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Research Article

Extremal Trees for the Exponential of Forgotten Topological Index

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Let F be the forgotten topological index of a graph G. The exponential of the forgotten topological index is defined as $e^F(G) = \sum_{(x,y)\in S} t_{x,y}(G)e^{(x^2+y^2)}$, where $t_{x,y}(G)$ is the number of edges joining vertices of degree x and y. Let $\mathbf{T}_n$ be the set of trees with n vertices; then, in this paper, we will show that the path P_n has the minimum value for e^F over $\mathbf{T}_n$.

1. Introduction

In this paper, let $V = V(G)$ and $E(G)$ be the vertex set and edge set, respectively. Let $d_v = d_G(v)$ be the degree of a vertex v in graph G. A vertex of degree one is a pendant vertex or a leaf. A branching vertex v of a tree T is a vertex of degree $d_v \geq 3$.

Let $P = w_0 w_1, \ldots, w_{q-1} w_q$ be the path graph with length of $r(P) = q$; if $d_T(w_0) \geq 3$, $d_T(w_1) = \ldots = d_T(w_{q-1}) = 2$ and $d_T(w_q) = 1$, then we called P is a pendant path.

Recently, topological indices have been considered by many researchers due to their many applications in various sciences. The forgotten topological index is defined in [1] as follows:

$$F(G) = \sum_{uv\in E(G)} d_u^2 + d_v^2. \tag{1}$$

For applications of the forgotten topological index, see [2–4].

Before starting a new definition, we consider the set $S = \{(x, y) \in N \times N : 1 \leq x \leq y \leq n - 1\}$, and let $t_{x,y}(G)$ be the number of edges joining vertices of degree x and y in a graph G. Therefore, the new definition will be as follows:

$$F = F(G) = \sum_{(x,y)\in S} t_{x,y}(G)\left(x^2 + y^2\right). \tag{2}$$

The exponential of the forgotten topological index F, denoted by e^F, is defined as

$$e^F = e^F(G) = \sum_{(x,y)\in S} t_{x,y}(G)e^{\left(x^2+y^2\right)}. \tag{3}$$

Recently, the exponential topological indices have attracted the attention of many researchers. In [5], the exponential Randić index is characterized. In [6], the authors have characterized the exponential atom bond connectivity and the exponential augmented Zagreb index. In [7], the problem maximal value of trees for the exponential second Zagreb index is solved. Then, in this paper, we solve the problem with finding the minimal value of e^F among trees.

2. Trees with Minimum Exponential of the Forgotten Topological Index

In this section, we will show that the path P_n has the minimal value of the exponential forgotten topological index among all trees.

Lemma 1. *Let T and T_1 be the trees in Figure 1 and A be a subtree of T. If $s \geq 3$, then $e^F(T) > e^F(T_1)$.*

Proof. By setting $k = d_T(u)$, hence, we can write

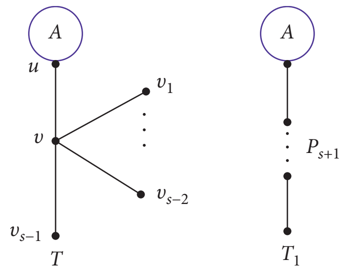

Figure 1: The trees T and T_1.

$$\begin{aligned}e^F(T)-e^F(T_1)&=\left(e^{s^2+k^2}+(s-1)e^{1+s^2}\right)\\&\quad-\left(e^{4+k^2}+(s-2)e^8+e^5\right)\\&=\left(e^{s^2+k^2}-e^{4+k^2}\right)+(s-2)\\&\quad\left(e^{1+s^2}-e^8\right)+\left(e^{1+s^2}-e^5\right)\\&\geq\left(e^{9+k^2}-e^{4+k^2}\right)+\left(e^{10}-e^8\right)+\left(e^{10}-e^5\right)\\&>\left(e^{9+k^2}-e^{4+k^2}\right)>0.\end{aligned}\tag{4}$$

□

Lemma 2. *Let T be a tree with minimum value of e^F in $\mathbf{T}_n$ and let u be a pendant vertex T, $v\in T$. If $uv\in E(T)$, then $d_T(v)=2$.*

Proof. Suppose $d_T(v)=g$ and P be the largest path of T and contains v. Let t be an end vertex of P and o a vertex in P, where $ot\in E(T)$; hence, by applying Lemma 1, we have $d_T(o)=2$.

