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Research Article

Persistence of Heteroclinic Cycles Connecting Repellers in Banach Spaces

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This paper is concerned with persistence of heteroclinic cycles connecting repellers in Banach spaces. It is proved that if a map with a regular and nondegenerate heteroclinic cycle connecting repellers undergoes a small perturbation, then the perturbed map can still have a regular and nondegenerate heteroclinic cycle connecting repellers. The perturbation rang is given by an explicit positive constant according to the properties of the original map. Hence, the perturbed map and the original map are simultaneously chaotic in the sense of both Devaney and Li-Yorke. Especially, the persistence of heteroclinic cycles connecting repellers is also discussed in the Euclidean space, where the repellers can expand in different norms. Finally, three examples are provided to illustrate the validity of the theoretical results.

1. Introduction

Chaos is a very important kind of dynamical behaviors in nonlinear systems and chaos problems have attracted a lot of attention from many scientists and sgd mathematicians. In 1975, the first mathematical definition of chaos and a famous result that "period three implies chaos" were given by Li and Yorke [1] in studying continuous interval maps. After that, different definitions of chaos from different points of view were proposed by researchers, one can see [2–4] for some related definitions of chaos. Among these mathematical definitions, chaos in the sense of Li-Yorke, Devaney or Wiggins are often used in the literature, see [5–8] for discussions of their relationships. Then, there appeared many works to study chaotic behaviors of multidimensional maps. A very famous work that "a snap-back repeller implies chaos" in the sense of Li-Yorke was proposed by Marotto [9], which is a generalization of Li and Yorke's result from one-dimensional maps to multidimensional maps. This result shows great power in studying chaos of finite dimensional maps. However, it is clear that there are many systems should be studied in infinite dimensional spaces, such as Banach spaces and metric spaces. Then, a lot of works have been done on chaotic behaviors of infinite dimensional maps. Some of these important results were given by Shi and her cooperators. In 2004, Shi and Chen [10] extended the concept of snap-back repeller to metric spaces and obtained several criteria of chaos. Later, Shi and her cooperators developed the coupled-expansion theory and used it to study chaos, see [11–14] and references therein.

Structural stability of chaotic maps is a very important and interesting question, and many results have been achieved. Marotto first studied perturbations of maps with snap-back repellers in [15, 16], and proved that if a scalar system with a snap-back repeller undergoes a small perturbation, then the perturbation system will have a transversal homoclinic point and thus has chaotic behaviors. Later, there appeared several results about multidimensional perturbations of chaotic systems, see [17–19]. In 2009, Li and Lyu [20] proved that if a map with a snap-back repeller in $\mathbf{R}^n$ undergoes a mall C^1 perturbation, then the perturbed map still has a snap-back repeller and consequently is chaotic in the sense of Li-Yorke. However, all the above perturbations of chaotic systems were made in finite dimensional spaces. In 2011, Chen et al. [21] studied the persistence of snap-back repellers under small C^1 perturbations in Banach spaces. In 2012, Zhang et al. [22] used a different method to study the persistence of snap-back repellers under small Lipschitz perturbations in Banach spaces. Moreover, Zhang and Shi [23] studied the persistence of coupled-expansion for time-varying systems under small time-varying perturbations in Banach spaces, and showed the persistence of snap-back repellers.

