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Research Article

Integral Equations Approach in Complex-Valued Generalized *b*-Metric Spaces

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In this paper, we study a rational type common fixed-point theorem (CFP theorem) in complex-valued generalized b-metric spaces (G_b-metric spaces) by using three self-mappings under the generalized contraction conditions. We find CFP and prove its uniqueness. To justify our result, we provide an illustrative example. Furthermore, we present a supportive application of the three Urysohn type integral equations (UTIEs) for the validity of our result. The UTIEs are

$$\begin{aligned}\gamma_1(q) &= \int_{k_1}^{k_2} Q_1(q,r,\gamma_1(r))dr + \hbar_1(q),\\ \gamma_2(q) &= \int_{k_1}^{k_2} Q_2(q,r,\gamma_2(r))dr + \hbar_2(q),\\ \gamma_3(q) &= \int_{k_1}^{k_2} Q_3(q,r,\gamma_3(r))dr + \hbar_3(q),\end{aligned}\tag{1}$$

where $q \in [k_1,k_2], \gamma_1,\gamma_2,\gamma_3,\hbar_1,\hbar_2,\hbar_3 \in \Upsilon$, where $\Upsilon = C([k_1,k_2],\mathbb{R}^n)$ is the set of all real-valued continuous functions defined on $[k_1,k_2]$ and $Q_1,Q_2,Q_3\colon [k_1, k_2]\times[k_1,k_2]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

1. Introduction

In 1922, Banach [1] proved a "Banach contraction principle" which is stated as "a single-valued contractive type mapping in a complete metric space has a unique FP." Later on, this principle generalized in many directions, and some researchers contributed to the theory of FP. In [2], Nazam et al. presented the idea of weakly increasing F-contractions and established some results in ordered partial-metric space with an application. Hu and Gu [3] presented a new idea of Menger probabilistic S-metric space by using the concepts of probabilistic and S-metric spaces. They investigated its topological characteristics and established some FP theorems with illustrative examples.

Bakhtin [4] introduced the idea of b-metric space. After that, Czerwik [5] established some FP results by using b-metric spaces. In [6], Boriceanu et al. extended the concept of fractal operator theory by introducing it in b-metric spaces and presented some generalized CFP results. While, Akkouchi [7] used the concept of an implicit relation in the said spaces and established CFP results for contractive type maps. Došenović at el. [8] discussed, complemented, improved, generalized, and enriched some FP results of $(\beta - \psi_1 - \psi_2)$ contraction in ordered b-metric spaces. They made a different approach of taking Picard's sequence which help shorten the proof comparing other previous studies in literature. They also complimented and enriched CFP results for $\beta_{q,\phi}$ s, ψ contraction maps.

Delfani et al. [9] contributed in ordered b-metric spaces by establishing FP theorems. In these results, they introduced the generalizations of $F - ts$ and $(\psi, \phi, F - ts)$ contractions. They also provided a suitable example to verify their FP result. In [10], Karapinar et al. contributed in b-metric spaces by generalizing it to prove FP results in view of (α, ψ)-Meir-Keeler type contractions. These results improved and unified some previous results. Abdeljawad et al. [11] extended the b-metric spaces to double controlled metric spaces by improving control functions $\alpha(\gamma_1, \gamma_2)$ and $\mu(\gamma_1, \gamma_2)$ on the right side of b-triangle, that is, $d(\gamma_1, \gamma_2) \leq \alpha(\gamma_1, \gamma_3)d\ \ (\gamma_1, \gamma_3) + \mu(\gamma_3, \gamma_2)d\ \ (\gamma_3, \gamma_2), \forall \gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. They also provided some examples in which two functions are not comparable.

Mustafa and Sims [12] introduced the generalized idea of metric space and established some FP theorems by using Dhage's theory. Later on, Mustafa et al. [13] proved some modified contractive-type FP results in this space. Abbas and Rhoades [14] started to investigate CFP results in the said spaces. In [15], Chugh et al. established the property P in G-metric space and proved some results. In this context, Mohanta et al. [16] contributed by establishing a CFP result which improved and supplemented some of the existing results. Mustafa introduced a mapping pair and obtained many CFP results for different contraction conditions. He supported these results through examples.

In 2014, Aghajani et al. [17] presented the idea of G_b-metric space and proved a weakly compatible CFP theorem. Aydi [18] improved, complemented, unified, and generalized some well-known existing results in said spaces and established some coupled and tripled coincidence point results. Gupta in [19] extended some existing results from literature and obtained various FP results in G_b-metric spaces. In [20], Makran et al. proved general CFP theorem by using multivalued maps and established its application. Mebawondu et al. [21] proved FP results for a different contraction type maps which involve C-class, α_s^δ-admissible, Suzuki-type maps in G_b-metric spaces.

Ege [22] gave the idea of complex-valued G_b-metric space and proved "Banach contraction principle and Kannan's contraction theorems for FP." Ege [23] used the idea of α-series to prove CFP results in said spaces. He also introduced $\alpha - \psi$ contraction maps to prove CFP results. Kumar and Vashistha [24] introduced the idea of coupled FP for mapping in the said space. They proved coupled FP results and supported it by providing an example satisfying their main result. Recently, Mehmood et al. [25] proved some CFP theorems by using compatible single-valued contractive type mappings on complex-valued b-metric spaces with an application.

This article presents a contraction theorem in complex-valued G_b-metric spaces by using three self-maps to establish a generalized CFP-theorem. This result improves, extends, and modifies some of the existing results (e.g, [22, 26]). We present an example to support our work. We also present an application of the three UTIEs for the existence of a common solution to support our work. This study is organized as follows: Section 2 consists of preliminary concepts. In Section 3, we present a CFP theorem with an illustrative example. While in Section 4, we present an application to support our main work. Finally, in Section 5, we discuss the conclusion of our work.

2. Preliminaries

Let the complex numbers be denoted by $\mathbb{C}$ and $z_i, z_{ii} \in \mathbb{C}$. Now, we define $\leq$ as $z_i \leq z_{ii}$, iff $R_e(z_i) \leq R_e(z_{ii})$ and $I_m(z_i) \leq I_m(z_{ii})$, where the real and imaginary parts of the complex number are denoted by R_e and I_m, respectively. Accordingly, $z_i \leq z_{ii}$, if any one of the following conditions holds:

(1) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,
(2) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,
(3) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$,
(4) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$.

Particularly, we can write $z_i \lesssim z_{ii}$ if $z_i \neq z_{ii}$ and one of (2), (3), and (4) is satisfied.

Remark 1 (see [27]). The following given properties can be held and verified if

(1) $\beta_1, \beta_2 \in \mathbb{R}$ and $\beta_1 \leq \beta_2 \Rightarrow \beta_1 y \leq \beta_2 y\ \forall y \in \mathbb{C}$,
(2) $0 \leq z_i \leq z_{ii} \Rightarrow |z_i| < |z_{ii}|$,
(3) $z_i \leq z_{ii}$ and $z_{ii} < z_{iii} \Rightarrow z_i < z_{iii}$.

Definition 1 (see [17]). Let $\Upsilon \neq \emptyset$ set and $b \geq 1$ be a given real number. A mapping $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+$ is called a G_b-metric if G holds the following axioms:

(1) $G(\gamma_1, \gamma_2, \gamma_3) = 0$ if $\gamma_1 = \gamma_2 = \gamma_3$
(2) $0 < G(\gamma_1, \gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Upsilon$ with $\gamma_1 \neq \gamma_2$
(3) $G(\gamma_1, \gamma_1, \gamma_2) \leq G(\gamma_1, \gamma_2, \gamma_3)$ for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ with $\gamma_2 \neq \gamma_3$
(4) $G(\gamma_1, \gamma_2, \gamma_3) = G(p\{\gamma_1, \gamma_2, \gamma_3\})$, where p is a permutation of $\gamma_1, \gamma_2, \gamma_3$
(5) $G(\gamma_1, \gamma_2, \gamma_3) \leq b[G(\gamma_1, a, a) + G(a, \gamma_2, \gamma_3)]$ for all $\gamma_1, \gamma_2, \gamma_3, a \in \Upsilon$

Then, a pair (Υ, G) is called a G_b-metric space.

Note that each G-metric space is a G_b-metric space with $b = 1$.

