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Research Article

On a Three-Sector Keynesian Model of Business Cycles

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In this study, we examined the effect of three-sector interaction on business cycles in Keynesian models. Specifically, we considered the durable, fast, and investment-goods sectors. We modified and extended Murakami's two-sector Keynesian model of business cycles to a three-sector Keynesian model of business cycles. We investigated the conditions for the existence of equilibrium points. Moreover, we studied the stability of the equilibrium points on the basis of the concept of Routh–Hurwitz stability criterion. Finally, to support the analysis and the results obtained, we performed a numerical simulation on a specific example.

1. Introduction

"Business cycles, also known as economic cycles or trade cycles, are the fluctuations of the gross domestic product around its long-term growth trend" [1]. The business cycle indicates the increase and decrease in production output of goods and services in an economy. For organizational policymakers and investors, it is vital to be on the lookout for the overall business cycle in the economy. In economic theory, business cycles and growth cycles are generally described by periodic orbits including limit cycles, and the main theme of the theory of business cycles is, in many cases, to establish the presence of a periodic orbit in dynamic models [2].

There are two main frameworks for modeling economic growth with capital accumulation in continuous time. These are Solow's one-sector growth model and Uzawa's two-sector growth model [3]. The Solow model is the starting point for almost all analyses of economic growth. As consumer behavior is not described by utility optimization in the Solow model, it does not have a rational mechanism to deal with issues related to optimal consumption over time. Ramsey's 1928 paper on optimal savings has influenced modeling of consumers' behavior since the mid 1960s [4, 5]. This approach assumes that utility is addable over time. It has become evident from extensive publications in the economic literature based on this approach in the past fifty years that even a simple model tends to lead to a complicated dynamic system.

Solow's one-sector growth model and Uzawa's two-sector growth model have played the role of key models in the neoclassical growth theory [6–8]. These two models and their various extensions and generalizations are fundamental for the development of new economic growth theories as well [5, 9]. Since Uzawa proposed the model in [6], many works have been published to extend and generalize the model from 1960s to today [10–17]. The Uzawa model extends the Solow model by breaking down the productive system into two sectors using capital and labor, one of which produces capital goods and the other consumption goods [7]. Ferrara and Guerrini [18] considered an extension of Uzawa's two-sector growth model, where capital goods are heterogeneous and the labor growth rate is nonconstant but variable over time. This setup led the model to be represented by a two-dimensional dynamical system in which one of its two equations can be explicitly solved.

The main goal of macroeconomic studies is to explain the mechanism of business cycles. Soon after the basis of macroeconomics was established by Keynes general theory, a lot of theories of business cycles were proposed from the late 1930s to the 1950s. For example, Kalecki (1935, 1937) and Kaldor (1940) put forward models of business cycles by synthesizing the Keynesian multiplier theory and the profit principle of investment, while Harrod (1936) and Samuelson (1939) initiated the so-called multiplier-accelerator model of business cycles by combining the multiplier theory and the acceleration principle of investment [19, 20]. These classical models of business cycles can be characterized by the following Keynesian features:

(i) Quantity or income adjustment governed by the principle of effective demand prevails
(ii) Variations in investment are the main source of business cycles

In their model, the mechanism of business cycle is explained as follows: investment is linked to aggregate income and capital stock, and the aggregate income and capital stock are varied through the multiplier process and capital formation induced by investment, respectively.

It is true that many models of business cycles, including those mentioned above, can describe some aspects of actual business cycles, but there is a certain important point of view missing, that is, the role of sectorial interactions in business cycles. This aspect is lacking in one-sector or one-commodity models of business cycles but should not be ignored when discussing actual business cycles. It goes without saying that the propagation's of shocks from one industry to another do enhance economic fluctuations in reality.

Murakami [19] proposed a two-sector model of business cycle by disaggregating the economy into two sectors, namely, the consumption-goods sector and the investment-goods sector. He examined the stability of equilibrium and the possibility of the existence of a periodic orbit in the two-sector model. As a result, he revealed that the counterpart of the Keynesian stability condition plays a key role in the stability of the two-sector model and that a periodic orbit may arise by way of a Hopf bifurcation if the stability condition is not satisfied. He also observed that the consumption-goods sector lags behind the investment-goods sector along the periodic orbit and that the interactions between these two sectors do play a significant role in business cycles. Furthermore, he numerically investigated the characteristics of a periodic orbit generated by a Hopf bifurcation in the two-sector model. The numerical simulations performed verified that a periodic orbit representing persistent business cycles is actually generated by a Hopf bifurcation in the two-sector model.

Murakami and Zimka [20] have examined the mathematical details of the two-sector Keynesian model proposed by Murakami [19]. In their analysis, they provided the criterion on the stability (or instability) of limit cycles that are generated by Hopf bifurcations. The numerical simulations imply that they are consistent with the theoretical results achieved. The formalization of the two sectors is a more appropriate description of macroeconomic systems than conventional and traditional one-sector ones, and they believe that their analysis contributes to a deeper understanding of real economies.

Murakami [19], considered two sectors, consumption goods and investment goods. In his study, only two kinds of goods that differ in property are considered and different stability conditions for these two sectors of the Keynesian model of business cycles were discussed. He sets up a two-sector model as follows:

$$\begin{cases} \dot{y}_c = \alpha_c\left[C\left(y_c, y_i\right) - y_c\right], \\ \dot{y}_i = \alpha_i\left[I_c\left(y_c, k_c\right) + I_i\left(y_i, k_i\right) - y_i\right], \\ \dot{k}_c = I_c\left(y_c, k_c\right) - \delta k_c, \\ \dot{k}_i = I_i\left(y_i, k_i\right) - \delta k_i, \end{cases} \tag{1}$$

where y_c stands for the level of income or output of the consumption-goods sectors, y_i stands for the level of income or output of the investment-goods sectors, k_c stands for the existing stocks of the consumption goods, k_i stands for the existing stocks of the investment goods (capital), C is the consumption function, and I_c and I_i are the (gross) investment functions for the consumption-goods and investment-goods sectors, respectively. Also, α_c, α_i, and δ are positive constants, where δ stands for the rate of capital depreciation and α_c and α_i are the parameters that measure the speed of the quantity adjustment processes for the consumption-goods and investment-goods sectors, respectively. The first two equations in (1) describe the quantity (or income) adjustment processes in the consumption-goods and investment-goods sectors, respectively. The last two equations in (1) represent the capital formation processes for the consumption-goods and investment-goods sectors, respectively.

