using (25) and (33), we concluded that

$$\left\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\right\| \le \left(\ell_1\left(\mathcal{C}_1+\mathcal{D}_1\right)+\ell_2\left(\mathcal{C}_2+\mathcal{D}_2\right)+\mathcal{C}_3+\mathcal{C}_4+\mathcal{D}_3+\mathcal{D}_4\right)\times\left(\left\|u_1-u_2\right\|+\left\|v_1-v_2\right\|\right). \tag{37}$$

As $\ell_1(\mathcal{C}_1+\mathcal{D}_1)+\ell_2(\mathcal{C}_2+\mathcal{D}_2)+\mathcal{C}_3+\mathcal{C}_4+\mathcal{D}_3+\mathcal{D}_4<1$, so the operator $\mathcal{Q}$ is a contraction and by applying Lemma 2, the operator $\mathcal{Q}$ has a unique solution which is the solution of the problem (3). The proof is finished. □

5. Existence Results

Two existence results are proved in this section.

5.1. Existence Result via Leray–Schauder Alternative. The Leray–Schauder alternative (Lemma 3) is used in the proof of our first existence result.

Theorem 2. *Let $\Theta\neq 0$ and f_1, f_2: $[c,d]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous functions. Assume that*

$[(H_1)]$ There exist real constants $u_i, v_i\ge 0$ for $i=1,2$ and $u_0, v_0>0$ such that for all $u,v\in\mathbb{R}$, we have

$$\begin{aligned}\left|f_1(z,u(z),v(z))\right| &\le u_0+u_1|u|+u_2|v|,\\ \left|f_2(z,u(z),v(z))\right| &\le v_0+v_1|u|+v_2|v|.\end{aligned} \tag{38}$$

If $(\mathcal{C}_1+\mathcal{D}_1)u_1+(\mathcal{C}_2+\mathcal{D}_2)v_1+\mathcal{C}_3+\mathcal{D}_3<1$ and $(\mathcal{C}_1+\mathcal{D}_1)u_2+(\mathcal{C}_2+\mathcal{D}_2)v_2+\mathcal{C}_4+\mathcal{D}_4<1$, where $\mathcal{C}_i$, $\mathcal{D}_i$ for $i=1,2,3,4$ are given by (23) and (24), respectively, then system (1.3) admits at least one solution on $[c,d]$.

Proof . The functions f_1, f_2 are continuous on $[c,d]\times\mathbb{R}^2$. Thus, the operator $\mathcal{Q}$ is continuous. Now, we will show that the operator $\mathcal{Q}$: $\mathcal{Y}\times\mathcal{Y}\longrightarrow\mathcal{Y}\times\mathcal{Y}$ is completely continuous. Let $B_r\subset\mathcal{Y}\times\mathcal{Y}$ be a bounded set, where $B_r=\{(u,v)\in\mathcal{Y}\times\mathcal{Y}\colon \|(u,v)\|\le r\}$. Then, for any $(u,v)\in B_r$, there exist positive real numbers P_1 and P_2 such that $|f_1(z,u(t),v(z))|\le P_1$ and $|f_2(z,u(z),v(z))|\le P_2$.