Definition 2 (see [22]). Let $\Upsilon \neq \emptyset$ set and $b \geq 1$ be a given real number. A mapping $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ is called a complex-valued G_b-metric if G holds the following axioms:

(1) $G(\gamma_1, \gamma_2, \gamma_3) = 0$ if $\gamma_1 = \gamma_2 = \gamma_3$
(2) $0 < G(\gamma_1, \gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Upsilon$ with $\gamma_1 \neq \gamma_2$
(3) $G(\gamma_1, \gamma_1, \gamma_2) \leq G(\gamma_1, \gamma_2, \gamma_3)$ for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ with $\gamma_2 \neq \gamma_3$

(4) $G(\gamma_1, \gamma_2, \gamma_3) = G(p\{\gamma_1, \gamma_2, \gamma_3\})$, where p is a permutation of $\gamma_1, \gamma_2, \gamma_3$

(5) $G(\gamma_1, \gamma_2, \gamma_3) \leq b[G(\gamma_1, a, a) + G(a, \gamma_2, \gamma_3)]$ for all $\gamma_1, \gamma_2, \gamma_3, a \in \Upsilon$

Then, a pair (Υ, G) is called a complex-valued G_b-metric space.

Example 1. Let $\Upsilon = 0, \infty)$ and a mapping $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ be defined as

$$G(\gamma_1, \gamma_2, \gamma_3) = \left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 + i\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2, \tag{2}$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Then, G is a complex-valued G_b-metric with $b = 2$, but it is not G-metric. To see this, let $\gamma_1 = 3, \gamma_2 = 5, \gamma_3 = 7, a = 7/2$, and we get $G(3, 5, 7) = 576/16 + 576/16i$, $G(3, 7/2, 7/2) = 9/16 + 9/16i$, and $G(7/2, 5, 7) = 441/16 + 441/16i$; thus,$G(3, 5, 7) = 576/16 + 576/16i \nleq 450/16 + 450/16i = G(3, 7/2, 7/2) + G(7/2, 5, 7)$.

Clearly, property (5) of definition (2.3) is satisfied with $b = 2$; hence,

$G(\gamma_1, \gamma_2, \gamma_3) = (3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2 + i(3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2$ is G_b-metric.

Proposition 1 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space. Then, $\forall \gamma_1, \gamma_2, \gamma_3 \in \Upsilon$.*

(1) $G(\gamma_1, \gamma_2, \gamma_3) \leq b(G(\gamma_1, \gamma_1, \gamma_2) + G(\gamma_1, \gamma_1, \gamma_3))$

(2) $G(\gamma_1, \gamma_2, \gamma_2) \leq 2bG(\gamma_1, \gamma_1, \gamma_2)$

Definition 3 (see [22]). Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ.

(1) $\{\gamma_j\}$ is complex-valued G_b-convergent to γ if for every $0 < a \in \mathbb{C}$, $\exists k \in \mathbb{N}$, such that $G(\gamma, \gamma_j, \gamma_m) < a, \forall j, m \geq k$.

(2) A sequence $\{\gamma_j\}$ is called complex-valued G_b-Cauchy if for every $0 < a \in \mathbb{C}$, $\exists k \in \mathbb{N}$, such that $G(\gamma_j, \gamma_m, \gamma_l) < a, \forall j, m, l \geq k$.

(3) If every complex-valued G_b-Cauchy sequence is complex-valued G_b-convergent in (Υ, G), then a pair (Υ, G) is called complex-valued G_b-complete.

Proposition 2 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ. Then, $\{\gamma_j\}$ is complex-valued G_b-convergent to γ if and only if $|G(\gamma, \gamma_j, \gamma_m)| \longrightarrow 0$ as $n, m \longrightarrow \infty$.*

Theorem 1 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space; then, for a sequence $\{\gamma_j\}$ in Υ and a point $\gamma \in \Upsilon$, the following are equivalent:*

(1) $\{\gamma_j\}$ is complex-valued G_b-convergent to γ

(2) $|G(\gamma_j, \gamma_j, \gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

(3) $|G(\gamma_j, \gamma, \gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

(4) $|G(\gamma_m, \gamma_j, \gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

Proposition 3 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ. Then, $\{\gamma_j\}$ is a complex-valued G_b-convergent to γ if and only if $|G(\gamma, \gamma_j, \gamma_m)| \longrightarrow 0$ as $j, m \longrightarrow \infty$.Proof: Assume that $\{\gamma_j\}$ is complex-valued G_b-convergent to γ and let*

$$a = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}, \forall \varepsilon > 0. \tag{3}$$

Then, we have $0 < a \in \mathbb{C}$, and there is a natural number k, such that $G(\gamma, \gamma_j, \gamma_m) < a$ for all $n, m \geq k$. Thus, $|G(\gamma, \gamma_j, \gamma_m)| < |a| = \varepsilon$ for all $j, m \geq k$, and so, $|G(\gamma, \gamma_j, \gamma_m)| \longrightarrow 0$ as $j, m \longrightarrow \infty$.

Suppose that $|G(\gamma, \gamma_j, \gamma_m)| \longrightarrow 0$ as $j, m \longrightarrow \infty$. For a given $a \in \mathbb{C}$ with $0 < a$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$,

$$|z| < \delta \Rightarrow z < a. \tag{4}$$

Considering δ, we have a natural number k, such that $|G(\gamma, \gamma_j, \gamma_m)| < \delta$ for all $j, m \geq k$. This means that $G(\gamma, \gamma_j, \gamma_m) < a$ for all $j, m \geq k$, i.e., $\{\gamma_j\}$ is complex-valued G_b-convergent to γ.

3. Main Result

Theorem 2. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3: \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3) + \beta_2 W(\gamma_1, \gamma_2, \gamma_3), \tag{5}$$

where

$$W(\gamma_1,\gamma_2,\gamma_3)=\max\left\{\begin{array}{c} G(\gamma_1,\gamma_2,F_2\gamma_2),G(F_1\gamma_1,F_1\gamma_1,\gamma_2),G(F_1\gamma_1,\gamma_2,\gamma_2),\\ G(F_2\gamma_2,F_2\gamma_2,\gamma_3),G(F_2\gamma_2,\gamma_3,\gamma_3),\\ \dfrac{G(\gamma_1,F_1\gamma_1,F_1\gamma_1)\cdot G(\gamma_2,F_2\gamma_2,F_2\gamma_2)}{1+G(\gamma_1,\gamma_2,\gamma_2)},\\ \dfrac{G(\gamma_2,F_2\gamma_2,F_2\gamma_2)\cdot G(\gamma_3,F_3\gamma_3,F_3\gamma_3)}{1+G(F_1\gamma_1,\gamma_3,\gamma_3)},\\ \dfrac{G(\gamma_3,F_3\gamma_3,F_3\gamma_3)\cdot G(\gamma_1,F_1\gamma_1,F_1\gamma_1)}{1+G(F_2\gamma_2,F_3\gamma_3,F_3\gamma_3)} \end{array}\right\}, \tag{6}$$

for all $\gamma_1,\gamma_2,\gamma_3\in\Upsilon$, $\beta_1,\beta_2\in 0,1/3)$, *such that* $(\beta_1+\beta_2)<1/3$, $b\beta_2<1/3$, *and* $(\beta_1+2b\beta_2)<2/3$. *Then,* F_1,F_2, *and* F_3 *have a unique CFP in* Υ.