We continue the proof with the following two cases. □

Case 1. If $d_T(v)=g\geq 4$.

Assuming that T_1 be the tree in Figure 2 and $A_v=\sum_{i=1}^{g-1}e^{g^2+y_i^2}$, where $y_1,\ldots,y_{g-1}$ are the degrees of the adjacent vertices to v different from u. Hence, we can write

$$\begin{aligned}e^F(T)-e^F(T_1)&=A_v+e^{1+g^2}+e^5-\sum_{i=1}^{g-1}e^{(g-1)^2+y_i^2}-e^8-e^5\\&=\left(A_v-\sum_{i=1}^{g-1}e^{(g-1)^2+y_i^2}\right)+\left(e^{1+g^2}-e^8\right)\\&\geq\left(A_v-\sum_{i=1}^{g-1}e^{(g-1)^2+y_i^2}\right)+\left(e^{17}-e^8\right)\\&>\left(A_v-\sum_{i=1}^{g-1}e^{(g-1)^2+y_i^2}\right)>0.\end{aligned}\tag{5}$$

This contradicts the minimality of T.

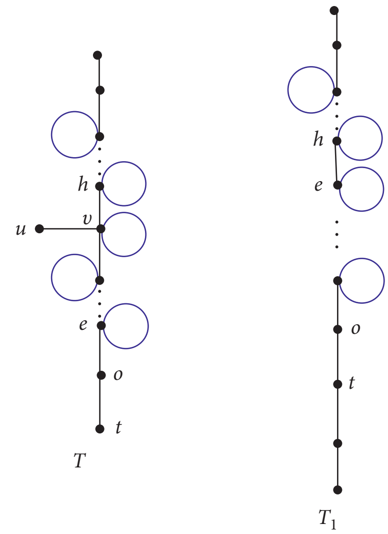

Figure 2: The trees T and T_1.

Case 2. If $d_T(v)=3$.

Suppose $h,e\in V(T)$ and $hv,ev\in E(T)$, where $h,e\neq u$, $d_T(h)=a$, and $d_T(e)=b$. By applying Lemma 1, we get $a\geq 2$ and $b\geq 2$. Let T_2 be the tree described in Figure 3. Therefore, we can write

$$\begin{aligned}e^F(T)-e^F(T_2)&=e^{9+a^2}+e^{9+b^2}+e^{10}-e^{a^2+b^2}-2e^8\\&=e^{9+a^2}+e^{9+b^2}-e^{a^2+b^2}+e^{10}-2e^8.\end{aligned}\tag{6}$$

Here, we show

$$f(a,b)=e^{9+a^2}+e^{9+b^2}+e^{10}>e^{a^2+b^2}+2e^8.\tag{7}$$

Since

$$f(a,2)=e^{9+a^2}+e^{13}+e^{10}>e^{a^2+4}+2e^8\tag{8}$$

and

$$f(a,3)=e^{9+a^2}+e^{18}+e^{10}>e^{a^2+9}+2e^8,\tag{9}$$

the above inequality holds for $a\geq 2$. Therefore, for $a,b\geq 2$, we have $f(a,b)>0$. Hence, we get $e^F(T)>e^F(T_2)$. That is a contradiction; hence, we get $d_T(v)=2$.

Let v be a branching vertex of degree y of a tree T; hence, T can be viewed as the coalescence of y subtrees of T at the vertex v. We call $T_1,\ldots,T_y$ are the y branches of T at v (see Figure 4).

Definition 1 (see [5]). A branching vertex x of tree T is an outer branching vertex of T if all branches of T at x except for possibly one are paths.

Lemma 3 (see [5]). *A tree $T\in\mathbf{T}_n$ has no outer branching vertex if and only if $T\cong P_n$.*

Lemma 4. *Let $T,T_1\in\mathbf{T}_n$ be the trees in Figure 5 and R be a subtree of T, $z\geq 3$ and $x=d_T(u)\geq 2$. Then, $e^F(T)>e^F(T_1)$.*

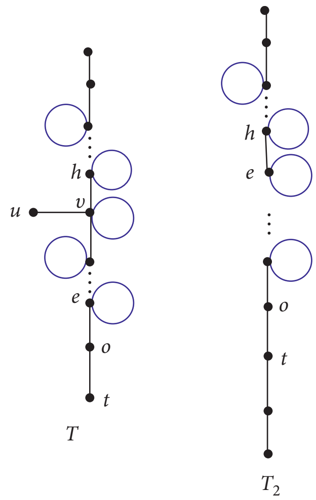

FIGURE 3: The trees T and T_2.