In 2006, Lin and Chen [24] gave a result that heteroclinical repellers imply chaos in the sense of Li-Yorke in $\mathbf{R}^n$. In their definition of heteroclinical repellers, there were some conditions given by the Jacobian matrices of a map. However, a map in a metric space may not have derivatives in general. In 2008, based on their work, Li et al. [25] grasped the essential meanings of the definition of heteroclinical repellers to extend it to general metric spaces without needing the continuity or continuous differentiability. For more intuitive to reflect the relationships of the repellers, they redefined it as a heteroclinic cycle connecting repellers and obtained several criteria of chaos. Later, they studied chaos induced by heteroclinic cycles connecting repellers in general Banach spaces [26], and used these results to study existence of chaos or chaotification problems [27]. This shows that the heteroclinic cycle connecting repellers has significant effects on chaos studying. Hence, it is worth studying whether a heteroclinic cycle connecting repellers has the persistence under small perturbations as that for a snap-back repeller. Recently, in 2020, Chen and Wu [28] studied the persistence of heteroclinic repellers in $\mathbf{R}^n$ for C^1 maps under small C^1 perturbations. Chen et al. [29] studied the persistence of heteroclinic repellers in Banach spaces for C^1 maps under small C^1 perturbations. It is noted that the definitions of heteroclinic repellers in [28, 29] both needed the differentiability of a map as that definition in [24]. In 2021, Wu [30] extended the concept of heteroclinic repellers in [24] to heteroclinic cycle connecting expanding periodic points in $\mathbf{R}^n$ and studied the persistence of it for C^1 maps under C^1 perturbations, where the maps needed to be continuously differentiable in the whole space. More recently, Chen and Luo [31] studied the persistence of regular nondegenerate snap-back repellers and heteroclinic cycles for continuous maps under small Lipschitz perturbations, where the maps were continuous in the whole Banach space. On the one hand, it should be pointed out that all the above results needed the maps to be continuous or continuously differentiable in the whole space. However, there are a lot of maps that may not be continuous or continuously differentiable in the whole space. On the other hand, it should be pointed out that all the above results needed the perturbations to be small enough and did not give a relatively explicit expression for the range of small perturbations, which is convenient and useful in applications to quickly check out whether the persistence is maintained. So, it is meaningful to study persistence of heteroclinic cycles connecting repellers for maps which are only continuous or continuously differentiable in some domains of the whole space, and it is also meaningful to study the explicit expression for the range of small perturbations.

The fixed point theory has become an essential tool to resolve some problems in nonlinear analysis, including fractional calculus, see [32, 33] and references therein for more details about this theory. Here, we will apply the Banach contractive mapping principle and the ideas used in [22, 23] to study the persistence of regular and nondegenerate heteroclinic cycles connecting repellers in Banach spaces, where the original maps are only continuous or continuously differentiable in some neighborhoods of points. An important result is that an explicit expression for the range of perturbations is given. It will be proved that if a map with a regular and nondegenerate heteroclinic cycle connecting repellers undergoes a small Lipschitz perturbation, then the perturbed map can still have a regular and nondegenerate heteroclinic cycle connecting repellers. So, the perturbed map and the original map are simultaneously chaotic in the sense of both Devaney and Li-Yorke. Particularly, the persistence of heteroclinic cycles connecting repellers is also discussed in $\mathbf{R}^n$. The significant difference between our result and those obtained in [28, 30] is that the repellers in our result expand in different norms, while the repellers in the latter expand in the single Euclidean norm. It is clear that different fixed points can expand in different norms in $\mathbf{R}^n$. So, our result is more general in practice.

The rest of the paper is organized as follows. Some concepts and lemmas are given in Section 2. Several theorems about perturbations of maps with heteroclinic cycles connecting repellers in general Banach spaces or the Euclidean space are given in Section 3. Three examples are provided to illustrate the validity the theoretical results in Section 4. Finally, conclusions are made in Section 5.

2. Preliminaries

Some definitions and lemmas are given in this section.

Two usually used definitions of chaos in the sense of Li-Yoke or Devaney are first introduced. Then, the concept of a heteroclinic cycle connecting repellers is introduced.

Definition 1 (see [1]). Let (X, d) be a metric space, $f\colon X \longrightarrow X$ be a map, and S be a set of X with at least two distinct points. Then, S is called a scrambled set of f if for any two distinct points $x, y \in S$,

$$\liminf_{n\longrightarrow\infty} d\left(f^n(x), f^n(y)\right) = 0, \quad \limsup_{n\longrightarrow\infty} d\left(f^n(x), f^n(y)\right) > 0. \tag{1}$$

The map f is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of f.

Remark 1. There are three conditions in the original characterization of chaos in Li-Yorke's theorem [1]. Since the third one is not essential, it is removed in Definition 1 in most literature.

Example 1. Consider the following Baker's equation

$$x_{n+1} = \begin{cases} 2x_n, & \text{for } 0 \le x_n \le \dfrac{1}{2}, \\[2ex] 2\left(1 - x_n\right), & \text{for } \dfrac{1}{2} < x_n \le 1, \end{cases} \tag{2}$$

which models the mixing of a dye spot on a strip of dough that is repeatedly stretched and folded over on itself. The iterative scheme (2) maps the interval $[0,1]$ into itself. It is easy to check that system (2) has a cycle of period three and hence is chaotic in the sense of Li-Yorke by the Li-Yorke theorem in [1]. This equation has been extensively discussed in the literature [7, 34] and references cited therein.