Proof. Fix $\gamma_0\in\Upsilon$, and $\{\gamma_j\}$ be a sequence in Υ, such that

$$\begin{aligned}\gamma_{3n+1}&=F_1\gamma_{3j},\\ \gamma_{3j+2}&=F_2\gamma_{3j+1},\\ \gamma_{3j+3}&=F_3\gamma_{3j+2}\forall n\geq 0.\end{aligned} \tag{7}$$

Now, by using (5),

$$\begin{aligned}G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)&=G\left(F_1\gamma_{3j},F_2\gamma_{3j+1},F_3\gamma_{3j+2}\right)\\ &\leq\beta_1 G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\\ &\quad+\beta_2 W\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right).\end{aligned} \tag{8}$$

This implies that

$$\begin{aligned}\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right|&\leq\beta_1\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\\ &\quad+\beta_2\left|W\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\end{aligned} \tag{9}$$

where

$$\begin{aligned}W\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)&=\max\left\{\begin{array}{c} G\left(\gamma_{3j},\gamma_{3j+1},F_2\gamma_{3j+1}\right),G\left(F_1\gamma_{3j},F_1\gamma_{3j},\gamma_{3j+1}\right),G\left(F_1\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(F_2\gamma_{3j+1},F_2\gamma_{3j+1},\gamma_{3j+2}\right),G\left(F_2\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)\cdot G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)}{1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)\cdot G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}{1+G\left(F_1\gamma_{3j},\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)}{1+G\left(F_2\gamma_{3j+1},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)} \end{array}\right\}\\ &=\max\left\{\begin{array}{c} G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right),G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right),G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\cdot G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)}{1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)}{1+G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\cdot G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)}{1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)} \end{array}\right\}.\end{aligned} \tag{10}$$

This implies that

$$\begin{aligned}
&\left|W\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| \\
&\leq \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|, \left|G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|, \left|G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|, \\
\left|G\left(\gamma_{3j+2}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|, \left|G\left(\gamma_{3j+2}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|, \\
\dfrac{\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right| \cdot \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}{\left|1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}, \\
\dfrac{\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right| \cdot \left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}{\left|1 + G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}, \\
\dfrac{\left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right| \cdot \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}{\left|1 + G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}
\end{array}\right\} \\
&\leq \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|, \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|, \\
\left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|, \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|
\end{array}\right\}.
\end{aligned} \tag{11}$$

By using Definition 2.3 (3) and after simplification, we get that

$$\left|W\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| \leq \max\left\{\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3}\right)\right|, \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|\right\}. \tag{12}$$

Now, there are two possibilities:

(i) If $|G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3})|$ is a maximum term in $\{|G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3})|, |G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})|\}$, then $|W(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})| = |G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3})|$; so, after simplification, (3.3) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq a_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|, \text{where} \quad a_1 = \beta_1/(1-\beta_2). \tag{13}$$

(ii) If $|G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})|$ is a maximum term in $\{|G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3})|, |G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})|\}$, then $|W(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})| = |G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2})|$; so, after simplification, (3.3) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq a_2\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|, \text{where} \quad a_2 = (\beta_1 + \beta_2). \tag{14}$$

Let $a\colon = \max\{a_1, a_2\} < 1/3$; then, from (13) and (14), for all $n \geq 0$, we have

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq a\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|. \tag{15}$$

Similarly,

$$\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq a\left|G\left(\gamma_{3j-1},\gamma_{3j},\gamma_{3j+1}\right)\right|. \tag{16}$$

Now, from (16) and (15) and by induction, we have that

$$\begin{aligned}\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right| &\leq a\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\\ &\leq a^2\left|G\left(\gamma_{3j-1},\gamma_{3j},\gamma_{3j+1}\right)\right.\\ &\leq \cdots \leq a^{3j+1}\left|G\left(\gamma_0,\gamma_1,\gamma_2\right)\right|.\end{aligned} \tag{17}$$

Let $m,n\in\mathbb{N}$ and $>n$, and we have that

$$\begin{aligned}\left|G\left(\gamma_j,\gamma_m,\gamma_m\right)\right| &\leq b\left|G\left(\gamma_j,\gamma_{n+1},\gamma_{n+1}\right)\right| + b\left|G\left(\gamma_{n+1},\gamma_m,\gamma_m\right)\right|\\ &\leq b\left|G\left(\gamma_j,\gamma_{n+1},\gamma_{n+1}\right)\right| + b^2\left|G\left(\gamma_{n+1},\gamma_{n+2},\gamma_{n+2}\right)\right| + \cdots + b^{m-n}\left|G\left(\gamma_{m-1},\gamma_m,\gamma_m\right)\right|\\ &\leq ba^n\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right| + b^2a^{n+1}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right| + \cdots + b^{m-n}a^{m-1}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &\leq \left[ba^n + b^2a^{n+1} + \cdots + b^{m-n}a^{m-1}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &= \left[ba^n + b^2a^{n+1} + \cdots + b^{m-n}a^{m-1}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &= ba^n\left[1 + ba + b^2a^2 \ldots + b^{m-(n+1)}a^{m-(n+1)}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &= ba^n\sum_{t=0}^{m-(n+1)} b^t a^t\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &\leq ba^n\sum_{t=0}^{\infty} b^t a^t\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &= \frac{ba^n}{1-ba}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right| \longrightarrow 0, as\, n\longrightarrow\infty.\end{aligned} \tag{18}$$

By Proposition 2.5 (1), we have $|G(\gamma_j,\gamma_m,\gamma_l)|\leq b\left\{|G(\gamma_j,\gamma_m,\gamma_m)| + |G(\gamma_l,\gamma_m,\gamma_m)|\right\}$ for $n,m,l\in\mathbb{N}$. If we take limit as $n,m,l\longrightarrow\infty$, we obtain $|G(\gamma_j,\gamma_m,\gamma_l)|\longrightarrow 0$. It implies $\left\{\gamma_j\right\}$ is a G_b-Cauchy sequence. Since, Υ is complete complex-valued G_b-metric space, $\exists\xi\in\Upsilon$, such that, $\gamma_j\longrightarrow\xi$, as $n\longrightarrow\infty$ or $\lim_{n\longrightarrow\infty}\gamma_j=\xi$. We have to show that $F_1\xi=\xi$; by contrary case, let $F_1\xi\neq\xi$. Now by (5),

$$\begin{aligned}G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right) &= G\left(F_1\xi,F_2\gamma_{3j+1},F_3\gamma_{3j+2}\right)\\ &\leq \beta_1 G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\\ &\quad + \beta_2 W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right).\end{aligned} \tag{19}$$

This implies that

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right| + \beta_2\left|W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|, \tag{20}$$

where

$$W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)=\max\left\{\begin{array}{c}G\left(\xi,\gamma_{3j+1},F_2\gamma_{3j+1}\right),G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right),G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(F_2\gamma_{3j+1},F_2\gamma_{3j+1},\gamma_{3j+2}\right),G\left(F_2\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G(\xi,F_1\xi,F_1\xi)\cdot G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)}{1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)\cdot G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}{1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)\cdot G(\xi,F_1\xi,F_1\xi)}{1+G\left(F_2\gamma_{3j+1},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}\end{array}\right\}$$
$$=\max\left\{\begin{array}{c}G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right),G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right),G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G(\xi,F_1\xi,F_1\xi)\cdot G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)}{1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)}{1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\cdot G(\xi,F_1\xi,F_1\xi)}{1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)}\end{array}\right\}. \tag{21}$$

This implies that

$$\left|W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq\max\left\{\begin{array}{c}\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right)\right|,\left|G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\ \left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\\ \dfrac{|G(\xi,F_1\xi,F_1\xi)|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot|G\xi,F_1\xi,F_1\xi|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}\end{array}\right\}. \tag{22}$$

After simplification, we get that

$$\left|W\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| \leq \max\left\{\begin{array}{c}\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|, \left|G\left(F_1\xi, F_1\xi, \gamma_{3j+1}\right)\right|, \left|G\left(F_1\xi, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|, \\ \dfrac{\left|G\left(\xi, F_1\xi, F_1\xi\right)\right| \cdot \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}{\left|1 + G\left(\xi, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}, \\ \dfrac{\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right| \cdot \left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}{\left|1 + G\left(F_1\xi, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}, \\ \dfrac{\left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right| \cdot \left|G\left(\xi, F_1\xi, F_1\xi\right)\right|}{\left|1 + G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}\end{array}\right\}. \tag{23}$$

Now, there are six possibilities:

(i) f $|G(\xi, \gamma_{3j+1}, \gamma_{3j+2})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq \beta_1\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| + \beta_2\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right|. \tag{24}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(F_1\xi, \xi, \xi\right)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2|G(\xi, \xi, \xi)|. \tag{25}$$

This implies that $|G(F_1\xi, \xi, \xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{26}$$

(ii) If $|G(F_1\xi, F_1\xi, \gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq \beta_1\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| + \beta_2\left|G\left(F_1\xi, F_1\xi, \gamma_{3j+1}\right)\right|. \tag{27}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\begin{aligned}\left|G\left(F_1\xi, \xi, \xi\right)\right| &\leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\left|G\left(F_1\xi, F_1\xi, \xi\right)\right| \\ &\leq 2b\beta_2\left|G\left(F_1\xi, \xi, \xi\right)\right| \because (\text{Definition 2.5 (2)}).\end{aligned} \tag{28}$$

This implies that $(1 - 2b\beta_2)|G(F_1\xi, \xi, \xi)| \leq 0$ is a contradiction, where $(1 - 2b\beta_2) \neq 0 \Rightarrow |G(F_1\xi, \xi, \xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{29}$$