The classification of types of goods into only two types does not explain well the interaction of different sectors in business cycles; on the contrary, modeling the interaction of every possible sector leads to higher dimensional differential equations.

To the best of our knowledge, the three-sector Keynesian model of business cycle has not been studied yet. Therefore, in this study, our aim is to extend the work of Murakami [19] to the three-sector Keynesian model of business cycles. We further split the consumption-goods sector into two: fast goods and durable goods. Thus, we investigate the effect of three-sector interactions on the business cycles.

The main contributions of this study are

(1) The mathematical model which describes the interaction of the three-sector Keynesian model of business cycles was formulated
(2) The conditions for the existence of equilibria of the three-sector Keynesian model of business cycles were given
(3) Different stability conditions for the dynamic system were analyzed and numerical simulation was conducted to support the theory

The study is organized as follows. Section 2 contains the model formulation of a three-sector Keynesian model of business cycles. In Section 3, we give the stability analysis for the formulated Keynesian model of business cycles. Moreover, we support the result by a numerical simulation. In Section 4, we put a conclusion along with some suggestions for a possible extension of this work.

2. Model Formulation

In this section, we extend the two-sector Keynesian model of business cycles presented in [19, 20] to three sectors. We consider three types of goods that differ in property, namely, durable goods, fast goods, and investment goods. The formulation of the model is based on the following main assumptions:

(1) Based on the Keynesian theory, we assume that aggregate consumption, denoted by C, is a function of all sectors' income:

$$C = C\left(y_d, y_f, y_i\right), \tag{2}$$

where C is twice continuously differentiable and all partial derivatives $(\partial C/\partial y_d)$, $(\partial C/\partial y_f)$, and $(\partial C/\partial y_i)$ lie between 0 and 1. In expression (2), y_d stands for the income or output of durable good, y_f stands for output of the fast good, and y_i stands for the income of the investment goods.

(2) The output (supply) of the fast good y_f is assumed to be varied in response to the existing excess demand or supply. Specifically, it is given by

$$\dot{y}_f = \alpha_f\left[C\left(y_d, y_f, y_i\right) - y_f\right], \tag{3}$$

where α_f is a positive parameter which stands for the speed of adjustment. Equation (3) expresses the Keynesian quantity adjustment process in the fast good sector.

(3) Next, we take a look at the demand side of the investment good, which is assumed to be dependent on the output of the sector and on the existing stock of capital of the sector. In particular, we assumed that the gross investment functions of the durable good sector I_d, the fast good sector I_f, and the investment-goods sector I_i are designated as

$$I_d = I_d\left(y_d, k_d\right), I_f = I_f\left(y_f, k_f\right), I_i = I_i\left(y_i, k_i\right), \tag{4}$$

where k_d, k_f, and k_i denote the existing stock of the investment function of durable good sector, fast good sector, and investment-goods sector, respectively.

(4) Similarly, the output (supply) of the investment goods, y_i, and the durable goods, y_d, is assumed to satisfy the following equations:

$$\begin{aligned} \dot{y}_d &= \alpha_d\left[I_d\left(y_d, k_d\right) + I_f\left(y_f, k_f\right) - y_d\right], \\ \dot{y}_i &= \alpha_i\left[I_f\left(y_f, k_f\right) + I_i\left(y_i, k_i\right) - y_i\right], \end{aligned} \tag{5}$$

where α_d and α_i are positive parameters which represent the speed of adjustment. The above equations are the Keynesian income adjustment process in the durable-goods sector and investment-goods sector, respectively. These equations mean that a change in the output of each good is proportional to the existing excess demand or supply and imply that the level of output is adjusted to meet the demand for each good.

(5) Finally, the demand for the investment goods of each sector I_d, I_f, and I_i is realized as the growth increment of the stock of the investment goods in the sector k_d, k_f, and k_i, respectively, and given in the following way:

$$\begin{aligned} \dot{k}_d = I_d\left(y_d, k_d\right) - \delta k_d, \dot{k}_f = I_f\left(y_f, k_f\right) \\ - \delta k_f, \dot{k}_i = I_i\left(y_i, k_i\right) - \delta k_i, \end{aligned} \tag{6}$$

where δ is a positive constant which stands for the rate of capital depreciation. These three equations represent the capital formation processes for the durable goods, fast goods, and investment-goods sectors, respectively.

Thus, the three-sector Keynesian model of business cycles is described by the following system of differential equations:

$$\begin{cases} \dot{y}_f = \alpha_f\left[C\left(y_d, y_f, y_i\right) - y_f\right], \\ \dot{y}_d = \alpha_d\left[I_d\left(y_d, k_d\right) + I_f\left(y_f, k_f\right) - y_d\right], \\ \dot{y}_i = \alpha_i\left[I_f\left(y_f, k_f\right) + I_i\left(y_i, k_i\right) - y_i\right], \\ \dot{k}_f = I_f\left(y_f, k_f\right) - \delta k_f, \\ \dot{k}_d = I_d\left(y_d, k_d\right) - \delta k_d, \\ \dot{k}_i = I_i\left(y_i, k_i\right) - \delta k_i. \end{cases} \tag{7}$$

In this model, the rate of depreciation is considered the same for the three sectors, but this assumption can be relaxed without much difficulty. For further analysis and numerical computations, we need to specify the functions C, I_d, I_f, and I_i.