Thus, for each $(u,v)\in B_r$, we have

$$\begin{aligned}&\left|\mathcal{Q}_1(u,v)(z)\right|\\ &\quad\le\left|\sigma_1\right|\int_c^z|u(s)|\mathrm{d}s+I^{\alpha_1}\left|f_1(z,u(z),v(z))\right|\\ &\quad+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left[\left|G_3\right|\left(\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d\left[\int_c^s|v(t)|\mathrm{d}t\right]\mathrm{d}H_1(s)\right.\right.\\ &\quad+\left|\lambda_1\right|\int_c^d\left|f_1(z,u(z),v(z))\right|\mathrm{d}H_1(s)+\left|\sigma_2\right|\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}\int_a^s|v(t)|\mathrm{d}t\mathrm{d}s\\ &\quad+\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}I^{\alpha_2}\left|f_2(s,u(s),v(s))\right|\mathrm{d}s\\ &\quad\left.+\left|\sigma_1\right|\int_c^d|u(s)|\mathrm{d}s+I^{\alpha_1}\left|f_1(d,u(d),v(d))\right|\right)\\ &\quad+\left|G_2\right|\left(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d\left[\int_c^s|u(t)\mathrm{d}t\right]\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^p\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}\left[\int_c^s|u(t)\mathrm{d}t\right]\mathrm{d}s\right.\\ &\quad+\left|\lambda_2\right|\int_c^d I^{\alpha_1}\left|f_1(s,u(s),v(s))\right|\mathrm{d}H_2(s)+\left|\sigma_2\right|\int_c^d|v(s)|\mathrm{d}s+I^{\alpha_2}\left|f_2(d,u(d),v(d))\right|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1}\left|f_1(s,u(s),v(s))\right|\mathrm{d}s\Bigg)\Bigg]\\
&\le r\Bigg[\left|\sigma_1\right|(d-c)+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d (s-c)\mathrm{d}H_1(s)\\
&+\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s+\left|\sigma_1\right|(d-c)\Bigg)\\
&+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\sigma_1\right\|\lambda_2\right|\int_c^d (s-c)\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\\
&+\left|\sigma_2\right|(d-c)\Bigg)\Bigg]+P_1\Bigg[\frac{1}{\Gamma(\alpha_1+1)}+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}\\
&+\left|G_2\right|\frac{(d-c)^{\gamma-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\lambda_2\right|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\\
&+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^{p}\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\Bigg)\Bigg]\\
&+P_2\Bigg[\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s\\
&+\frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg)+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\Bigg]\\
&=P_1\mathscr{C}_1+P_2\mathscr{C}_2+r(\mathscr{C}_3+\mathscr{C}_4),
\end{aligned}
\tag{39}
$$

which yields

$$\left\|\mathscr{Q}_1(u,v)\right\|\le\mathscr{C}_1P_1+\mathscr{C}_2P_2+r(\mathscr{C}_3+\mathscr{C}_4). \tag{40}$$

Similarly, we obtain that

$$\left\|\mathscr{Q}_2(u,v)\right\|\le\mathscr{D}_1P_1+\mathscr{D}_2P_2+r(\mathscr{D}_3+\mathscr{D}_4). \tag{41}$$

Hence, from the above inequalities, we get that the operator $\mathscr{Q}$ is uniformly bounded, since

$$
\begin{aligned}
\|\mathscr{Q}(u,v)\|\le{}&(\mathscr{C}_1+\mathscr{D}_1)P_1+(\mathscr{C}_2+\mathscr{D}_2)P_2\\
&+r(\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4).
\end{aligned}
\tag{42}
$$

Next, we are going to prove that the operator $\mathscr{Q}$ is equicontinuous. Let $\tau_1,\tau_2\in[c,d]$ with $\tau_1<\tau_2$. Then, we have

$$
\begin{aligned}
&\left|\mathscr{Q}_1(u,v)(\tau_2)-\mathscr{Q}_1(u,v)(\tau_1)\right|\\
&\le\left|\sigma_1\right|r(\tau_2-\tau_1)+P_1\int_c^{\tau_1}\frac{\left[(\tau_2-c)^{\alpha_1-1}-(\tau_2-c)^{\alpha_1-1}\right]}{\Gamma(\alpha_1)}\mathrm{d}s+P_1\int_{\tau_1}^{\tau_2}\frac{(\tau_2-c)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{d}s\\
&+\frac{\left[(\tau_2-c)^{\gamma_1-1}-(\tau_2-c)^{\gamma_1-1}\right]}{|\Theta|\Gamma(\gamma_1)}\Bigg[r\Bigg(\left|G_3\right|\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d (s-c)\mathrm{d}H_1(s)\\
&+\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s+\left|\sigma_1\right|(d-c)\Bigg)\\
&+\left|G_2 r\right|\Bigg(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d (s-c)\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s
\end{aligned}
$$

$$+|\sigma_2|(d-c)\Big)+P_1\Bigg[|G_3|\frac{1}{\Gamma(\alpha_1+1)}+|G_2|\Bigg(|\lambda_2|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d(s-c)^{\alpha_1}\mathrm{d}H_2(s)$$
$$+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^{p}|\nu_r|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\Bigg)\Bigg]$$
$$+P_2\Bigg[|G_3|\Bigg(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{n}|\mu_i|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s$$
$$+\frac{|\lambda_1|}{\Gamma(\alpha_2+1)}\int_c^d(s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg)+|G_2|\frac{1}{\Gamma(\alpha_2+1)}\Bigg]\Bigg]. \tag{43}$$

Therefore, we obtain

$$\left|\mathcal{Q}_1(u,v)(\tau_2)-\mathcal{Q}_1(u,v)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{44}$$

Analogously, we can get the following inequality:

$$\left|\mathcal{Q}_2(u,v)(\tau_2)-\mathcal{Q}_2(u,v)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{45}$$

Hence, the set $\mathcal{Q}\mathcal{B}_r$ is equicontinuous. Accordingly, Arzelá–Ascoli theorem implies that the operator $\mathcal{Q}$ is completely continuous.