(iii) If $|G(F_1\xi, \gamma_{3j+1}, \gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq \beta_1\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| + \beta_2\left|G\left(F_1\xi, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|. \tag{30}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(F_1\xi, \xi, \xi\right)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\left|G\left(F_1\xi, \xi, \xi\right)\right|. \tag{31}$$

This implies that $(1 - \beta_2)|G(F_1\xi, \xi, \xi)| \leq 0$ is a contradiction, where $(1 - \beta_2) \neq 0 \Rightarrow |G(F_1\xi, \xi, \xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{32}$$

(iv) If $|G(\xi, F_1\xi, F_1\xi)| \cdot |G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})|/|1 + G(\xi, \gamma_{3j+1}, \gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi, \gamma_{3j+2}, \gamma_{3j+3}\right)\right| \leq \beta_1\left|G\left(\xi, \gamma_{3j+1}, \gamma_{3j+2}\right)\right| + \beta_2\frac{\left|G\left(\xi, F_1\xi, F_1\xi\right)\right| \cdot \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}{\left|1 + G\left(\xi, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}. \tag{33}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(F_1\xi,\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2\frac{|G(\xi,F_1\xi,F_1\xi)|\cdot|G(\xi,\xi,\xi)|}{|1+G(\xi,\xi,\xi)|}. \tag{34}$$

This implies that $|G(F_1\xi,\xi,\xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{35}$$

(v) If $|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})|\cdot|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|/|1+G(F_1\xi,\gamma_{3j+2},\gamma_{3j+2})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right| \le \beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)\right|} \tag{36}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(F_1\xi,\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2\frac{|G(\xi,\xi,\xi)|\cdot|G(\xi,\xi,\xi)|}{\left|1+G(F_1\xi,\xi,\xi)\right|}. \tag{37}$$

This implies that $|G(F_1\xi,\xi,\xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{38}$$

(vi) If $|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|\cdot|G(\xi,F_1\xi,F_1\xi)|/|1+G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right| \le \beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot\left|G(\xi,F_1\xi,F_1\xi)\right|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}. \tag{39}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(F_1\xi,\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2\frac{|G(\xi,\xi,\xi)|\cdot\left|G(\xi,F_1\xi,F_1\xi)\right|}{|1+G(\xi,\xi,\xi)|}. \tag{40}$$

This implies that $|G(F_1\xi,\xi,\xi)| = 0$; thus,

$$F_1\xi = \xi. \tag{41}$$

From (26)–(41), we get that ξ is a FP of F_1, that is,

$$F_1\xi = \xi. \tag{42}$$

Next, we have to show that $F_2\xi = \xi$; by contrary case, let $F_2\xi \ne \xi$. Now by (5),

$$\begin{aligned} G\left(\gamma_{3j+1},F_2\xi,\gamma_{3j+3}\right) &= G\left(F_1\gamma_{3j},F_2\xi,F_3\gamma_{3j+2}\right) \\ &\le \beta_1 G\left(\gamma_{3j},\xi,\gamma_{3j+2}\right) + \beta_2 W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right). \end{aligned} \tag{43}$$

This implies that

$$\left|G\left(\gamma_{3j+1},F_2\xi,\gamma_{3j+3}\right)\right| \le \beta_1\left|G\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)\right| + \beta_2\left|W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)\right|, \tag{44}$$

where

$$W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right) = \max\left\{\begin{array}{c} G\left(\gamma_{3j},\xi,F_2\xi\right), G\left(F_1\gamma_{3j},F_1\gamma_{3j},\xi\right), G\left(F_1\gamma_{3j},\xi,\xi\right), \\ G\left(F_2\xi,F_2\xi,\gamma_{3j+2}\right), G\left(F_2\xi,\gamma_{3j+2},\gamma_{3j+2}\right), \\ \dfrac{G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)\cdot G(\xi,F_2\xi,F_2\xi)}{1+G\left(\gamma_{3j},\xi,\xi\right)}, \\ \dfrac{G(\xi,F_2\xi,F_2\xi)\cdot G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}{1+G\left(F_1\gamma_{3j},\gamma_{3j+2},\gamma_{3j+2}\right)}, \\ \dfrac{G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)}{1+G\left(F_2\xi,F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} G(\gamma_{3j},\xi,F_2\xi), G(\gamma_{3j+1},\gamma_{3j+1},\xi), G(\gamma_{3j+1},\xi,\xi), \\ G(F_2\xi,F_2\xi,\gamma_{3j+2}), G(F_2\xi,\gamma_{3j+2},\gamma_{3j+2}), \\ \dfrac{G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\cdot G(\xi,F_2\xi,F_2\xi)}{1+G(\gamma_{3j},\xi,\xi)}, \\ \dfrac{G(\xi,F_2\xi,F_2\xi)\cdot G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})}{1+G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})}, \\ \dfrac{G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\cdot G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})}{1+G(F_2\xi,\gamma_{3j+3},\gamma_{3j+3})} \end{array}\right\}. \tag{45}$$

This implies that

$$\left|W(\gamma_{3j},\xi,\gamma_{3j+2})\right| = \max\left\{\begin{array}{c} \left|G(\gamma_{3j},\xi,F_2\xi)\right|, \left|G(\gamma_{3j+1},\gamma_{3j+1},\xi)\right|, \left|G(\gamma_{3j+1},\xi,\xi)\right|, \\ \left|G(F_2\xi,F_2\xi,\gamma_{3j+2})\right|, \left|G(F_2\xi,\gamma_{3j+2},\gamma_{3j+2})\right|, \\ \dfrac{\left|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\right|\cdot\left|G(\xi,F_2\xi,F_2\xi)\right|}{\left|1+G(\gamma_{3j},\xi,\xi)\right|}, \\ \dfrac{\left|G(\xi,F_2\xi,F_2\xi)\right|\cdot\left|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\right|}{\left|1+G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})\right|}, \\ \dfrac{\left|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\right|\cdot\left|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\right|}{\left|1+G(F_2\xi,\gamma_{3j+3},\gamma_{3j+3})\right|}. \end{array}\right\} \tag{46}$$

Now, there are eight possibilities:

(i) If $|G(\gamma_{3j},\xi,F_2\xi)|$ is a maximum term in (46), then after simplification, (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j},\xi,F_2\xi)\right|. \tag{47}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2\left|G(\xi,\xi,F_2\xi)\right|. \tag{48}$$

This implies that $(1-\beta_2)|(G\xi,F_2\xi,\xi)| \le 0$ is a contradiction, where $(1-\beta_2)\neq 0 \Rightarrow |G(\xi,F_2\xi,\xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{49}$$

(ii) If $|G(\gamma_{3j+1},\gamma_{3j+1},\xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j+1},\gamma_{3j+1},\xi)\right|. \tag{50}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2|G(\xi,\xi,\xi)|. \tag{51}$$

This implies that $|G(\xi,F_2\xi,\xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{52}$$

(iii) If $|G(\gamma_{3j+1},\xi,\xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j+1},\xi,\xi)\right|. \tag{53}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2|G(\xi,\xi,\xi)|. \tag{54}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \quad (55)$$

(iv) If $|G(F_2\xi, F_2\xi, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \left|G\left(F_2\xi, F_2\xi, \gamma_{3j+2}\right)\right|. \quad (56)$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\begin{aligned}\left|G(\xi, F_2\xi, \xi)\right| &\leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \left|G(F_2\xi, F_2\xi, \xi)\right| \\ &\leq 2b\beta_2 \left|G(\xi, F_2\xi, \xi)\right| \because (\text{Proposition 2.5 (2)}).\end{aligned} \quad (57)$$

This implies that $(1 - 2b\beta_2)|G(\xi, F_2\xi, \xi)| \leq 0$ is a contradiction, where $(1 - 2b\beta_2) \neq 0 \Rightarrow |G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \quad (58)$$

(v) If $|G(F_2\xi, \gamma_{3j+2}, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \left|G\left(F_2\xi, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|. \quad (59)$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \left|G(F_2\xi, \xi, \xi)\right|. \quad (60)$$

This implies that $(1 - \beta_2)|G(\xi, F_2\xi, \xi)| \leq 0$ is a contradiction, where $(1 - \beta_2) \neq 0 \Rightarrow |G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \quad (61)$$