2.1. The Consumption Function, C. Keynes was interested in the level of total spending in general [21]. He was particularly concerned about consumption, which was a major concern because it is by far the largest slice of the total spending pie. He made three basic points about consumption:

(i) Consumption depends on disposable income (income minus taxes)

(ii) Consumption and disposable income move in the same direction

(iii) When disposable income changes, consumption changes by less

The above three points give us a specific statement about the relationship between consumption and disposable income. The consumption function can be written as

$$C\left(y_d, y_f, y_i\right) = c_0 + c\left(y_d + y_f + y_i\right), \tag{8}$$

where

(i) The expression $y_d + y_f + y_i$ represents disposable income

(ii) The constant c stands for marginal propensity to consume

(iii) The parameter c_0 is autonomous consumption, which does not change as disposable income changes, but rather due to other factors

The formula for the marginal propensity to consume is given by

$$c = \frac{\Delta C}{\Delta\left(y_d + y_f + y_i\right)}. \tag{9}$$

Based on Keynes points (ii) and (iii) above, c is always a positive number between 0 and 1.

2.2. The Investment Functions, I_d, I_f, and I_i. Assume that, at each period, each firm puts its rate of (gross) capital formation to one of the two alternative levels: higher and lower ones, say, $\delta + \mu_j$ and $\delta - \lambda_j \geq 0$ with $0 < \lambda_j \leq \delta$ (the restriction of $\lambda_j \leq \delta$ is, of course, imposed for the rate of gross capital formation to be nonnegative for each sector) and $\mu_j > 0$ (it can differ from sector to sector). Since we can safely suppose that the share of firms choosing the higher level of investment (among each sector), say p_j, increases as the output-capital ratio, which is proportionate to the rate of profit under the assumption of constant capital share, rises; the ratio p_j can be related to the output-capital ratio (y_j/k_j) in the following way:

$$\ln\left(\frac{p_j}{1-p_j}\right) = \beta_j \frac{y_j}{k_j} - \beta_{0j}. \tag{10}$$

Moreover,

$$\frac{I_j}{k_j} = p_j\left(\delta + \mu_j\right) + \left(1 - p_j\right)\left(\delta - \lambda_j\right). \tag{11}$$

By substituting the first relationships into the second, we obtain the following logistic investment function:

$$I_j\left(y_j, k_j\right) = \left[\delta + \frac{\mu_j e^{\left(\beta_j\left(y_j/k_j\right) - \beta_{0j}\right)} - \lambda_j}{1 + e^{\left(\beta_j\left(y_j/k_j\right) - \beta_{0j}\right)}}\right] k_j, \tag{12}$$

where $\beta_j > 0$ and $\beta_{0j} > 0$ are measures the sensitivity of capital formation to changes in the output-capital ratio and $\lambda_j > 0$ and $\mu_j > 0$ are constants that determine the minimum and maximum rates of gross capital formation, respectively, for $j = d, f, i$ [20].

Thus, keeping all above descriptions, we get the following functions:

$$C\left(y_d, y_f, y_i\right) = c_0 + c\left(y_d + y_f + y_i\right), \tag{13}$$

$$I_f\left(y_f, k_f\right) = \left[\delta + \frac{\mu_f e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)}}\right] k_f, \tag{14}$$

$$I_d\left(y_d, k_d\right) = \left[\delta + \frac{\mu_d e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)} - \lambda_d}{1 + e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)}}\right] k_d, \tag{15}$$

$$I_i\left(y_i, k_i\right) = \left[\delta + \frac{\mu_i e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)} - \lambda_i}{1 + e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)}}\right] k_i. \tag{16}$$

3. Analysis

In this section, we provide a stability analysis for a three-sector Keynesian model of business cycles and numerical examples.

3.1. The Equilibrium Point. An equilibrium point of system (5) is defined as a point at which

$$\dot{y}_f = \dot{y}_d = \dot{y}_i = \dot{k}_d = \dot{k}_f = \dot{k}_i = 0. \tag{17}$$

That means, it is the solution of the following simultaneous equation:

$$\begin{aligned}
&\alpha_f\left[c_0 - (1-c)y_f + c\left(y_d + y_i\right)\right] = 0,\\
&\alpha_d\left\{\left[\delta + \frac{\mu_f e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)} - \lambda_d}{1 + e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)}}\right] k_d\right.\\
&\qquad\left. + \left[\delta + \frac{\mu_f e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)}}\right] k_f - y_d\right\} = 0,\\
&\alpha_i\left\{\left[\delta + \frac{\mu_f e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)}}\right] k_f\right.\\
&\qquad\left. + \left[\delta + \frac{\mu_i e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)} - \lambda_i}{1 + e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)}}\right] k_i - y_i\right\} = 0,\\
&\frac{\mu_f e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f/k_f\right) - \beta_{0f}\right)}} k_f = 0,\\
&\frac{\mu_d e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)} - \lambda_d}{1 + e^{\left(\beta_d\left(y_d/k_d\right) - \beta_{0d}\right)}} k_d = 0,\\
&\frac{\mu_i e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)} - \lambda_i}{1 + e^{\left(\beta_i\left(y_i/k_i\right) - \beta_{0i}\right)}} k_i = 0.
\end{aligned} \tag{18}$$

Since the case of $k_d = 0$, $k_f = 0$, or $k_i = 0$ has no impact from economic point of view, we give emphasis to the cases when $k_d \neq 0$, $k_f \neq 0$, and $k_i \neq 0$. Denote an equilibrium point for the given system by $(y_f^*, y_d^*, y_i^*, k_f^*, k_d^*, k_i^*)$. Then, the equilibrium point of the system is given by in the following form:

$$\left(y_f^*, y_d^*, y_i^*, k_f^*, k_d^*, k_i^*\right) = \left(\frac{c_0}{1-c} + \frac{c}{1-c}\left(y_d^* + y_i^*\right), \frac{\delta\sigma_f}{(1-c)(1-\delta\sigma_d) - c\delta\sigma_f}c_0, \frac{\delta\sigma_f}{(1-c)(1-\delta\sigma_i) - c\delta\sigma_f}c_0, \sigma_f y_f^*, \sigma_d y_d^*, \sigma_i y_i^*\right), \tag{19}$$

where

$$\sigma_f = \frac{\beta_f}{\beta_{0f} + \ln\lambda_f - \ln\mu_f} > 0,$$

$$\sigma_d = \frac{\beta_d}{\beta_{0d} + \ln\lambda_d - \ln\mu_d} > 0,$$

$$\sigma_i = \frac{\beta_i}{\beta_{0i} + \ln\lambda_i - \ln\mu_i} > 0,$$

$$(1-c)(1-\delta\sigma_d) - c\delta\sigma_f > 0, \text{ and } (1-c)(1-\delta\sigma_i) - c\delta\sigma_f > 0. \tag{20}$$

We assumed all are positive to ensure that the unique equilibrium $(y_d^*, y_f^*, y_i^*, k_d^*, k_f^*, k_i^*)$ lies in the economically meaningful domain $\mathbb{R}_+^6$. These conditions are satisfied when $\beta_{0\,d}$, β_{0f}, and β_{0i} are sufficiently large and δ is sufficiently small.