Definition 2 (see [4]). Let (X, d) be a metric space. A map $f\colon V \subset X \longrightarrow V$ is said to be chaotic on V in the sense of Devaney if

(i) The set of the periodic points of f is dense in V
(ii) f is topologically transitive in V
(iii) f has sensitive dependence on initial conditions in V

Remark 2. In 1992, Banks et al. [5] proved that conditions (i) and (ii) together imply condition (iii) if f is continuous in V. So, condition (iii) is redundant in the above definition in this case. It has been proved by [6] that chaos in the sense of Devaney is stronger than chaos in the sense of Li-Yorke under some conditions.

Example 2. Let

$$\sum\nolimits_2^+ := \left\{ s = (s_0, s_1, s_2, \ldots) \colon s_j = 0 \text{ or } 1 \right\}, \tag{3}$$

with the distance

$$\rho(s,t) := \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}, \tag{4}$$

where $s = (s_0, s_1, s_2, \ldots)$ and $t = (t_0, t_1, t_2, \ldots)$. Then $(\sum_2^+ \rho)$ is a complete metric space and a Cantor set, see Lemma 2.5 in [10]. The shift map $\sigma\colon \Sigma_2^+ \longrightarrow \Sigma_2^+$ defined by $\sigma((s_0, s_1, s_2, \ldots)) = (s_1, s_2, \ldots)$ is continuous. The dynamical system defined by σ is called a one-sided symbolic dynamical system. It follows from [[4], Part 1, Proposition 6.6] that σ has the following properties:

(i) $\mathrm{Card}\,\mathrm{Per}_n(\sigma) = 2^n$
(ii) $\mathrm{Per}(\sigma)$ is dense in Σ_2^+
(iii) there exists a dense orbit of σ in Σ_2^+

Here, $\mathrm{Card}\,\mathrm{Per}_n(\sigma)$ denotes the number of periodic points of period n for σ. It is clear that property (iii) implies that σ is transitive. Therefore, the symbolic dynamical system is chaotic in the sense of Devaney. See [3, 4] for more discussions about this symbolic dynamical system.

Definition 3 (see [26], Definition 2.5). Let (X, d) be a metric space and $f\colon X \longrightarrow X$ be a map with $k(\geq 2)$ fixed points $z_1, \ldots, z_k \in X$.

(I) Suppose that, for each $i\,(1 \leq i \leq k)$, z_i is an expanding fixed point of f in $\overline{B}_{r_i}(z_i)$, and there exist a point $x_{i0} \in B_{r_i}(z_i)$, $x_{i0} \neq z_i$, and a positive integer $m_i \geq 1$ such that $f^{m_i}(x_{i0}) = z_{t(i)}$, and z_i is the limit for the backward orbit of x_{i0}, where $\overline{B}_{r_i}(z_i)$ and $\overline{B}_{r_i}(z_i)$ are the closed and open balls of radius r_i centered at z_i, $t(i) = [i \bmod k] + 1$. Then all the points $x_{i0}\,(1 \leq i \leq k)$, together with their backward and forward orbits consist of a set, which is called a k-heteroclinic cycle connecting repellers $z_1, \ldots, z_k$.

(II) Suppose that f has a k-heteroclinic cycle connecting repellers $z_1, \ldots, z_k$. For each point x_0 on the cycle, if there exists a positive constant r_0 such that for each positive constant $r \leq r_0$, $f(x_0)$ is an interior point of $f(B_r(x_0))$, then the cycle is called regular; if there exist positive constants r_1 and μ such that $d(f(x), f(y)) \geq \mu\, d(x,y), \forall x, y \in \overline{B}_{r_1}(x_0)$, then the cycle is called nondegenerate.

Remark 3. It is pointed out that the necessary and sufficient condition for a heteroclinic cycle connecting repellers is used to give the definition (I) for simplicity, see (1) of Remark 2.2 in [26]. In addition, it does not need the continuity or continuous differentiability in this definition, while some similar definitions need them, see [24–26, 28–31] for more details about this concept.

For convenience, some notations are given in the following. The continuously differentiable maps in a set U of a Banach space X are denoted by $C^1(U, X)$. The derivative of a map f at a point $x \in X$ is denoted by $Df(x)$. In addition, for a linear map $L\colon X \longrightarrow X$, denote

$$\begin{aligned} \|L\| &:= \sup\{\|Lx\| \colon x \in X, \|x\| = 1\}, \\ \|L\|^0 &:= \inf\{\|Lx\| \colon x \in X, \|x\| = 1\}. \end{aligned} \tag{5}$$

If a bounded linear map L has a bounded inverse, then L is said to be an invertible linear map, see Definition 4.17 in [35]. The following four lemmas will be used in the paper.