Finally, we shall show the boundedness of the set $Z=\{(u,v)\in\mathcal{Y}\times\mathcal{Y}\colon\ (u,v)=\mu\mathcal{Q}(u,v),\ 0\le\mu\le 1\}$. Let any $(u,v)\in Z$, then $(u,v)=\mu\mathcal{Q}(u,v)$. We have, for all $z\in[c,d]$,

$$\begin{aligned}u(z)&=\mu\mathcal{Q}_1(u,v)(z),\\ v(z)&=\mu\mathcal{Q}_2(u,v)(z).\end{aligned} \tag{46}$$

Then, we get

$$\begin{aligned}\|u\|&\le(u_0+u_1\|u\|+u_2\|v\|)\mathcal{C}_1+(v_0+v_1\|u\|+v_2\|v\|)\mathcal{C}_2+\|u\|\mathcal{C}_3+\|v\|\mathcal{C}_4,\\ \|v\|&\le(u_0+u_1\|u\|+u_2\|v\|)\mathcal{D}_1+(v_0+v_1\|u\|+v_2\|v\|)\mathcal{D}_2+\|u\|\mathcal{D}_3+\|v\|\mathcal{D}_4,\end{aligned} \tag{47}$$

which imply that

$$\begin{aligned}\|u\|+\|v\|\le{}&(\mathcal{C}_1+\mathcal{D}_1)u_0+(\mathcal{C}_2+\mathcal{D}_2)v_0+[(\mathcal{C}_1+\mathcal{D}_1)u_1+(\mathcal{C}_2+\mathcal{D}_2)v_1+\mathcal{C}_3+\mathcal{D}_3]\|u\|\\ &+[(\mathcal{C}_1+\mathcal{D}_1)u_2+(\mathcal{C}_2+\mathcal{D}_2))v_2+\mathcal{C}_4+\mathcal{D}_4]\|v\|.\end{aligned} \tag{48}$$

Thus, we obtain

$$\|(u,v)\|\le\frac{(\mathcal{C}_1+\mathcal{D}_1)u_0+(\mathcal{C}_2+\mathcal{D}_2)v_0}{M^*}, \tag{49}$$

where $M^*=\min\{1-(\mathcal{C}_1+\mathcal{D}_1)u_1-(\mathcal{C}_2+\mathcal{D}_2)v_1-(\mathcal{C}_3+\mathcal{D}_3),1-(\mathcal{C}_1+\mathcal{D}_1)u_2-(\mathcal{C}_2+\mathcal{D}_2)v_2-(\mathcal{C}_4+\mathcal{D}_4)\}$, which shows that the set Z is bounded. Therefore, by Leray–Schauder alternative (Lemma 3), the operator $\mathcal{Q}$ has at least one fixed point. Hence, we deduce that problem (3) admits a solution on $[c,d]$, which completes the proof. □

5.2. Existence Result via Krasnosel'skiĭ's Fixed-Point Theorem. Now, Krasnosel'skiĭ's fixed-point theorem (Lemma 4) is applied to prove our second existence result.

Theorem 3. *Assume that $\Theta\neq 0$ and $f_1,f_2\colon[c,d]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions satisfying condition (4.8) in Theorem 4.1. Furthermore, suppose that there exist positive constants R_1 and R_2 such that for all $z\in[c,d]$ and $u,v\in\mathbb{R}$, we have*

$$\begin{aligned}|f_1(z,u,v)|&\le R_1,\\ |f_2(z,u,v)|&\le R_2.\end{aligned} \tag{50}$$

If $\mathcal{C}_3+\mathcal{C}_4<1$, $\mathcal{D}_3+\mathcal{D}_4<1$ and $((d-c)^{\alpha_1}/\Gamma(\alpha_1+1))l_1+((d-c)^{\alpha_2}/(d-c)^{\alpha_2})\ell_2<1$, then problem (1.3) admits a solution on $[c,d]$.