(vi) If $|G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| \cdot |G(\xi, F_2\xi, F_2\xi)|/|1 + G(\gamma_{3j}, \xi, \xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right| \cdot \left|G(\xi, F_2\xi, F_2\xi)\right|}{\left|1 + G\left(\gamma_{3j}, \xi, \xi\right)\right|}. \quad (62)$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \frac{|G(\xi, \xi, \xi)| \cdot \left|G(\xi, F_2\xi, F_2\xi)\right|}{|1 + G(\xi, \xi, \xi)|}. \quad (63)$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \quad (64)$$

(vii) If $|G(\xi, F_2\xi, F_2\xi)| \cdot |G(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3})|/|1 + G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G(\xi, F_2\xi, F_2\xi)\right| \cdot \left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}{\left|1 + G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}. \quad (65)$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$|G(\xi, F_2\xi, \xi)| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \frac{|G(\xi, F_2\xi, F_2\xi)| \cdot |G(\xi, \xi, \xi)|}{|1 + G(\xi, \xi, \xi)|}. \quad (66)$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \quad (67)$$

(viii) If $|G(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3})| \cdot |G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| /|1 + G(F_2\xi, \gamma_{3j+3}, \gamma_{3j+3})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right| \cdot \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}{\left|1 + G\left(F_2\xi, {}_{3j+3}, \gamma_{3j+3}\right)\right|}. \quad (68)$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \frac{|G(\xi, \xi, \xi)| \cdot |G(\xi, \xi, \xi)|}{\left|1 + G(F_2\xi, \xi, \xi)\right|}. \tag{69}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{70}$$

From (49) to (70), we find that ξ is a FP of F_2, that is,

$$F_2\xi = \xi. \tag{71}$$

Now, we have to show that $F_3\xi = \xi$; by contrary case, let $F_3\xi \neq \xi$. Now, by (5),

$$\begin{aligned} G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right) &= G\left(F_1\gamma_{3j}, F_2\gamma_{3j+1}, F_3\xi\right) \\ &\leq \beta_1 G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right) + \beta_2 W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right). \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned} \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1 &\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| \\ &+ \beta_2 \left|W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right|, \end{aligned} \tag{73}$$

where

$$\begin{aligned} &W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right) \\ &= \max\left\{\begin{array}{c} G\left(\gamma_{3j}, \gamma_{3j+1}, F_2\gamma_{3j+1}\right), G\left(F_1\gamma_{3j}, F_1\gamma_{3j}, \gamma_{3j+1}\right), G\left(F_1\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right), \\ G\left(F_2\gamma_{3j+1}, F_2\gamma_{3j+1}, \xi\right), G\left(F_2\gamma_{3j+1}, \xi, \xi\right), \\ \dfrac{G\left(\gamma_{3j}, F_1\gamma_{3j}, F_1\gamma_{3j}\right) \cdot G\left(\gamma_{3j+1}, F_2\gamma_{3j+1}, F_2\gamma_{3j+1}\right)}{1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}, \\ \dfrac{G\left(\gamma_{3j+1}, F_2\gamma_{3j+1}, F_2\gamma_{3j+1}\right) \cdot G(\xi, F_3\xi, F_3\xi)}{1 + G\left(F_1\gamma_{3j}, \xi, \xi\right)}, \\ \dfrac{G(\xi, F_3\xi, F_3\xi) \cdot G\left(\gamma_{3j}, F_1\gamma_{3j}, F_1\gamma_{3j}\right)}{1 + G\left(F_2\gamma_{3j+1}, F_3\xi, F_3\xi\right)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right), G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right), G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right), \\ G\left(\gamma_{3j+2}, \gamma_{3j+2}, \xi\right), G\left(\gamma_{3j+2}, \xi, \xi\right), \\ \dfrac{G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right) \cdot G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)}{1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}, \\ \dfrac{G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right) \cdot G(\xi, F_3\xi, F_3\xi)}{1 + G\left(\gamma_{3j+1}, \xi, \xi\right)}, \\ \dfrac{G(\xi, F_3\xi, F_3\xi) \cdot G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}{1 + G\left(\gamma_{3j+2}, F_3\xi, F_3\xi\right)} \end{array}\right\}. \end{aligned} \tag{74}$$

This implies that

$$\begin{aligned}
&\left|W\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right| \\
&= \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\
\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|,\left|G\left(\gamma_{3j+2},\xi,\xi\right)\right|,\\
\dfrac{\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|}{\left|1+G\left(\gamma_{3j+1},\xi,\xi\right)\right|},\\
\dfrac{\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|\cdot\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|}{\left|1+G\left(\gamma_{3j+2},F_3\xi,F_3\xi\right)\right|}
\end{array}\right\}
\end{aligned}\tag{75}$$

After simplification, we get that

$$\begin{aligned}
&\left|W\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right| \\
&= \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|,\left|G\left(\gamma_{3j+2},\xi,\xi\right)\right|,\\
\dfrac{\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|}{\left|1+G\left(\gamma_{3j+1},\xi,\xi\right)\right|},\\
\dfrac{\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|\cdot\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|}{\left|1+G\left(\gamma_{3j+2},F_3\xi,F_3\xi\right)\right|}
\end{array}\right\}.
\end{aligned}\tag{76}$$

Now, there are six possibilities:

(i) If $|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|$ is a maximum term in (76), then (73) can be written as

$$\begin{aligned}\left|G\left(\gamma_{3j+1},\gamma_{3j+2},F_3\xi\right)\right| \leq{}& \beta_1\left|G\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right|\\ &+\beta_2\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|.\end{aligned}\tag{77}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(\xi,\xi,F_3\xi\right)\right| \leq \beta_1|G(\xi,\xi,\xi)|+\beta_2|G(\xi,\xi,\xi)|.\tag{78}$$

This implies that $|G(\xi,\xi,F_3\xi)|=0$; thus,

$$F_3\xi=\xi.\tag{79}$$

(ii) If $|G(\gamma_{3j+2},\gamma_{3j+2},\xi)|$ is a maximum term in (76), then (73) can be written as

$$\begin{aligned}\left|G\left(\gamma_{3j+1},\gamma_{3j+2},F_3\xi\right)\right| \leq{}& \beta_1\left|G\left(\gamma_{3j},\gamma_{3j+1}\xi\right)\right|\\ &+\beta_2\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|.\end{aligned}\tag{80}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(\xi,\xi,F_3\xi\right)\right| \leq \beta_1|G(\xi,\xi,\xi)|+\beta_2|G(\xi,\xi,\xi)|.\tag{81}$$

This implies that $|G(\xi,\xi,F_3\xi)|=0$; thus,

$$F_3\xi=\xi.\tag{82}$$

(iii) If $|G(\gamma_{3j+2}, \xi, \xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\left|G\left(\gamma_{3j+2}, \xi, \xi\right)\right|, \tag{83}$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\left|G(\xi, \xi, F_3\xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2|G(\xi, \xi, \xi)|. \tag{84}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{85}$$

(iv) If $|G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| \cdot |G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})| / |1 + G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right| \cdot \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}{\left|1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}. \tag{86}$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$|G(\xi, \xi, F_3\xi)| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{|G(\xi, \xi, \xi)| \cdot |G(\xi, \xi, \xi)|}{|1 + G(\xi, \xi, \xi)|}. \tag{87}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{88}$$

(v) If $|G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})| / |1 + G(\gamma_{3j+1}, \xi, \xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right| \cdot |G(\xi, F_3\xi, F_3\xi)|}{\left|1 + G\left(\gamma_{3j+1}, \xi, \xi\right)\right|}. \tag{89}$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$|G(\xi, \xi, F_3\xi)| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{|G(\xi, \xi, \xi)| \cdot |G(\xi, F_3\xi, F_3\xi)|}{|1 + G(\xi, \xi, \xi)|}. \tag{90}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{91}$$

(vi) If $|G(\xi, F_3\xi, F_3\xi)| / |1 + G(\gamma_{3j+2}, F_3\xi, F_3\xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{|G(\xi, F_3\xi, F_3\xi)| \cdot \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}{\left|1 + G\left(\gamma_{3j+2}, F_3\xi, F_3\xi\right)\right|}. \tag{92}$$

Applying $\lim_{n \longrightarrow \infty}$ on both sides, we get

$$\begin{aligned}\left|G(\xi, \xi, F_3\xi)\right| &\leq \beta_1|G(\xi, \xi, \xi)| \\ &+ \beta_2\frac{\left|G(\xi, F_3\xi, F_3\xi)\right| \cdot |G(\xi, \xi, \xi)|}{\left|1 + G(\xi, F_3\xi, F_3\xi)\right|}.\end{aligned} \tag{93}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{94}$$