For further analysis, we transform our system by introducing new variables as follows:

$$\begin{aligned} \tilde{y}_f &= y_f - y_f^*, \tilde{y}_d = y_d - y_d^*, \tilde{y}_i = y_i - y_i^*, \\ \tilde{k}_f &= k_f - k_f^*, \tilde{k}_d = k_d - k_d^*, \tilde{k}_i = k_i - k_i^*. \end{aligned} \tag{21}$$

The following system of equations are obtained after plugging the above new variables in (7):

$$\begin{cases} \dot{\tilde{y}}_f = \alpha_f\left[c_0 - (1-c)\tilde{y}_f + c\left(\tilde{y}_d + \tilde{y}_i\right)\right], \\ \dot{\tilde{y}}_d = \alpha_d\left[\delta + \dfrac{\mu_d e^{\left(\beta_d\left(y_d^* + \tilde{y}_d/k_d^* + \tilde{k}_d\right) - \beta_{0d}\right)} - \lambda_d}{1 + e^{\left(\beta_d\left(y_d^* + \tilde{y}_d/k_d^* + \tilde{k}_d\right) - \beta_{0d}\right)}}\right]\left(k_d^* + \tilde{k}_d\right) \\ +\alpha_d\left\{\left[\delta + \dfrac{\mu_f e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)}}\right]\left(k_f^* + \tilde{k}_f\right) - \left(y_d^* + \tilde{y}_d\right)\right\}, \\ \dot{\tilde{y}}_i = \alpha_i\left[\delta + \dfrac{\mu_f e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)}}\right]\left(k_f^* + \tilde{k}_f\right) \\ +\alpha_i\left\{\left[\delta + \dfrac{\mu_i e^{\left(\beta_i\left(y_i^* + \tilde{y}_i/k_i^* + \tilde{k}_i\right) - \beta_{0i}\right)} - \lambda_i}{1 + e^{\left(\beta_i\left(y_i^* + \tilde{y}_i/k_i^* + \tilde{k}_i\right) - \beta_{0i}\right)}}\right]\left(k_i^* + \tilde{k}_i\right) - \left(y_i^* + \tilde{y}_i\right)\right\}, \\ \dot{\tilde{k}}_f = \left[\dfrac{\mu_f e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)} - \lambda_f}{1 + e^{\left(\beta_f\left(y_f^* + \tilde{y}_f/k_f^* + \tilde{k}_f\right) - \beta_{0f}\right)}}\right]\left(k_f^* + \tilde{k}_f\right), \\ \dot{\tilde{k}}_d = \left[\dfrac{\mu_d e^{\left(\beta_d\left(y_d^* + \tilde{y}_d/k_d^* + \tilde{k}_d\right) - \beta_{0d}\right)} - \lambda_d}{1 + e^{\left(\beta_d\left(y_d^* + \tilde{y}_d/k_d^* + \tilde{k}_d\right) - \beta_{0d}\right)}}\right]\left(k_d^* + \tilde{k}_d\right), \\ \dot{\tilde{k}}_i = \left[\dfrac{\mu_i e^{\left(\beta_i\left(y_i^* + \tilde{y}_i/k_i^* + \tilde{k}_i\right) - \beta_{0i}\right)} - \lambda_i}{1 + e^{\left(\beta_c\left(y_i^* + \tilde{y}_i/k_i^* + \tilde{k}_i\right) - \beta_{0c}\right)}}\right]\left(k_i^* + \tilde{k}_i\right). \end{cases} \tag{22}$$

Table 1: Routh's table.

a_6	a_4	a_2	a_0
a_5	a_3	a_1	0
b_1	b_2	b_3	0
c_1	c_2	c_3	0
d_1	d_2	0	0
e_1	0	0	0
f_1	0	0	0

The newly obtained system has a unique equilibrium point $(0,0,0,0,0,0)$.

3.2. Stability and Instability Analysis

3.2. Stability and Instability Analysis. Before we investigate the stability and instability conditions, we have to linearize the system at the unique equilibrium point. Computation of the Jacobian matrix at this unique equilibrium point gives

$$J(0,0,0,0,0,0)=\begin{bmatrix} \alpha_f c & -\alpha_f(1-c) & \alpha_f c & 0 & 0 & 0 \\ \alpha_d A_d & \alpha_d A_f & 0 & \alpha_d(\delta-B_d) & \alpha_d(\delta-B_f) & 0 \\ \alpha_i A_d & 0 & -\alpha_i(1-A_i) & \alpha_i(\delta-B_d) & 0 & \alpha_i(\delta-B_i) \\ 0 & A_f & 0 & 0 & -B_f & 0 \\ A_d & 0 & 0 & -B_d & 0 & 0 \\ 0 & 0 & A_i & 0 & 0 & -B_i \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} A_f &= \frac{\beta_f\lambda_f\mu_f}{\lambda_f+\mu_f}>0, A_d=\frac{\beta_d\lambda_d\mu_d}{\lambda_d+\mu_d}>0, \\ A_i &= \frac{\beta_i\lambda_i\mu_i}{\lambda_i+\mu_i}>0, B_f=\frac{\left(\beta_{0f}+\ln\lambda_f-\ln\mu_f\right)\lambda_f\mu_f}{\lambda_f+\mu_f}>0, \\ B_d &= \frac{(\beta_{0\,d}+\ln\lambda_d-\ln\mu_d)\lambda_d\mu_d}{\lambda_d+\mu_d}>0 \text{ and } B_i \\ &= \frac{(\beta_{0i}+\ln\lambda_i-\ln\mu_i)\lambda_i\mu_i}{\lambda_i+\mu_i}>0. \end{aligned} \tag{24}$$