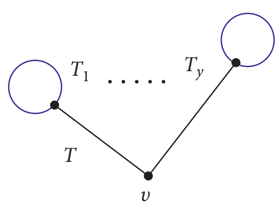

FIGURE 4: Branches of the tree T at v.

Proof. By direct calculation, it is not difficult to see $3 \le z \le 8$. Here, we let $z \ge 9$; therefore, we have

$$\begin{aligned} e^F(T) - e^F(T_1) &= e^{(z+1)^2+4} + (z-1)\left(e^{(z+1)^2+4} - e^{z^2+4}\right) \\ &\quad + \left(e^{(z+1)^2+x^2} - e^{z^2+x^2}\right) + \left(e^5 - 2e^8\right) \\ &> e^{(z+1)^2+4} + \left(e^5 - 2e^8\right) \ge e^{104} + \left(e^5 - 2e^8\right) > 0. \end{aligned} \tag{10}$$

□

Lemma 5. *Let $T, T_1 \in \mathbf{T}_n$ be the trees in Figure 6 and $t \ge 3$. Then, $e^F(T) > e^F(T_1)$, where $s(P_u) = -2 + \sum_{i=1}^{t} s(P_i)$.*

Proof. We describe the graph in Figure 7; let P_t be the path of length $s(P_t) = -2t + 2 + \sum_{i=1}^{t} s(P_i)$. It is not difficult to see $e^F(T) = e^F(T_3)$. Then, by repeated of Lemma 4, we get $e^F(T) > e^F(T_1)$. □

Corollary 1. *Let $T \in \mathbf{T}_n$ be a tree with a unique outer branching vertex and every pendant path has length at least 2. Then, $e^F(T) > e^F(P_n)$.*

Proof. If T has a unique outer branching vertex, hence, T has the form trees in Figure 6. Let M be the tree in Figure 8. If $s = 2$, then $e^F(T) = e^F(M)$. If $s \ge 3$, then by using Lemma 5, we have $e^F(T) > e^F(M)$. Hence, we can write

$$\begin{aligned} e^F(T) - e^F(P_n) &= 3e^5 + 3e^{13} + (n-7)e^8 - 2e^5 - (n-3)e^8 \\ &= e^5 + 3e^{13} - 4e^8 > 0. \end{aligned} \tag{11}$$

□

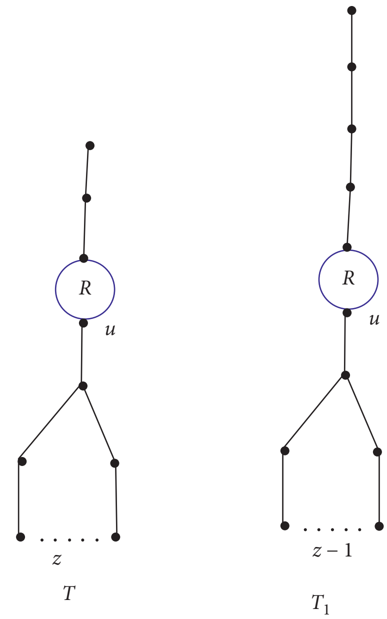

FIGURE 5: The trees T and T_1.

Lemma 6. *Let $T, T_1 \in \mathbf{T}_n$ be the trees in Figure 9, such that $1 \le d_T(v) \le 3$; then, $e^F(T) > e^F(T_1)$.*

Proof. By setting $x = d_T(v)$, we can write

$$\begin{aligned} e^F(T) - e^F(T_1) &= 2e^5 + 2e^{13} + e^{9+x^2} - e^5 - 3e^8 - e^{4+x^2} \\ &= e^5 + 2e^{13} - 3e^8 + \left(e^{9+x^2} - e^{4+x^2}\right) \\ &> e^5 + 2e^{13} - 3e^8 > 0. \end{aligned} \tag{12}$$

□

Lemma 7. *Let $T \in \mathbf{T}_n$ be the tree in Figure 10, $s \ge 1$ and $z \ge 0$; then, T is not minimal in $\mathbf{T}_n$.*

Proof. Set $x = d_T(u) \ge 2$, and let T_1 be tree in Figure 11. Hence, we have $e^F(T) > e^F(T_1)$ if the following conditions are hold:

(1) $s \ge 1, z \ge 2$.