Proof . First, we decompose the operator $\mathcal{Q}$ defined by (1) into four operators as

$$\begin{aligned}
\mathcal{S}_1(u,v)(z) &= -\sigma_1\int_c^z u(s)\mathrm{d}s + \frac{(z-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}\Bigg[G_3\Bigg(-\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s)\\
&\quad+\lambda_1\int_c^d I^{\alpha_2}f_2(s,u(s),v(s))\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}I^{\alpha_2}f_2(s,u(s),v(s))\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}f_1(d,u(d),v(d))\Bigg)\\
&\quad+G_2\Bigg(-\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1}f_1(s,u(s),v(s))\mathrm{d}H_2(s)\\
&\quad-\sigma_1\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}I^{\alpha_1}f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad+\sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2}f_2(d,u(d),v(d))\Bigg)\Bigg],\\
\mathcal{S}_2(u,v)(z) &= I^{\alpha_1}f_1(z,u(z),v(z))\,I^{\alpha_1}f_1u,v(z),\\
\mathcal{S}_3(u,v)(z) &= -\sigma_2\int_c^z v(s)\mathrm{d}s + \frac{(z-c)^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Bigg[G_1\Bigg(-\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s)\\
&\quad+\lambda_2\int_c^d I^{\alpha_1}f_1(s,u(s),v(s))\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}I^{\alpha_1}f_1(s,u(s),v(s))\mathrm{d}s + \sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2}f_2(d,u(d),v(d))\Bigg)\\
&\quad+G_4\Bigg(-\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2}f_2(s,u(s),v(s))\mathrm{d}H_1(s)\\
&\quad-\sigma_2\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}I^{\alpha_2}f_2(s,u(s),v(s))\mathrm{d}s\\
&\quad+\sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}f_1(d,u(d),v(d))\Bigg)\Bigg],\\
\mathcal{S}_4(u,v)(z) &= I^{\alpha_2}f_2(z,u(z),v(z))\,I^{\alpha_2}f_2u,v(z).
\end{aligned} \tag{51}$$

Accordingly, $\mathcal{Q}_1(u,v)(z) = \mathcal{S}_1(u,v)(z) + \mathcal{S}_2(u,v)(z)$ and $\mathcal{Q}_2(u,v)(z) = \mathcal{S}_3(u,v)(z) + \mathcal{S}_4(u,v)(z)$. Let $B_\varepsilon = \{(u,v)\in\mathcal{Y}\times\mathcal{Y};\|(u,v)\|\le\varepsilon\}$ with

$$\varepsilon \ge \max\left\{\frac{\mathcal{C}_1R_1+\mathcal{C}_2R_2}{1-(\mathcal{C}_3+\mathcal{C}_4)}, \frac{\mathcal{D}_1R_1+\mathcal{D}_2R_2}{1-(\mathcal{D}_3+\mathcal{D}_4)}\right\}. \tag{52}$$

First, it is showed that $\mathcal{Q}_1(x,y)+\mathcal{Q}_2(u,v)\in B_\varepsilon$ for all $(x,y),(u,v)\in B_\varepsilon$. According to the proof of Theorem 1, we get

$$\begin{aligned}
&\mathcal{S}_2(u,v)(z)I^{\alpha_1}f_1(z,u(z),v(z)),\\
&\mathcal{S}_4(u,v)(z)I^{\alpha_2}f_2(z,u(z),v(z)).
\end{aligned} \tag{53}$$