Lemma 1 (see [22], Lemma 2.4). *Let $(X, \|\cdot\|)$ be a Banach space, $z \in X$, and $f\colon \overline{B}_r(z) \longrightarrow f(\overline{B}_r(z))$ be a continuous map. Assume that $f(B_r(z))$ is an open set of X and*

$$\|f(x) - f(y)\| \geq \mu\|x - y\|, \quad \forall x, y \in \overline{B}_r(z), \tag{6}$$

for some constant $\mu > 0$, then

$$B_{(\mu - L)r}(F(z)) \subset F(B_r(z)), \tag{7}$$

where $F = f + g$ and g is a Lipschitz map in $\overline{B}_r(z)$ with Lipschitz constant $L < \mu$.

Lemma 2 (see [25], Theorem 3.4). *Let (X, d) be a complete metric space and $f\colon X \longrightarrow X$ be a map. Assume that*

(i) f has a regular and nondegenerate k-heteroclinic cycle connecting repellers $z_1, \ldots, z_k \in X$, $k \geq 2$
(ii) f is continuous in some neighborhood of each point on the cycle

Then there exists an uncountable, perfect, bounded, and closed set V such that $f(V) = V$ and f is chaotic on V in the sense of Devaney as well as in the sense of Li-Yorke.

Lemma 3 (see [26], Lemma 2.2; [22], Lemma 2.3). *Let $(X, \|\cdot\|)$ be a Banach space and $f\colon X \longrightarrow X$ be a map. Assume that f has a heteroclinic cycle connecting repellers*

$z_1, \ldots, z_k \in X$, $k \geq 2$, and for each point x_0 on the cycle f is continuously differentiable in some neighborhood of x_0 and satisfies that $Df(x_0)$ is an invertible linear map, then the cycle is regular and nondegenerate.

Lemma 4 (see [11], Lemma 2.2). *Let $(X, \|\cdot\|)$ be a Banach space. Suppose that a map $f\colon X \longrightarrow X$ is continuously differentiable in $B_{r_0}(x_0)$ for some $x_0 \in X$ and some $r_0 > 0$, and satisfies that $\lambda_0 = \|Df(x_0)\|^0 > 0$, then for each $\varepsilon \in (0, \lambda_0)$, there exists a positive constant $r_1 < r_0$ such that*

$$\|f(x) - f(y)\| \geq (\lambda_0 - \varepsilon)\|x - y\|, \quad \forall x, y \in \overline{B}_{r_1}(x_0). \tag{8}$$

3. Persistence of Heteroclinic Cycles Connecting Repellers in Banach Spaces

In this section, we will study persistence of heteroclinic cycles connecting repellers in Banach spaces. Assume that $(X, \|\cdot\|)$ is a Banach space, f, g: $X \longrightarrow X$ are two maps, and f has a regular and nondegenerate heteroclinic cycle connecting repellers and is continuous in some neighborhoods of interest points. Here, we study the following system:

$$x_{n+1} = f(x_n) + g(x_n), \quad n \geq 0, \tag{9}$$

where g is viewed as a mall perturbation. It is proved that there still has a regular and nondegenerate heteroclinic cycle connecting repellers in (9) when g satisfies some conditions. Consequently, the perturbed system (9) is chaotic in the sense of both Devaney and Li-Yorke.

Theorem 1. *Suppose that $(X, \|\cdot\|)$ is a Banach space and $f\colon X \longrightarrow X$ is a map with $k(\geq 2)$ different fixed points $z_1, \ldots, z_k \in X$ and satisfies the following:*

(i) For each i $(1 \leq i \leq k)$, z_i is a regular expanding fixed point of f in $\overline{B}_{r_i}(z_i)$ with expanding coefficient λ_{i0} for some constant $r_i > 0$. Furthermore, there exist a point $x_{i0} \in B_{r_i}(z_i)$, $x_{i0} \neq z_i$, and a positive integer $m_i \geq 1$ such that $f^{m_i}(x_{i0}) = z_{t(i)}$, where $t(i) = [i \bmod k] + 1$. Consequently, f has a heteroclinic cycle Γ connecting repellers $z_1, \ldots, z_k$.