(2) $s \ge 4, z \ge 0$.

It is not difficult to see that our result holds for $s + z \le 11$. Therefore, we let $z + s \ge 12$. Then,

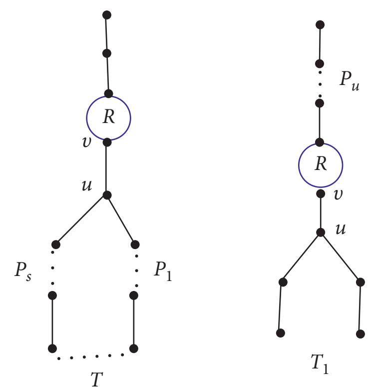

Figure 6: The trees T and T_1.

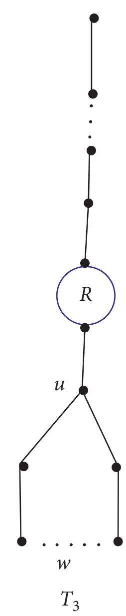

Figure 7: The tree T_3.

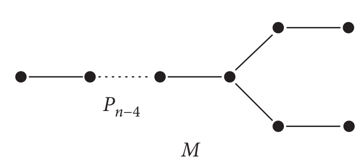

Figure 8: The tree M.

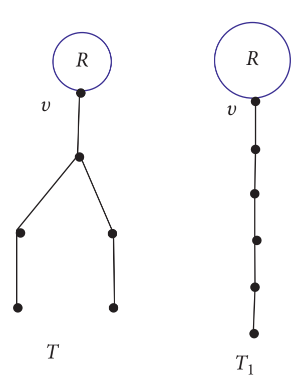

Figure 9: The trees T and T_1.

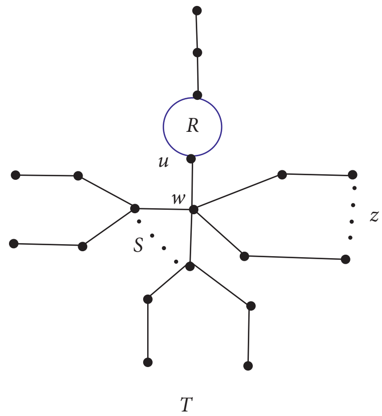

Figure 10: The tree T.

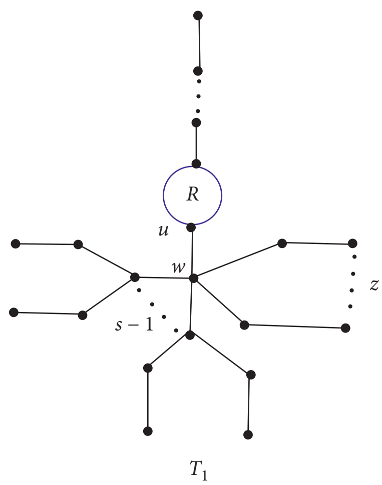

Figure 11: The tree T_1.

$$\begin{aligned} e^F(T) - e^F(T_1) = &\left(2e^5 + 2e^{13} - 5e^8\right) + (s-1) \\ &\left(e^{9+(s+z+1)^2} - e^{9+(s+z)^2}\right) \\ &+ z\left(e^{4+(s+z+1)^2} - e^{4+(s+z)^2}\right) \\ &+\left(e^{x^2+(s+z+1)^2} - e^{x^2+(s+z)^2}\right) + e^{9+(s+z+1)^2} \\ &> \left(2e^5 + 2e^{13} - 5e^8\right) > 0. \end{aligned} \tag{13}$$

To continue the proof, we must consider the following conditions:

(3) $s = 1$ and $z = 0$.
(4) $s = 1$ and $z = 1$.
(5) $s = 2$ and $z = 0$.
(6) $s = 3$ and $z = 1$.
(7) $s = 2$ and $z = 1$.
(8) $s = 3$ and $z = 0$.

Note that, in (3), (4), and (5),, we have $2 \leq d_T(w) \leq 3$; therefore, by Lemma 6, we can obtain trees with the minimum value of e^F.