From (79)–(94), we find that ξ is a FP of F_3, that is,

$$F_3\xi = \xi. \tag{95}$$

Hence proved that ξ is a CFP of F_1, F_2 and F_3, that is,

$$F_1\xi = F_2\xi = F_3\xi = \xi. \tag{96}$$

Uniqueness: assume that $\xi^* \in \Upsilon$ is another CFP of the mappings F_1, F_2, and F_3, such that

$$\begin{aligned}F_1\xi = F_2\xi = F_3\xi = \xi, \\ F_1\xi^* = F_2\xi^* = F_3\xi^* = \xi^*.\end{aligned} \tag{97}$$

Then, from (5), we have that

$$\begin{aligned}G(\xi, \xi^*, \xi^*) &= G(F_1\xi, F_2\xi^*, F_3\xi^*) \\ &\leq \beta_1 G(\xi, \xi^*, \xi^*) + \beta_2 W(\xi, \xi^*, \xi^*).\end{aligned} \tag{98}$$

This implies that

$$\left|G\left(\xi,\xi^*,\xi^*\right)\right|\le\beta_1\left|G\left(\xi,\xi^*,\xi^*\right)\right|+\beta_2\left|W\left(\xi,\xi^*,\xi^*\right)\right|,\tag{99}$$

where

$$\begin{aligned}W\left(\xi,\xi^*,\xi^*\right)&=\max\left\{\begin{array}{c}G\left(\xi,\xi^*,F_2\xi^*\right),G\left(F_1\xi,F_1\xi,\xi^*\right),G\left(F_1\xi,\xi^*,\xi^*\right),\\ G\left(F_2\xi^*,F_2\xi^*,\xi^*\right),G\left(F_2\xi^*,\xi^*,\xi^*\right),\\ \dfrac{G\left(\xi,F_1\xi,F_1\xi\right)\cdot G\left(\xi^*,F_2\xi^*,F_2\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)},\\ \dfrac{G\left(\xi^*,F_2\xi^*,F_2\xi^*\right)\cdot G\left(\xi^*,F_3\xi^*,F_3\xi^*\right)}{\left(1+G\left(F_1\xi,\xi^*,\xi^*\right)\right)},\\ \dfrac{G\left(\xi^*,F_3\xi^*,F_3\xi^*\right)\cdot G\left(\xi,F_1\xi,F_1\xi\right)}{\left(1+G\left(F_2\xi^*,F_3\xi^*,F_3\xi^*\right)\right)}\end{array}\right\}\\ &=\max\left\{\begin{array}{c}G\left(\xi,\xi^*,\xi^*\right),G\left(\xi,\xi,\xi^*\right),G\left(\xi,\xi^*,\xi^*\right),\\ G\left(\xi^*,\xi^*,\xi^*\right),G\left(\xi^*,\xi^*,\xi^*\right),\\ \dfrac{G\left(\xi,\xi,\xi\right)\cdot G\left(\xi^*,\xi^*,\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)},\\ \dfrac{G\left(\xi^*,\xi^*,\xi^*\right)\cdot G\left(\xi^*,\xi^*,\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)},\\ \dfrac{G\left(\xi^*,\xi^*,\xi^*\right)\cdot G\left(\xi,\xi,\xi\right)}{\left(1+G\left(\xi^*,\xi^*,\xi^*\right)\right)}\end{array}\right\}.\end{aligned}\tag{100}$$

This implies that

$$\left|W\left(\xi,\xi^*,\xi^*\right)\right|=\max\left\{\begin{array}{c}\left|G\left(\xi,\xi^*,\xi^*\right)\right|,\left|G\left(\xi,\xi,\xi^*\right)\right|,\left|G\left(\xi,\xi^*,\xi^*\right)\right|,\\ \left|G\left(\xi^*,\xi^*,\xi^*\right)\right|,\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|,\\ \dfrac{\left|G\left(\xi,\xi,\xi\right)\right|\cdot\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|}{\left|1+G\left(\xi,\xi^*,\xi^*\right)\right|},\\ \dfrac{\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|\cdot\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|}{\left|1+G\left(\xi,\xi^*,\xi^*\right)\right|},\\ \dfrac{\left|\left(\xi^*,\xi^*,\xi^*\right)\right|\cdot\left|G\left(\xi,\xi,\xi\right)\right|}{\left|1+G\left(\xi^*,\xi^*,\xi^*\right)\right|}\end{array}\right\}.\tag{101}$$

Now, by using Proposition 2.5 (2) and after simplifying, we get that

$$\left|W(\xi,\xi^*,\xi^*)\right| \leq \max\left|G(\xi,\xi^*,\xi^*)\right|, 2b\left|G(\xi,\xi^*,\xi^*)\right|. \quad (102)$$

Clearly, $2b|G(\xi,\xi^*,\xi^*)|$ is a maximum term in (102), so (99) can be written as

$$\left|G(\xi,\xi^*,\xi^*)\right| \leq \beta_1\left|G(\xi,\xi^*,\xi^*)\right| + 2b\beta_2\left|G(\xi,\xi^*,\xi^*)\right|. \quad (103)$$

This implies that $(1-\beta_1-2b\beta_2)|G(\xi,\xi^*,\xi^*)| \leq 0$ is a contradiction, where $(1-\beta_1-2b\beta_2) \neq 0 \Rightarrow |G(\xi,\xi^*,\xi^*)| = 0 \Rightarrow \xi = \xi^*$. Hence proved that F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 1. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1,\gamma_2,\gamma_3) + \beta_2 W(\gamma_1,\gamma_2,\gamma_3), \quad (104)$$

where

$$W(\gamma_1,\gamma_2,\gamma_3) = \max\left\{\begin{array}{c} G(\gamma_1,\gamma_2,F_2\gamma_2), G(F_1\gamma_1,F_1\gamma_1,\gamma_2), G(F_1\gamma_1,\gamma_2,\gamma_2), \\ G(F_2\gamma_2,F_2\gamma_2,\gamma_3), G(F_2\gamma_2,\gamma_3,\gamma_3), \\ \dfrac{G(\gamma_1,F_1\gamma_1,F_1\gamma_1)\cdot G(\gamma_2,F_2\gamma_2,F_2\gamma_2)}{1+G(\gamma_1,\gamma_2,\gamma_2)}, \end{array}\right\}, \quad (105)$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$, $\beta_1,\beta_2 \in 0, 1/3)$, such that $(\beta_1+\beta_2) < 1/3$, $b\beta_2 < 1/3$, and $(\beta_1 + 2b\beta_2) < 2/3$; then, F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 2. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1,\gamma_2,\gamma_3) + \beta_2 W(\gamma_1,\gamma_2,\gamma_3), \quad (106)$$

where

$$W(\gamma_1,\gamma_2,\gamma_3) = \max\left\{\begin{array}{c} G(\gamma_1,\gamma_2,F_2\gamma_2), G(F_1\gamma_1,F_1\gamma_1,\gamma_2), G(F_1\gamma_1,\gamma_2,\gamma_2), \\ G(F_2\gamma_2,F_2\gamma_2,\gamma_3), G(F_2\gamma_2,\gamma_3,\gamma_3), \\ \dfrac{G(\gamma_2,F_2\gamma_2,F_2\gamma_2)\cdot G(\gamma_3,F_3\gamma_3,F_3\gamma_3)}{1+G(F_1\gamma_1,\gamma_3,\gamma_3)}, \end{array}\right\}, \quad (107)$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$, $\beta_1,\beta_2 \in 0, 1/3)$, such that $(\beta_1+\beta_2) < 1/3$, $b\beta_2 < 1/3$, and $(\beta_1 + 2b\beta_2) < 2/3$; then, F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 3. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1,\gamma_2,\gamma_3) + \beta_2 W(\gamma_1,\gamma_2,\gamma_3), \quad (108)$$

where

$$W(\gamma_1,\gamma_2,\gamma_3) = \max\left\{\begin{array}{c} G(\gamma_1,\gamma_2,F_2\gamma_2), G(F_1\gamma_1,F_1\gamma_1,\gamma_2), G(F_1\gamma_1,\gamma_2,\gamma_2), \\ G(F_2\gamma_2,F_2\gamma_2,\gamma_3), G(F_2\gamma_2,\gamma_3,\gamma_3), \\ \dfrac{G(\gamma_3,F_3\gamma_3,F_3\gamma_3)\cdot G(\gamma_1,F_1\gamma_1,F_1\gamma_1)}{1+G(F_2\gamma_2,F_3\gamma_3,F_3\gamma_3)} \end{array}\right\}, \quad (109)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$, $\beta_1, \beta_2 \in 0, 1/3)$, *such that* $(\beta_1 + \beta_2) < 1/3$, $b\beta_2 < 1/3$, *and* $(\beta_1 + 2b\beta_2) < 2/3$; *then,* F_1, F_2, *and* F_3 *have a unique CFP in* Υ.