The characteristic equation associated with the Jacobian matrix (11) is given by

$$a_6r^6+a_5r^5+a_4r^4+a_3r^3+a_2r^2+a_1r+a_0=0. \tag{25}$$

To find the coefficients, a_i $(i=0,1,2,3,4,5,6)$, let us use the following notation for the entries of the Jacobian matrix:

$$\begin{aligned} &c_1=-\alpha_d(1-c), c_2=\alpha_d c, \\ &c_3=\alpha_f A_d, c_4 \\ &c_5=\alpha_f(\delta-B_d), c_6=\alpha_f\left(\delta-B_f\right), \\ &c_7=\alpha_i A_d, c_8=-\alpha_i(1-A_i), \\ &c_9=\alpha_i(\delta-B_d), c_{10}=\alpha_i(\delta-B_i). \end{aligned} \tag{26}$$

After using the symbolic calculation in MATLAB R2021a with an appropriate code, we get the coefficients as follows:

$$\begin{aligned}
a_6 &= 1,\\
a_5 &= B_i - c1 - c4 - c8,\\
a_4 &= c_1c_4 - A_ic_{10} - B_ic_1 - B_ic_4 - B_ic_8 - A_fc_5 - c_2c_3 - c_1c_7 + c_1c_8 + c_4c_8 - B_dB_f,\\
a_3 &= A_fB_dc_6 - B_dB_fB_i - A_fB_ic_5 + B_dB_fc_1 + B_dB_fc_4 + B_dB_fc_8 -\\
&\quad A_dc_2c_6 + A_fc_1c_5 + A_fc_5c_8 + A_ic_1c_{10} + A_ic_4c_{10} + B_ic_1c_4 - B_ic_2c_3 -\\
&\quad B_ic_1c_7 + B_ic_1c_8 + B_ic_4c_8 + c_1c_4c_7 - c_1c_4c_8 + c_2c_3c_8,\\
a_2 &= A_fB_dB_ic_6 + A_iB_dB_fc_{10} + B_dB_fB_ic_1 + B_dB_fB_ic_4 + B_dB_fB_ic_8\\
&\quad + A_fA_ic_5c_{10} + A_dB_fc_2c_5 - A_fB_dc_1c_6 + A_dB_fc_1c_9\\
&\quad - A_fB_dc_6c_8 - A_dB_ic_2c_6 + A_fB_ic_1c_5 + A_fB_ic_5c_8 - B_dB_fc_1c_4\\
&\quad + B_dB_fc_2c_3 + B_dB_fc_1c_7 - B_dB_fc_1c_8 - B_dB_fc_4c_8 + A_dc_2c_6c_8\\
&\quad - A_fc_1c_3c_9 + A_fc_1c_5c_7 - A_fc_1c_5c_8 - A_ic_1c_4c_{10}\\
&\quad + A_ic_2c_3c_{10} + B_ic_1c_4c_7 - B_ic_1c_4c_8 + B_ic_2c_3c_8,\\
a_1 &= A_dA_ic_2c_6c_{10} - A_dA_fc_1c_6c_9 - A_fA_ic_1c_5c_{10} - A_dB_fc_1c_4c_9\\
&\quad - A_fB_dc_1c_6c_7 - A_dB_fc_2c_5c_8 + A_fB_dc_1c_6c_8 + A_dB_ic_2c_6c_8\\
&\quad - A_fB_ic_1c_3c_9 + A_fB_ic_1c_5c_7 - A_fB_ic_1c_5c_8 - B_dB_fc_1c_4c_7\\
&\quad + B_dB_fc_1c_4c_8 - B_dB_fc_2c_3c_8 - A_fA_iB_dc_6c_{10} + A_dB_fB_ic_2c_5\\
&\quad - A_fB_dB_ic_1c_6 + A_dB_fB_ic_1c_9 - A_iB_dB_fc_1c_{10} - A_fB_dB_ic_6c_8\\
&\quad - A_iB_dB_fc_4c_{10} - B_dB_fB_ic_1c_4 + B_dB_fB_ic_2c_3 + B_dB_fB_ic_1c_7\\
&\quad - B_dB_fB_ic_1c_8 - B_dB_fB_ic_4c_8,\\
a_0 &= A_fA_iB_dc_1c_6c_{10} - A_dA_iB_fc_2c_5c_{10} - A_dA_fB_ic_1c_6c_9\\
&\quad - A_dB_fB_ic_1c_4c_9 - A_fB_dB_ic_1c_6c_7 - A_dB_fB_ic_2c_5c_8\\
&\quad + A_fB_dB_ic_1c_6c_8 + A_iB_dB_fc_1c_4c_{10} - A_iB_dB_fc_2c_3c_{10}\\
&\quad - B_dB_fB_ic_1c_4c_7 + B_dB_fB_ic_1c_4c_8 - B_dB_fB_ic_2c_3c_8.
\end{aligned} \tag{27}$$

Table 1 shows Routh's tabular method.

In Table 1,

$$\begin{aligned}
b_1 &= \frac{a_5a_4 - a_6a_3}{a_5}, b_2 = \frac{a_5a_2 - a_6a_1}{a_5}, b_3 = \frac{a_5a_0 - a_6\times 0}{a_5} = a_0,\\
c_1 &= \frac{b_1a_3 - b_2a_5}{b_1}, c_2 = \frac{b_1a_1 - b_3a_5}{b_1}, c_3 = \frac{b_1\times 0 - a_5\times 0}{b_1} = 0,\\
d_1 &= \frac{c_1b_2 - b_1c_2}{c_1}, d_2 = \frac{c_1b_3 - 0\times b_1}{c_1} = b_3,\\
e_1 &= \frac{d_1c_2 - d_2c_1}{d_1}, f_1 = \frac{e_1d_2 - d_1\times 0}{e_1} = d_2.
\end{aligned} \tag{28}$$

To investigate the stability criterion, as an example, let us take the following parameters specified in [20] for the corresponding two-sector model. The calculated numbers are indicated in Table 2:

$$\begin{aligned}
c &= 0.53, c_0 = 1, \lambda_d = \lambda_f = \lambda_i = 0.09,\\
\mu_d &= \mu_f = \mu_i = 0.21, \beta_d = \beta_f = \beta_i = 9.4,\\
\beta_{0\,d} &= \beta_{0f} = \beta_{0i} = 5.8,\\
\delta &= 0.09, \alpha_d = \alpha_f = \alpha_i = 2.55336.
\end{aligned} \tag{29}$$

After evaluating the above values and substituting them into their respective places in Table 1, we get the values indicated in Table 2.