Consequently, $\mathcal{Q}_1(x,y)+\mathcal{Q}_2(u,v)\in B_\varepsilon$ and we conclude the condition (i) of Lemma 4. Now, it is indicated that the operator $(\mathcal{S}_2,\mathcal{S}_4)$ is a contraction mapping. For $(x_1,y_1),(x_2,y_2)\in B_\varepsilon$, we infer that

$$\begin{aligned}
\left|\mathcal{S}_2(x_1,y_1)(z)-\mathcal{S}_2(x_2,y_2)(z)\right| &\le I^{\alpha_1}\left|f_1x_1,y_1-f_1x_2,y_2\right|(z)\\
&\le \ell_1\left(\|x_1-x_2\|+\|y_1-y_2\|\right)I^{\alpha_1}(1)(d) \le \ell_1\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\left(\|x_1-x_2\|+\|y_1-y_2\|\right),
\end{aligned} \tag{54}$$

$$\begin{aligned}
\left|\mathcal{S}_4(x_1,y_1)(z)-\mathcal{S}_4(x_2,y_2)(z)\right|\,| &\le I^{\alpha_2}\left|f_2x_1,y_1-f_2x_2,y_2\right|(z) \le \ell_2\left(\|x_1-x_2\|+\|y_1-y_2\|\right)I^{\alpha_2}(1)(d)\\
&\le \ell_2\frac{(d-c)^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\|x_1-x_2\|+\|y_1-y_2\|\right).
\end{aligned} \tag{55}$$

As $((d-c)^{\alpha_1}/\Gamma(\alpha_1+1))\ell_1 + (d-c)^{\alpha_2}/\Gamma(\alpha_2+1)\ell_2 < 1$, the operator $(\mathcal{S}_2, \mathcal{S}_4)$ is a contraction and the condition *(iii)* of Lemma 4 is concluded. In the final step, the condition *(ii)* of Lemma 4 is verified for the operator $(\mathcal{S}_1, \mathcal{S}_3)$. As the functions f_1, f_2 are continuous, one can see that the operator $(\mathcal{S}_1, \mathcal{S}_3)$ is continuous. Furthermore, for $(u, v) \in B_\varepsilon$, as in the proof of Theorem 1, we have

$$\begin{aligned}
\left|\mathcal{S}_1(u,v)(z)\right| &\leq \left(\mathscr{C}_1 - \frac{1}{\Gamma(\alpha_1+1)}\right)R_1 + \mathscr{C}_2 R_2 + \left(\mathscr{C}_3 + \mathscr{C}_4\right)\varepsilon = P^*,\\
\left|\mathcal{S}_3(u,v)(z)\right| &\leq \mathscr{D}_1 R_1 + \left(\mathscr{D}_2 - \frac{1}{\Gamma(\alpha_2+1)}\right)R_2 + \left(\mathscr{D}_3 + \mathscr{D}_4\right)\varepsilon = Q^*.
\end{aligned} \tag{56}$$

Hence, $\|(\mathcal{S}_1, \mathcal{S}_2)(u,v)\| \leq P^* + Q^*$, which implies that $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is uniformly bounded. Now, we claim that the set $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is equicontinuous. For this aim, let $\tau_1, \tau_2 \in [c, d]$ with $\tau_1 < \tau_2$. For any $(u, v) \in B_\varepsilon$, similar to the proofs of equicontinuous for the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in 2, we can show that $|\mathcal{S}_1(u,v)(\tau_2) - \mathcal{S}_1(u,v)(\tau_1)|$, $|\mathcal{S}_3(u,v)(\tau_2) - \mathcal{S}_3(u,v)(\tau_1)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Consequently, the set $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is equicontinuous and by applying Arzelá-Ascoli theorem, the operator $(\mathcal{S}_1, \mathcal{S}_3)$ will be compact on B_ε. Therefore, by applying Lemma 4, problem (3) has at least one solution on $[c, d]$. This completes the proof. □

Remark 1. In Theorem 2, the functions f_1, f_2 are bounded by linear planes in three-dimension space. While, in Theorem 3, f_1, f_2 are bounded by fixed constants and also satisfied Lipschitz condition in (25).

6. Examples

Now, we present some examples to show the benefits of our results.