Remark 2. If we put $\beta_2 = 0$ and $F_1 = F_2 = F_3$ in Theorem 2, we get (Theorem 3.7 [22]).

Example 2. Let (Υ, G) be a complex-valued G_b-metric space, where $\Upsilon = [0, 1]$ and $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$, with $G(\gamma_1, \gamma_2, \gamma_3) = (3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2 + i\,(3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2$, for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Now, we define $F_1, F_2, F_3: \Upsilon \longrightarrow \Upsilon$ as

$$F_1\gamma = F_2\gamma = F_3\gamma = \ln\left(1 + \frac{\gamma}{5 + \gamma}\right). \tag{110}$$

Notice that

$$|G(\gamma_1, \gamma_2, \gamma_3)|, |W(\gamma_1, \gamma_2, \gamma_3)| \geq 0. \tag{111}$$

In all regards, it is enough to show that $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$, for all $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$ and $\beta_1, \beta_2 \in 0, 1/3)$, with $\beta_1 + \beta_2 < 1/3$, $b\beta_2 < 1/3$, and $\beta_1 + 2b\beta_2 < 2/3$.

$$\begin{aligned}
G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) &= \left(\frac{3}{4}|F_1\gamma_1 - F_2\gamma_2| + \frac{3}{4}|F_2\gamma_2 - F_3\gamma_3| + \frac{3}{4}|F_3\gamma_3 - F_1\gamma_1|\right)^2 \\
&\quad + i\left(\frac{3}{4}|F_1\gamma_1 - F_2\gamma_2| + \frac{3}{4}|F_2\gamma_2 - F_3\gamma_3| + \frac{3}{4}|F_3\gamma_3 - F_1\gamma_1|\right)^2 \\
&= \left(\begin{array}{c} \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_1}{5+\gamma_1}\right) - \ln\left(1 + \frac{\gamma_2}{5+\gamma_2}\right)\right| + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_2}{5+\gamma_2}\right) - \ln\left(1 + \frac{\gamma_3}{5+\gamma_3}\right)\right| \\ + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_3}{5+\gamma_3}\right) - \ln\left(1 + \frac{\gamma_1}{5+\gamma_1}\right)\right| \end{array}\right)^2 \\
&\quad + i\left(\begin{array}{c} \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_1}{5+\gamma_1}\right) - \ln\left(1 + \frac{\gamma_2}{5+\gamma_2}\right)\right| + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_2}{5+\gamma_2}\right) - \ln\left(1 + \frac{\gamma_3}{5+\gamma_3}\right)\right| \\ + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_3}{5+\gamma_3}\right) - \ln\left(1 + \frac{\gamma_1}{5+\gamma_1}\right)\right| \end{array}\right)^2 \\
&\leq \left(\frac{3}{4}\left|\frac{\gamma_1}{5+\gamma_1} - \frac{\gamma_2}{5+\gamma_2}\right| + \frac{3}{4}\left|\frac{\gamma_2}{5+\gamma_2} - \frac{\gamma_3}{5+\gamma_3}\right| + \frac{3}{4}\left|\frac{\gamma_3}{5+\gamma_3} - \frac{\gamma_1}{5+\gamma_1}\right|\right)^2 \\
&\quad + i\left(\frac{3}{4}\left|\frac{\gamma_1}{5+\gamma_1} - \frac{\gamma_2}{5+\gamma_2}\right| + \frac{3}{4}\left|\frac{\gamma_2}{5+\gamma_2} - \frac{\gamma_3}{5+\gamma_3}\right| + \frac{3}{4}\left|\frac{\gamma_3}{5+\gamma_3} - \frac{\gamma_1}{5+\gamma_1}\right|\right)^2 \\
&\leq \left(\frac{3}{4}\left|\frac{5\gamma_1 - 5\gamma_2}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_2 - 5\gamma_3}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_3 - 5\gamma_1}{25}\right|\right)^2 \\
&\quad + i\left(\frac{3}{4}\left|\frac{5\gamma_1 - 5\gamma_2}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_2 - 5\gamma_3}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_3 - 5\gamma_1}{25}\right|\right)^2 \\
&= \frac{1}{25}\left[\begin{array}{c} \left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \\ + i\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \end{array}\right].
\end{aligned} \tag{112}$$

$$G(\gamma_1, \gamma_2, \gamma_3) = \left[\begin{array}{c} \left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \\ + i\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \end{array}\right]. \tag{113}$$

For $\gamma_1, \gamma_2, \gamma_3 \in [0,1]$, we discuss different cases with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$. Hence,

$$\beta_1 + \beta_2 = \frac{1}{10} + \frac{1}{20} < \frac{1}{3}, b\beta_2 = 2\left(\frac{1}{20}\right) < \frac{1}{3}, \tag{114}$$

$$\beta_1 + 2b\beta_2 = \frac{1}{10} + 2(2)\frac{1}{20} < \frac{2}{3}. \tag{115}$$

Case 1: let $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0$; then from (112) and (113), directly we get that $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$. Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 2: let $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 1/2$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \\ +i\left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \\ +i\left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{400} + i\frac{9}{400} \leq \frac{9}{160} + i\frac{9}{160}. \tag{116}$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 3: let $\gamma_1 = 1/2, \gamma_2 = 1/3, \gamma_3 = 1/4$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{1600} + i\frac{9}{1600} \leq \frac{9}{640} + i\frac{9}{640}. \tag{117}$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 4: let $\gamma_1 = 1/2, \gamma_2 = 1/2, \gamma_3 = 1$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{400} + i\frac{9}{400} \leq \frac{9}{160} + i\frac{9}{160}. \tag{118}$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$. Thus, all the conditions of Theorem 2 are satisfied with noticing that the point $0 \in \Upsilon$, which remains fixed under mappings F_1, F_2, and F_3, is indeed unique.

4. Applications

In this section, we present an application of the three UTIEs to support our main work. Let $\Upsilon = C([k_1, k_2], \mathbb{R}^n)$ be the set of all real-valued continuous functions defined on $[k_1, k_2]$. Now, we state and prove a result based on the three UTIEs to uplift our work.

Theorem 3. *Let $\Upsilon = C([k_1, k_2], \mathbb{R}^n)$, where $[k_1, k_2] \subseteq \mathbb{R}$ and $G\colon \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ are defined as*

$$G(\gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} \|\gamma_1(q) - \gamma_2(q)\| \\ + \gamma_2(q) - \gamma_3(q)\| \\ + \gamma_3(q) - \gamma_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (119)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ and $q \in [k_1, k_2]$. Consider the UTIEs are

$$\begin{aligned} \gamma_1(q) &= \int_{k_1}^{k_2} Q_1(q, r, \gamma_1(r)) dr + \hbar_1(q), \\ \gamma_2(q) &= \int_{k_1}^{k_2} Q_2(q, r, \gamma_2(r)) dr + \hbar_2(q), \\ \gamma_3(q) &= \int_{k_1}^{k_2} Q_3(q, r, \gamma_3(r)) dr + \hbar_3(q), \end{aligned} \quad (120)$$

where $r \in [k_1, k_2]$. Let $Q_1, Q_2, Q_3\colon [k_1, k_2] \times [k_1, k_2] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are such that $D_{\gamma_1}, E_{\gamma_2}, F_{\gamma_3} \in \Upsilon$; for every $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$, we have that

$$\begin{aligned} D_{\gamma_1}(q) &= \int_{k_1}^{k_2} Q_1(q, r, \gamma_1(r)) dr, \\ E_{\gamma_2}(q) &= \int_{k_1}^{k_2} Q_2(q, r, \gamma_2(r)) dr, \\ F_{\gamma_3}(q) &= \int_{k_1}^{k_2} Q_3(q, r, \gamma_3(r)) dr. \end{aligned} \quad (121)$$