(ii) The heteroclinic cycle Γ connecting repellers is regular and nondegenerate, and f is continuous in $\overline{B}_{r_i}(z_i)$ and some neighborhood U_{ij} of x_{ij}, where $x_{ij} = f^j(x_{i0})$ for $1 \leq i \leq k$, $1 \leq j \leq m_i - 1$.

Then, there exists a constant $\varepsilon_0 > 0$ such that for any Lipschitz map g in each set of $\overline{B}_{r_i}(z_i)$ and U_{ij}, $1 \leq i \leq k$, $1 \leq j \leq m_i - 1$, with the Lipschitz constant L satisfying

$$\max\left\{L, \|g(z_i)\|, \left\|g\left(x_{ij}\right)\right\|, 1 \leq i \leq k, 0 \leq j \leq m_i - 1\right\} < \varepsilon_0. \tag{10}$$

The perturbed system (9) also has a regular and nondegenerate heteroclinic cycle Γ' connecting repellers, and consequently there exists an uncountable, perfect, bounded, and closed set V such that system (9) is chaotic on V in the sense of both Devaney and Li-Yorke.

Proof. Without loss of generality and for simplicity, we only show that Theorem 1 is true for $k = 2$. When $k > 2$, one can use a similar method to prove it. For convenience, let $F(x) := f(x) + g(x)$ in the rest of this paper and $i = 1$ or 2 in the rest of this proof.

Without loss of generality, we can suppose that $\overline{B}_{r_1}(z_1) \cap \overline{B}_{r_2}(z_2) = \varnothing$, and $f(x_{i0}) \notin B_{r_i}(z_i)$. Otherwise, one can see the third paragraph in the proof of Theorem 3.1 in [25].

Since z_i is a regular expanding fixed point of f in $\overline{B}_{r_i}(z_i)$ with an expanding coefficient λ_{i0}, we get that

$$\|f(x) - f(y)\| \geq \lambda_{i0}\|x - y\|, \quad \forall x, y \in \overline{B}_{r_i}(z_i), \tag{11}$$

$f\colon \overline{B}_{r_i}(z_i) \longrightarrow f(\overline{B}_{r_i}(z_i))$ is a homeomorphism and $f(B_{r_i}(z_i))$ is open, $f(D)$ is open for any open set $D \subset B_{r_i}(z_i)$. Take a constant

$$\delta_{i0} < \frac{r_i - \|z_i - x_{i0}\|}{2}, \tag{12}$$

such that $\overline{B}_{\delta_{i0}}(x_{i0}) \subset B_{r_i}(z_i)$. Then, it follows form (11) that $f\colon B_{\delta_{i0}}(x_{i0}) \longrightarrow f(B_{\delta_{i0}}(x_{i0}))$ is also a homeomorphism.

From assumption (ii), it follows that there exist positive constants μ_{ij} and δ_{ij} such that

$$\|f(x) - f(y)\| \geq \mu_{ij}\|x - y\|, \quad \forall x, y \in \overline{B}_{\delta_{ij}}\left(x_{ij}\right), \tag{13}$$

$f\colon B_{\delta_{ij}}(x_{ij}) \longrightarrow f(B_{\delta_{ij}}(x_{ij}))$ is homeomorphic, and $f(B_{\delta_{ij}}(x_{ij}))$ is open for $1 \leq j \leq m_i - 1$, where δ_{ij} satisfies the following conditions

$$\delta_{i1} < \lambda_{i0}\delta_{i0}, \quad \delta_{i,j+1} < \mu_{ij}\delta_{ij}, \quad \text{for } 1 \leq j \leq m_i - 2, \tag{14}$$

$\overline{B}_{\delta_{ij}}(x_{ij})$ are disjoint subsets of U_{ij} and $\overline{B}_{\delta_{ij}}(x_{ij}) \cap B_{r_i}(z_i) = \varnothing$ for fixed i and $1 \leq j \leq m_i - 1$.

In the following, we will show that the map F satisfies the conditions in Lemma 2. It will be finished by the following three steps. □

Step 1. It is to prove that F has two regular expanding fixed points z_1^* and z_2^* when g satisfies some conditions.