As we can see from Table 2, the first column of Routh's array, there are two sign changes, one is from 530.3539 to −2422.1 and the other is from −2422.1 to 2487.6. Therefore, system (10) is unstable for the specified parameters. Moreover, since sign changes twice in the first column, the characteristic polynomial has two roots with positive real parts. In general, system (10) is stable whenever all entries in the first column under the Routh array are positive, i.e.,

Table 2: Ruth's Table with parameter calculated.

1	−1.5696	−2.1324	5544.8
1.0413	−6.5034	2508.2	0
4.6761	−2410.9	5544.8	0
530.3539	1273.5	0	0
−2422.1	5544.8	0	0
2487.6	0	0	0
5544.8	0	0	0

$$a_6, a_5, b_1, c_1, d_1, e_1, f_1 > 0. \tag{30}$$

4. Numerical Simulations

To assess the validity of our analysis, we perform numerical simulations taking the following parameters specified in [20]

$$\begin{aligned} &c = 0.53, c_0 = 1, \lambda_d = \lambda_f = \lambda_i = 0.09, \\ &\mu_d = \mu_f = \mu_i = 0.21, \beta_d = \beta_f = \beta_i = 9.4, \\ &\beta_{0\,d} = \beta_{0f} = \beta_{0i} = 5.8, \\ &\delta = 0.09, \alpha_d = \alpha_f = \alpha_i = 2.55336. \end{aligned} \tag{31}$$

After plugging the above specific values in place of each parameters in system (5), we have

$$\begin{aligned} \dot{y}_f &= 1.3532808\left[y_i + y_d\right] - 1.2000792 y_f + 2.55336, \\ \dot{y}_d &= \left[0.2298024 + \frac{0.5362056 e^{9.4\left(y_d/k_d\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_d/k_d\right)-5.8}}\right] k_d \\ &\quad + \left[0.2298024 + \frac{0.5362056 e^{9.4\left(y_f/k_f\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_f/k_f\right)-5.8}}\right] k_f - 2.55336 y_d, \\ \dot{y}_i &= \left[0.2298024 + \frac{0.5362056 e^{9.4\left(y_f/k_f\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_f/k_f\right)-5.8}}\right] k_f \\ &\quad + \left[0.2298024 + \frac{0.5362056 e^{9.4\left(y_i/k_i\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_i/k_i\right)-5.8}}\right] \times k_i - 2.55336 y_i, \\ \dot{k}_f &= \frac{0.5362056 e^{9.4\left(y_f/k_f\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_f/k_f\right)-5.8}} k_f, \\ \dot{k}_d &= \frac{0.5362056 e^{9.4\left(y_d/k_d\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_d/k_d\right)-5.8}} k_d, \\ \dot{k}_i &= \frac{0.5362056 e^{9.4\left(y_i/k_i\right)-5.8} - 0.2298024}{1 + e^{9.4\left(y_i/k_i\right)-5.8}} k_i. \end{aligned} \tag{32}$$

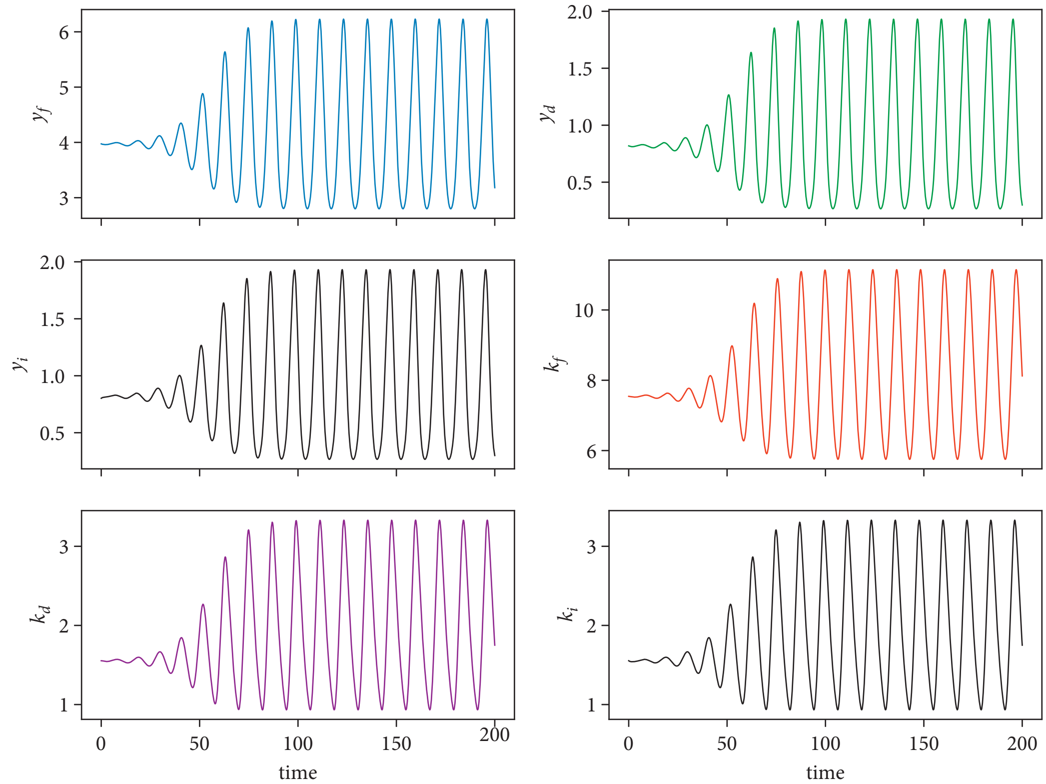


Figure 1: The plots of the trajectories for y_f, y_d, y_i, k_f, k_d, and k_i.