Here, if (6) holds, then we consider graph T_2 in Figure 12. Hence, we can write

$$\begin{aligned} e^F(T) - e^F(T_2) = &\left(e^5 - 2e^8\right) + 3\left(e^{34} - e^{25}\right) + \left(e^{29} - e^{20}\right) \\ &+ \left(e^{x^2+25} - e^{x^2+16}\right) + e^{20} \\ &> \left(e^5 - 2e^8\right) + 3\left(e^{34} - e^{25}\right) + \left(e^{29} - e^{20}\right) \\ &+ e^{20} > \left(e^5 - 2e^8\right) + e^{20} > 0. \end{aligned} \tag{14}$$

If (7) holds, then we consider graph B in Figure 13. So, we have

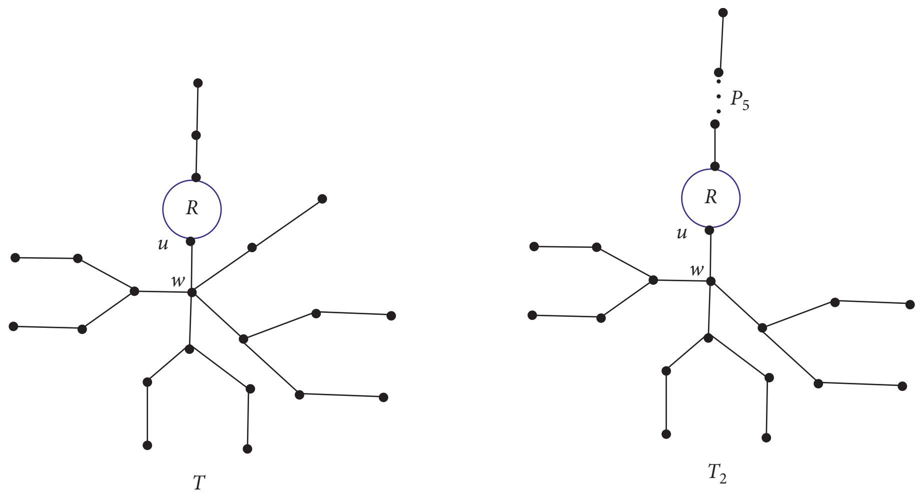

Figure 12: The trees T and T_2.

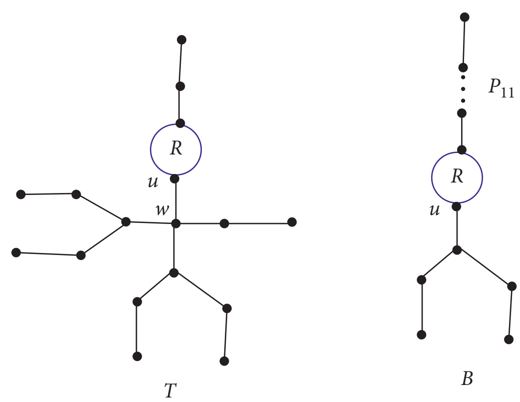

Figure 13: The trees T and B.

$$\begin{aligned} e^F(T) - e^F(B) &= 6e^5 + 4e^{13} + 2e^{25} + e^{20} + e^{x^2+16} - 3e^5 \\ &\quad - 2e^{13} - 8e^8 - e^{x^2+9} \\ &= \left(e^{x^2+16} - e^{x^2+9}\right) + 3e^5 + 2e^{13} + 2e^{25} \\ &\quad + e^{20} - 8e^8 > e^{20} - 8e^8 > 0. \end{aligned} \tag{15}$$

Finally, if (8) holds, then we consider graph C in Figure 14. Hence, we can write

$$\begin{aligned} e^F(T) - e^F(C) &= 7e^5 + 6e^{13} + 3e^{25} + e^{x^2+16} - 3e^5 - 2e^{13} \\ &\quad - 11e^8 - e^{x^2+9} \\ &= \left(e^{x^2+16} - e^{x^2+9}\right) + 4e^5 + 4e^{13} + 3e^{25} \\ &\quad - 11e^8 > 4e^5 + 4e^{13} + 3e^{25} - 11e^8 > 0. \end{aligned} \tag{16}$$

Therefore, T is not minimal in $\mathbf{T}_n$. □

Theorem 1. *Let $T \in \mathbf{T}_n$ and $T \not\equiv P_n$; then, T is not minimal for e^F.*