If there exists $\mu \in (0, 1)$, such that for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$,

$$\begin{pmatrix} \|D_{\gamma_1}(q) - E_{\gamma_2}(q) + \hbar_1(q) - \hbar_2(q)\| \\ + E_{\gamma_2}(q) - F_{\gamma_3}(q) + \hbar_2(q) - \hbar_3(q)\| \\ + F_{\gamma_3}(q) - D_{\gamma_1}(q) + \hbar_3(q) - \hbar_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \leq \mu M(\gamma_1, \gamma_2, \gamma_3), \quad (122)$$

where

$$M(\gamma_1, \gamma_2, \gamma_3) = \max\{A_1(\gamma_1, \gamma_2, \gamma_3)(q), A_2(\gamma_1, \gamma_2, \gamma_3)(q)\}, \quad (123)$$

with

$$A_1(\gamma_1, \gamma_2, \gamma_3)(q) = \begin{pmatrix} \|\gamma_1(q) - \gamma_2(q)\| \\ + \gamma_2(q) - \gamma_3(q)\| \\ + \gamma_3(q) - \gamma_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (124)$$

$$A_2(\gamma_1, \gamma_2, \gamma_3)(q) = \max \left\{ \begin{array}{c} a_1(\gamma_1, \gamma_2, \gamma_3)(q), a_2(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_3(\gamma_1, \gamma_2, \gamma_3)(q), a_4(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_5(\gamma_1, \gamma_2, \gamma_3)(q), a_6(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_7(\gamma_1, \gamma_2, \gamma_3)(q), a_8(\gamma_1, \gamma_2, \gamma_3)(q) \end{array} \right\}, \quad (125)$$

where

$$
\begin{aligned}
a_1(\gamma_1,\gamma_2,\gamma_3)(q) &= \begin{pmatrix} \gamma_1(q)-\gamma_2(q) \\ +E_{\gamma_2}(q)+\hbar_2(q)-\gamma_2(q) \\ +E_{\gamma_2}(q)+\hbar_2(q)-\gamma_1(q) \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1},\\
a_2(\gamma_1,\gamma_2,\gamma_3)(q) &= \left(2D_{\gamma_1}(q)+\hbar_1(q)-\gamma_2(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_3(\gamma_1,\gamma_2,\gamma_3)(q) &= \left(2D_{\gamma_1}(q)+\hbar_1(q)-\gamma_2(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_4(\gamma_1,\gamma_2,\gamma_3)(q) &= \left(2E_{\gamma_2}(q)+\hbar_2(q)-\gamma_3(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_5(\gamma_1,\gamma_2,\gamma_3)(q) &= \left(2E_{\gamma_2}(q)+\hbar_2(q)-\gamma_3(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1},\\
a_6(\gamma_1,\gamma_2,\gamma_3)(q) &= \frac{\begin{pmatrix} 4D_{\gamma_1}(q)+\hbar_1(q)-\gamma_1(q) \\ \cdot E_{\gamma_2}(q)+\hbar_2(q)-\gamma_2(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1+\left(2\gamma_1(q)-\gamma_2(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1}},\\
a_7(\gamma_1,\gamma_2,\gamma_3)(q) &= \frac{\begin{pmatrix} 4E_{\gamma_2}(q)+\hbar_2(q)-\gamma_2(q) \\ \cdot F_{\gamma_3}(q)+\hbar_3(q)-\gamma_3(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1+\left(2D_{\gamma_1}(q)+\hbar_1(q)-\gamma_3(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1}},\\
a_8(\gamma_1,\gamma_2,\gamma_3)(q) &= \frac{\begin{pmatrix} 4F_{\gamma_3}(q)+\hbar_3(q)-\gamma_3(q) \\ \cdot D_{\gamma_1}(q)+\hbar_1(q)-\gamma_1(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1+\left(2E_{\gamma_2}(q)+\hbar_2(q)-F_{\gamma_3}(q)-\hbar_3(q)\right)^2\sqrt{1+k_1^2}e^{i\cot k_1}}.
\end{aligned}
\tag{126}
$$

Then, the three UTIEs, i.e., (120) have a unique common solution.

Proof 1. Define $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ as

$$
\begin{aligned}
F_1\gamma_1 &= F_1\gamma_1(q) = D_{\gamma_1}(q)+\hbar_1(q) = D_{\gamma_1}+\hbar_1, \gamma_1(q) = \gamma_1,\\
F_2\gamma_2 &= F_2\gamma_2(q) = E_{\gamma_2}(q)+\hbar_2(q) = E_{\gamma_2}+\hbar_2, \gamma_2(q) = \gamma_2,\\
F_3\gamma_3 &= F_3\gamma_3(q) = F_{\gamma_3}(q)+\hbar_3(q) = F_{\gamma_3}+\hbar_3, \gamma_3(q) = \gamma_3.
\end{aligned}
\tag{127}
$$

Then, we have the following main two cases:

(1) If $A_1(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in $\{A_1(\gamma_1,\gamma_2,\gamma_3)(q), A_2(\gamma_1,\gamma_2,\gamma_3)(q)\}$, then from (122), (123), and (127), we have that

$$
G(F_1\gamma_1,F_2\gamma_2,F_3\gamma_3) \le \mu \begin{pmatrix} \|\gamma_1-\gamma_2\| \\ +\gamma_2-\gamma_3\| \\ +\gamma_3-\gamma_1\| \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1},
\tag{128}
$$

For all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_1$ and $\beta_2 = 0$ in (31). Then, the given three UTIEs i.e., (4.1) have a unique common solution in Υ.

(2) If $A_2(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in $\{A_1(\gamma_1,\gamma_2,\gamma_3)(q), A_2(\gamma_1,\gamma_2,\gamma_3)(q)\}$, then from (123), we have that

$$
M(\gamma_1,\gamma_2,\gamma_3) = A_2(\gamma_1,\gamma_2,\gamma_3)(q).
\tag{129}
$$

Then, there are furthermore eight subcases arising:

(i) If $a_1(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1,\gamma_2,\gamma_3)(q) = a_1(\gamma_1,\gamma_2,\gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$
G(F_1\gamma_1,F_2\gamma_2,F_3\gamma_3) \le \mu \begin{pmatrix} \|\gamma_1-\gamma_2\| \\ +E_{\gamma_2}+\hbar_2-\gamma_2\| \\ +E_{\gamma_2}+\hbar_2-\gamma_1\| \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1},
\tag{130}
$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (31). Then, the given three UTIEs, i.e., (4.1) have a unique common solution in Υ.

(ii) If $a_2(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_2(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2D_{\gamma_1} + \hbar_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{131}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(iii) If $a_3(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_3(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2D_{\gamma_1} + \hbar_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{132}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(iv) If $a_4(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_4(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2E_{\gamma_2} + \hbar_2 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{133}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(v) If $a_5(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_5(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2E_{\gamma_2} + \hbar_2 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{134}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(vi) If $a_6(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_6(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4D_{\gamma_1} + \hbar_1 - \gamma_1\| \\ \cdot E_{\gamma_2} + \hbar_2 - \gamma_2\| \end{pmatrix}^2 \left(\sqrt{1 + k_1^2}\, e^{i \cot k_1}\right)^2}{1 + \left(2\gamma_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \tag{135}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(vii) If $a_7(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_7(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4E_{\gamma_2} + \hbar_2 - \gamma_2\| \\ \cdot F_{\gamma_3} + \hbar_3 - \gamma_3\| \end{pmatrix}^2 \left(\sqrt{1 + k_1^2}\, e^{i \cot k_1}\right)^2}{1 + \left(2D_{\gamma_1} + \hbar_1 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \tag{136}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(viii) If $a_8(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_8(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4F_{\gamma_3} + \hbar_3 - \gamma_3\| \\ \cdot D_{\gamma_1} + \hbar_1 - \gamma_1\| \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\, e^{i \cot k_1}\right)^2}{1 + \left(2E_{\gamma_2} + \hbar_2 - F_{\gamma_3} - \hbar_3\|\right)^2 \sqrt{1+k_1^2}\, e^{i \cot k_1}}, \quad (137)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ. □

5. Conclusions

We have established a generalized CFP-theorem in complex-valued G_b-metric spaces for three self-mappings. In this result, we have used a generalized rational contraction condition and proved a unique CFP-theorem. To justify our result, we presented an illustrative example in the said space by using three self-maps. Also, we present an application of integral equations to get the existing result for a common solution to support our work. By using this concept, one can prove different contractive-type FP and CFP results for many self-mappings in complex-valued G_b-metric spaces with different types of integral operators.

Data Availability

No datasets were generated or analyzed to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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