Example 1. Consider the following coupled system of Hilfer type sequential fractional differential equations involving Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{aligned}
&\left({}^{H}D^{(5/4)(2/3)} + \frac{1}{20}{}^{H}D^{(1/4)(2/3)}\right)u(z) = f_1(z, u(z), v(z)), \quad z \in [(1/8), (13/8)],\\
&\left({}^{H}D^{(7/4)(1/3)} + \frac{1}{15}{}^{H}D^{(3/4)(1/3)}\right)v(z) = f_2(z, u(z), v(z)), \quad z \in [(1/8), (13/8)],\\
&u\left(\frac{1}{8}\right) = 0,\\
&v\left(\frac{1}{8}\right) = 0,\\
&u\left(\frac{13}{8}\right) = \frac{1}{4}\int_{1/8}^{13/8} v(s)\mathrm{d}\left(e^{-2s}\right) + \frac{2}{7}\int_{1/2}^{5/8} v(s)\mathrm{d}s + \frac{3}{11}\int_{9/8}^{5/4} v(s)\mathrm{d}s,\\
&v\left(\frac{13}{8}\right) = \frac{1}{5}\int_{1/8}^{13/8} u(s)\mathrm{d}\left(e^{-3s}\right) + \frac{4}{13}\int_{1/4}^{3/8} u(s)\mathrm{d}s + \frac{5}{17}\int_{3/4}^{7/8} u(s)\mathrm{d}s + \frac{6}{19}\int_{11/8}^{3/2} u(s)\mathrm{d}s.
\end{aligned} \tag{57}$$

Here, $\alpha_1 = 5/4$, $\alpha_2 = 7/4$, $\beta_1 = 2/3$, $\beta_2 = 1/3$, $c = 1/8$, $d = 13/8$, $\lambda_1 = 1/4$, $\lambda_2 = 1/5$, $H_1(t) = e^{-2t}$, $H_2(t) = e^{-3t}$, $n = 2$, $\mu_1 = 2/7$, $\mu_2 = 3/11$, $\eta_1 = 1/2$, $\eta_2 = 9/8$, $\xi_1 = 5/8$, $\xi_2 = 5/4$, $p = 3$, $\nu_1 = 4/13$, $\nu_2 = 5/17$, $\nu_3 = 6/19$, $\zeta_1 = 1/4$, $\zeta_2 = 3/4$, $\zeta_3 = 11/8$, $\theta_1 = 3/8$, $\theta_2 = 7/8$, and $\theta_3 = 3/2$. Then, we can compute that $\gamma_1 = 7/4$, $\gamma_2 = 11/6$, $G_1 \approx 1.474766913$, $G_2 \approx -0.03380798224$, $G_3 \approx 1.490431261$, $G_4 \approx 0.03704876432$, $\Theta \approx 2.199291254$, $\mathscr{C}_1 \approx 1.765659740$, $\mathscr{C}_2 \approx 0.006128241272$, $\mathscr{C}_3 \approx 0.1499796921$, $\mathscr{C}_4 \approx 0.000526941741$, $\mathscr{D}_1 \approx 1.242948896$, $\mathscr{D}_2 \approx 0.06369559511$, $\mathscr{D}_3 \approx 0.1998340850$, and $\mathscr{D}_4 \approx 0.003945721441$.

(i) The Lipschitzian functions $f_1, f_2\colon [(1/8), (13/8)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given by

$$f_1(t, u, v) = \frac{1}{8t+4}\left(\frac{u^2 + 2|u|}{2(1+|u|)}\right) + \frac{\sin|v|}{8t+5} + \frac{1}{2}\cos 2\pi t, \tag{58}$$

$$\begin{aligned}
f_2(t, u, v) = {} & \frac{1}{8t+3}\tan^{-1}|u|\\
& + \frac{e^{(1-8t)}}{8t+7}\left(\frac{4v^2 + 5|v|}{5(1+|v|)}\right) + \frac{1}{4}\log_e t.
\end{aligned} \tag{59}$$

From direct computation to (59)-(60), we get

$$\begin{aligned}|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)|&\le\frac{1}{5}|u_1-u_2|+\frac{1}{6}|v_1-v_2|,\\|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)|&\le\frac{1}{4}|u_1-u_2|+\frac{1}{8}|v_1-v_2|,\end{aligned}\tag{60}$$

for $u_1,u_2,v_1,v_2\in\mathbb{R}$. Setting $\ell_1=1/5$ and $\ell_2=1/4$, we obtain $\ell_1(\mathscr{C}_1+\mathscr{D}_1)+\ell_2\ (\mathscr{C}_2+\mathscr{D}_2)+\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4\approx 0.9734641265<1$. By application of our Theorem 1, the problem of Hilfer type sequential fractional differential system involving Riemann–Stieltjes integral multistrip boundary conditions (58) with (59)-(60) has a unique solution on $[(1/8),(13/8)]$.