Proof. By using Lemma 3, we know that T has an outer branching vertex u. Using Lemmas 2, 5, and 6, we let all pendant paths of T have length at least 2 and T has the form in Figure 15, such that $d_T(v) \geq 4$, and otherwise, T is not minimal. If u is the unique outer branching vertex of T, then the result obtained by Corollary 1. Otherwise, among all outer branching vertices of T, choose u as the farthest from u. From Lemma 5, we let T is the form in Figure 16, such that $d_T(v_1) \geq 4$. Note that u_1 is the farthest outer branching vertex from u; it is clear if T_i is not a path; then, w_i is an outer branching vertex of T, and by Lemma 5, we let T_i have the form in Figure 17. Therefore, we get $e^F(T) = e^F(E)$, where E is described in Figure 18 and $s + z = q + 1$. The result follows from Lemma 7. □

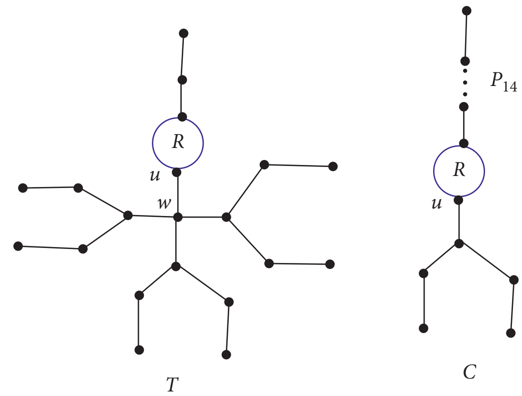

Figure 14: The trees T and C.

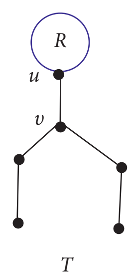

Figure 15: The tree T.

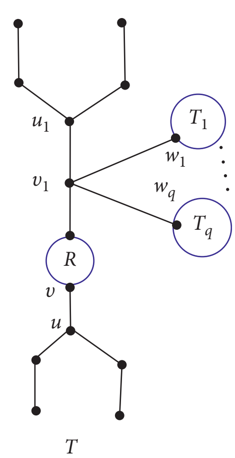

Figure 16: The tree T.

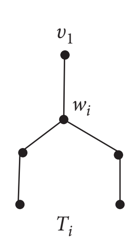

Figure 17: The tree T_i.

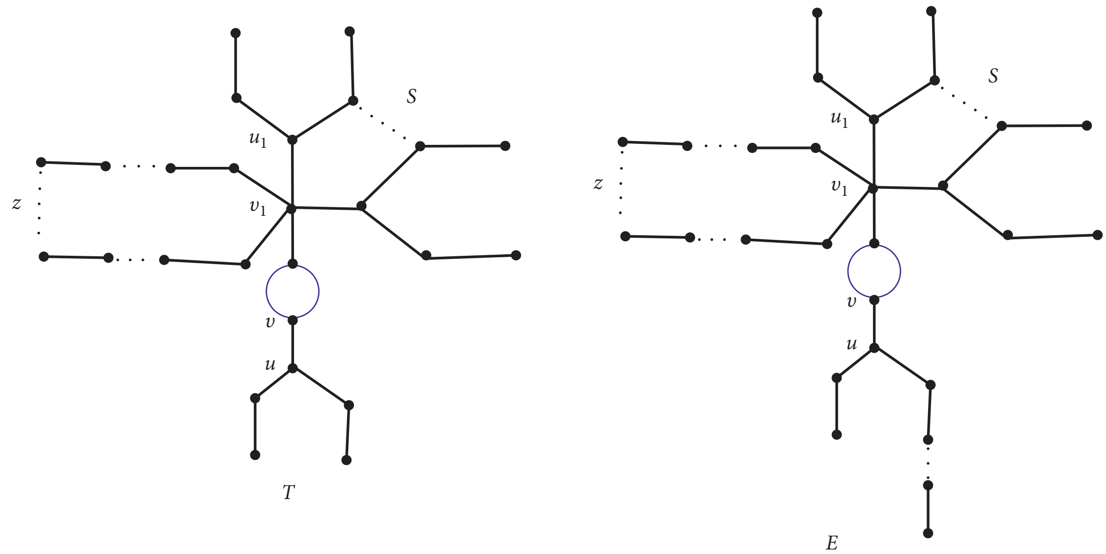

FIGURE 18: The tree T and E.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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