Making the right side of all equations in the above system equal to 0, we get the following equilibrium point:

$$\left(y_f^*, y_d^*, y_i^*, k_f^*, k_d^*, k_i^*\right) = (3.9740, 0.8187, 0.8187, 7.5425, 1.5538, 1.5538). \tag{33}$$

We set the initial condition for our simulation as follows:

$$\begin{aligned}\left(y_f^*(0), y_d^*(0), y_i^*(0), k_f^*(0), k_d^*(0), k_i^*(0)\right) &= \left(y_f^*, y_d^*, 0.98 y_i^*, k_f^*, k_d^*, k_i^*\right) \\ &= (3.9740, 0.8187, 0.8023, 7.5425, 1.5538, 1.5538).\end{aligned} \tag{34}$$

Figure 1 corresponds to the solution path of the system $(y_f(t), y_d(t), y_i(t), k_f(t), k_d(t), k_i(t))$. One can see from the simulation that the solution paths are periodic, describing the fluctuations of durable, fast, and investment-goods sectors. These periodic orbits may be interpreted to represent persistent business cycles, and they do not tell us about the stability of the periodic orbit. It is well known that there is a criterion to recognize whether the periodic orbit is stable or not, that is, whether a Hopf bifurcation is supercritical or subcritical. However, it is usually hard to derive economic implications from this criterion because it requires third-order partial derivatives of the relevant functions, which usually do not have economic meanings, and so we do not analytically investigate the periodic stability.

Figure 2 depicts the 3D phase portrait of the dynamical system. Figure 2(a) is the phase portrait projected in the $y_f - y_d - y_i$ spaces in black and the $y_f - y_f^*, y_d - y_d^*, y_i - y_i^*$ space in blue. Figure 2(b) is the phase portrait for the $k_f - k_d - k_i$ spaces in black and the $k_f - k_f^*, k_d - k_d^*, k_i - k_i^*$ space in blue. The plots show that the periodic orbit is a stable limit cycle.

In Figure 3, the time paths for the state variables $y_f - y_f^*, y_d - y_d^*$, and $y_i - y_i^*$ shown in the Figure 3(a) and $k_f - k_f^*, k_d - k_d^*$, and $k_i - k_i^*$ shown in Figure 3(b). We can observe the lead and lag relationships between y_f and y_d, y_i shown in Figure 3(a) and between k_f and k_d, k_i. We also observe that the paths of y_d, y_i and k_d, k_i are indistinguishable.

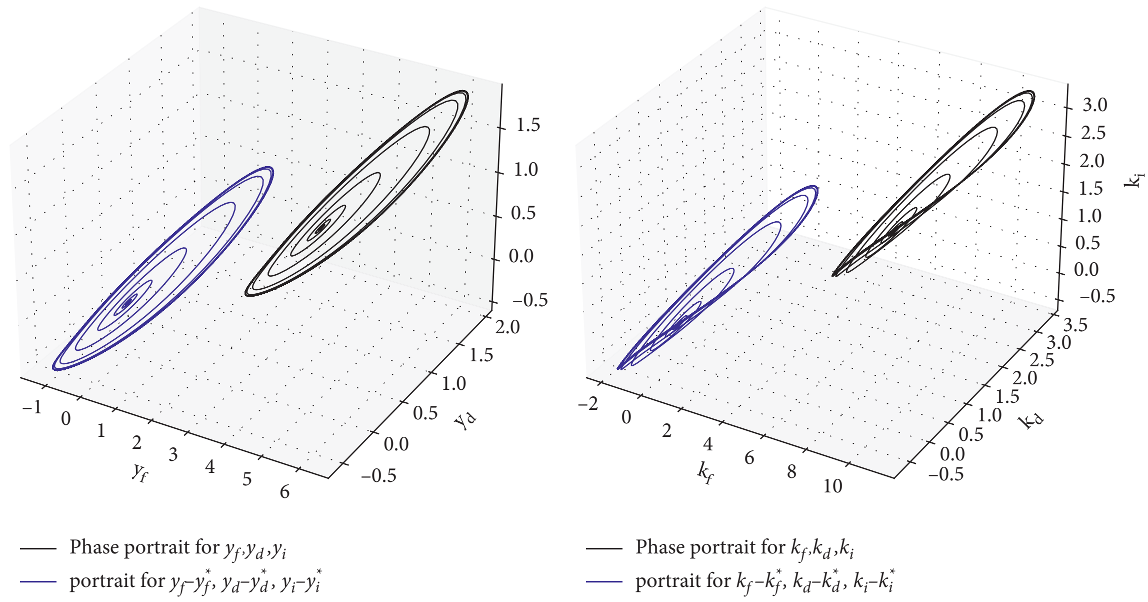
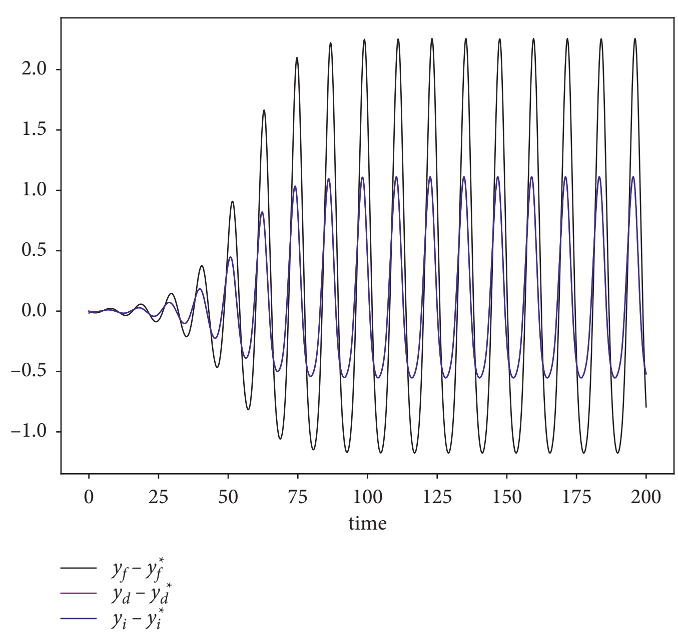
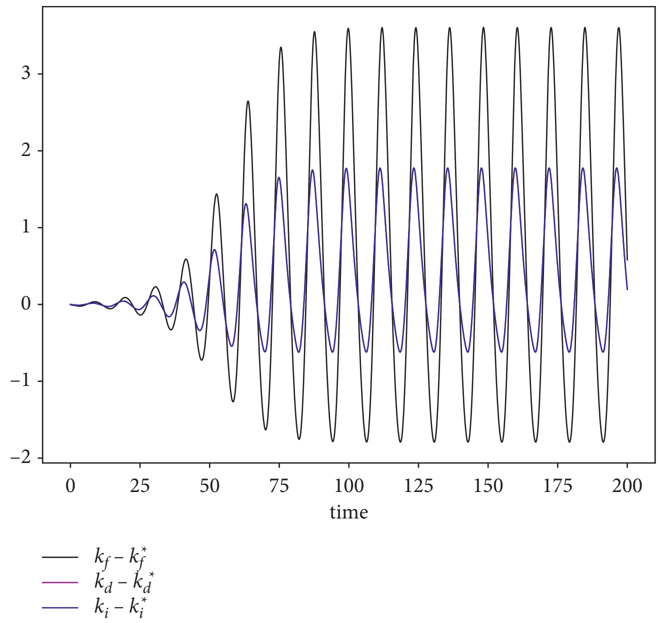