(ii) Let the nonlinear functions $f_1,f_2\colon[(1/8),(13/8)]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be defined by

$$f_1(t,u,v)=\frac{1}{2}\cos^2\pi t+\frac{1}{5}ue^{-v^4}+\frac{|v|^{15}\sin^4 u}{3\left(1+v^{14}\right)},\tag{61}$$

$$f_2(t,u,v)=\frac{1}{3}\sin^2\pi t+\frac{u^{18}\cos^{12}v}{4\left(1+|u|^{17}\right)}+\frac{v}{\pi}\tan^{-1}u.\tag{62}$$

We remark that $|f_1(t,u,v)|\le(1/2)+\ (1/5)|u|+(1/3)|v|$ and $|f_2(t,u,v)|\le(1/3)+(1/4)|u|+(1/2)|v|$. Now, we choose the constants as in Theorem 2 by $u_0=1/2$, $u_1=1/5$, $u_2=1/3$, $v_0=1/3$, $v_1=1/4$, and $v_2=1/2$. Then, we can find that $(\mathscr{C}_1+\mathscr{D}_1)u_1+(\mathscr{C}_2+\mathscr{D}_2)v_1+\mathscr{C}_3+\mathscr{D}_3\approx 0.9748101164<1$ and $(\mathscr{C}_1+\mathscr{D}_1)u_2+(\mathscr{C}_2+\mathscr{D}_2)v_2+\mathscr{C}_4+\mathscr{D}_4\approx 0.7915367403<1$. The benefit of Theorem 2 can be used to conclude that the coupled system of Hilfer type sequential fractional differential equations subject to boundary conditions (58) with (62)-(63) has at least one solution on $[(1/8),(13/8)]$.

(iii) Suppose that two Lipschitzian functions $f_1,f_2\colon[(1/8),(13/8)]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are stated by

$$f_1(t,u,v)=\frac{1}{8t+1}+\frac{1}{8t+2}\left(\frac{|u|}{1+|u|}\right)+\frac{1}{8t+3}\tan^{-1}|v|,\tag{63}$$

$$f_2(t,u,v)=\frac{1}{8t+4}+\frac{1}{3}e^{1-8t}\ \sin|u|+\frac{1}{5}e^{-(1-8t)^2}\left(\frac{|v|}{1+|v|}\right).\tag{64}$$

Actually, we can compute the bounds of the above two functions by $|f_1(t,u,v)|\le(5/6)+(\pi/8)$, $|f_2(t,u,v)|\le(47/60)$, for all $u,v\in\mathbb{R}$. In addition, we can find that $|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)|\le(1/3)|u_1-u_2|+(1/4)|v_1-v_2|$ and $|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)|\le(1/3)\ \ |u_1-u_2|+(1/5)|v_1-v_2|$, and thus we can set $\ell_1=1/3$ and $\ell_2=1/3$ satisfying condition (4.8) in Theorem 1. Then, we obtain $\mathscr{C}_3+\mathscr{C}_4\approx 0.1505066338<1$, $\mathscr{D}_3+\mathscr{D}_4\approx 0.2037798064<1$, and $((d-c)^{\alpha_1}/\Gamma(\alpha_1+1))\ell_1+((d-c)^{\alpha_1}/\ \Gamma(\alpha_2+1))\ell_2\approx 0.9097463067<1$ that all conditions in Theorem 3 are fulfilled. In this step, we conclude that the problem (58) with (64) has at least one solution on $[(1/8),(13/8)]$. Finally, we observe that the uniqueness result cannot be obtained because $\ell_1(\mathscr{C}_1+\mathscr{D}_1)+\ell_2\ (\mathscr{C}_2+\mathscr{D}_2)+\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4\approx 1.380430597>1$.

7. Conclusions

In the present research, we studied a coupled system of Hilfer type sequential fractional differential equations supplemented with Riemann–Stieltjes integral multistrip boundary conditions. First, an auxiliary lemma, concerning a linear variant of the considered problem, has been proved which is pivotal to converting the coupled system into a fixed point problem. Then, existence and uniqueness results are established via standard fixed point theorems. Thus, the classical Banach fixed point theorem is applied to obtain a uniqueness result, while Leray–Schauder alternative and Krasnosel'skiĭ's fixed point theorem are applied to present the existence results. Numerical examples are also constructed to illustrate the obtained results. The obtained results are new and enrich the existing literature on coupled systems of Hilfer type sequential fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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