FIGURE 2: The 3D phase portrait. (a) Phase portrait projected on the $y_f - y_d - y_i$ space in black and the transformed one in blue. (b) Phase portrait projected on the $k_f - k_d - k_i$ space in black and the transformed one in blue.



(a)

FIGURE 3: Continued.



(b)

Figure 3: Lead and lag plots. (a) $(y_f - y_f^*, y_d - y_d^*, y_i - y_i^*)$. (b) $(k_f - k_f^*, k_d - k_d^*, k_i - k_i^*)$.

5. Conclusions and Outlook

In this study, we have described the mathematical details on a three-sector Keynesian model of business cycles, which is a modification of Murakami's two-sector Keynesian model of business cycles. We consider three types of goods that differ in property, namely, durable goods, fast goods, and investment goods. We formalized the consumption and investment functions from Keynesian perspectives and then set up a dynamical system composed of differential equations. Moreover, we have investigated the periodic nature of the solution paths and the performed numerical simulations appear that they are reliable with the accomplished hypothetical coming about. Furthermore, we would like to suggest researchers who are interested in this area to use this study as a reference and conduct further research on the multi-sectoral Keynesian model of business cycles by paying attention to the effect of changes in parameter values and providing the existence and stability of a limit cycle.

Data Availability

All the data are available in the article and cited wherever required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors have equally contributed to this manuscript.

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References

[1] V. Zarnowitz, *Recent Work on Business Cycles in Historical Perspective: Review of Theories and Evidence*, NBER, Cambridge, UK, 1984.

[2] H. Murakami, "Monetary policy in the unique growth cycle of post keynesian systems," *Structural Change and Economic Dynamics*, vol. 52, pp. 39–49, 2020.

[3] W.-B. Zhang, "Economic growth and the taylor rule in the solow-tobin model," *Asian Business Review*, vol. 9, no. 2, pp. 49–56, 2019.

[4] F. P. Ramsey, "A mathematical theory of saving," *The Economic Journal*, vol. 38, no. 152, pp. 543–559, 1928.

[5] R. J. Barro and X. Sala-i-Martin, "Economic growth," *Citeseer*, 1995.

[6] H. Uzawa, "On a two-sector model of economic growth ii," *The Review of Economic Studies*, vol. 30, no. 2, pp. 105–118, 1963.

[7] R. M. Solow, "Note on uzawa's two-sector model of economic growth," in *Readings in the Theory of Growth,* Springer, Berlin, Germany, 1971.

[8] E. Burmeister and R. Dobell, "Mathematical theories of economic growth," 1970.

[9] P. Aghion, P. Howitt, P. W. Howitt, M. Brant-Collett, and C. García-Peñalosa, *Endogenous Growth Theory*, MIT Press, Cambridge, UK, 1998.

[10] E. M. Drandakis, "Factor substitution in the two-sector growth model," *The Review of Economic Studies*, vol. 30, no. 3, pp. 217–228, 1963.

[11] P. A. Diamond, "Disembodied technical change in a two-sector model," *The Review of Economic Studies*, vol. 32, no. 2, pp. 161–168, 1965.

[12] C.-C. Von Weizsäcker, "Tentative notes on a two sector model with induced technical progress," *The Review of Economic Studies*, vol. 33, no. 3, pp. 245–251, 1966.

[13] W. Corden, "The two sector growth model with fixed coefficients," *The Review of Economic Studies*, vol. 33, no. 3, pp. 253–262, 1966.

[14] J. E. Stiglitz, "A two-sector two class model of economic growth," *The Review of Economic Studies*, vol. 34, no. 2, pp. 227–238, 1967.

[15] H. N. Gram, "Two-sector models in the theory of capital and growth," *The American Economic Review*, vol. 66, no. 5, pp. 891–903, 1976.

[16] K. Mino, "Analysis of a two-sector model of endogenous growth with capital income taxation," *International Economic Review*, pp. 227–251, 1996.

[17] J.-P. Drugeon and A. Venditti, "Intersectoral external effects, multiplicities & indeterminacies," *Journal of Economic Dynamics and Control*, vol. 25, no. 5, pp. 765–787, 2001.

[18] M. Ferrara and L. Guerrini, "On the dynamics of a three-sector growth model," *International Review of Economics*, vol. 55, no. 3, pp. 275–283, 2008.

[19] H. Murakami, "A two-sector keynesian model of business cycles," *Metroeconomica*, vol. 69, no. 2, pp. 444–472, 2018.

[20] H. Murakami and R. Zimka, "On dynamics in a two-sector keynesian model of business cycles," *Chaos, Solitons & Fractals*, vol. 130, Article ID 109419, 2020.

[21] R. A. Arnold, "Economics," *Cengage Learning*, 2015.

[22] A. Tihtina, "Three-sector keynesian model of business cycles," *MSc Thesis*, 2021.