For proving the existence of z_1^*, we take two positive constants δ_{2,m_2} and ε_1 such that

$$\delta_{2,m_2} < \begin{cases} \min\left\{\lambda_{20}\delta_{20}, \dfrac{r_1 - \|z_1 - x_{10}\|}{2} - \delta_{10}\right\}, & \text{if } m_2 = 1; \\ \min\left\{\mu_{2,m_2-1}\delta_{2,m_2-1}, \dfrac{r_1 - \|z_1 - x_{10}\|}{2} - \delta_{10}\right\}, & \text{if } m_2 > 1. \end{cases}$$

$$\varepsilon_1 = \frac{(\lambda_{10} - 1)\delta_{2,m_2}}{1 + \delta_{2,m_2}}. \tag{15}$$

Consider the following equation

$$F(x) = x, \quad x \in \overline{B}_{\delta_{2,m_2}}(z_1), \tag{16}$$

which is equivalent to the following equation:

$$f(x) = x - g(x), \quad x \in \overline{B}_{\delta_{2,m_2}}(z_1). \tag{17}$$

It follows from the first relation of (15) that $\overline{B}_{\delta_{2,m_2}}(z_1) \subset B_{r_1}(z_1)$. By assumption (i) and (11), we get that $f\colon \overline{B}_{\delta_{2,m_2}}(z_1) \longrightarrow f(\overline{B}_{\delta_{2,m_2}}(z_1))$ is homeomorphic. Then we obtain that $f(B_{\delta_{2,m_2}}(z_1))$ is an open set and the inverse map $f^{-1}\colon f(\overline{B}_{\delta_{2,m_2}}(z_1)) \longrightarrow \overline{B}_{\delta_{2,m_2}}(z_1)$ satisfies the following:

$$\|f^{-1}(x) - f^{-1}(y)\| \le \lambda_{10}^{-1}\|x - y\|, \quad \forall x, y \in f\left(\overline{B}_{\delta_{2,m_2}}(z_1)\right). \tag{18}$$

Hence, equation (17) is translated into the following:

$$f^{-1}(x - g(x)) = x. \tag{19}$$

Here, it should prove that

$$x - g(x) \in f\left(\overline{B}_{\delta_{2,m_2}}(z_1)\right), \quad \forall x \in \overline{B}_{\delta_{2,m_2}}(z_1). \tag{20}$$

On the one hand, for any $x \in \overline{B}_{\delta_{2,m_2}}(z_1)$, we have

$$\begin{aligned}\|g(x)\| &= \left\|g(x) - g(z_1) + g(z_1)\right\| \le \left\|g(x) - g(z_1)\right\| \\ &\quad + \left\|g(z_1)\right\| \le L\delta_{2,m_2} + \left\|g(z_1)\right\|.\end{aligned} \tag{21}$$

Suppose that the map g satisfies

$$\max\left\{L, \left\|g(z_1)\right\|\right\} < \varepsilon_1, \tag{22}$$

then,

$$L\delta_{2,m_2} + \left\|g(z_1)\right\| < \varepsilon_1\delta_{2,m_2} + \varepsilon_1 = (\lambda_{10} - 1)\delta_{2,m_2}. \tag{23}$$

Therefore, it follows from (21) and (23) that, for any $x \in \overline{B}_{\delta_{2,m_2}}(z_1)$,

$$\left\|x - g(x) - z_1\right\| \le \|g(x)\| + \left\|x - z_1\right\| < (\lambda_{10} - 1)\delta_{2,m_2} + \delta_{2,m_2} = \lambda_{10}\delta_{2,m_2}. \tag{24}$$

On the other hand, for any $x \in \partial B_{\delta_{2,m_2}}(z_1)$,

$$\left\|f(x) - z_1\right\| = \left\|f(x) - f(z_1)\right\| \ge \lambda_{10}\left\|x - z_1\right\| = \lambda_{10}\delta_{2,m_2}. \tag{25}$$

Since $z_1 \in f(B_{\delta_{2,m_2}}(z_1))$ and $f(B_{\delta_{2,m_2}}(z_1))$ is an open set, it follows from (24) and (25) that (20) is true.

According to the above discussion, we can define a map

$$h_1(x) = f^{-1}(x - g(x)), \quad x \in \overline{B}_{\delta_{2,m_2}}(z_1). \tag{26}$$

For any $x \in \overline{B}_{\delta_{2,m_2}}(z_1)$, it follows from (18) and (24) that

$$\left\|h_1(x) - z_1\right\| = \left\|f^{-1}(x - g(x)) - f^{-1}(z_1)\right\| \le \lambda_{10}^{-1}\left\|x - g(x) - z_1\right\| < \delta_{2,m_2}, \tag{27}$$

which implies that h_1 maps $\overline{B}_{\delta_{2,m_2}}(z_1)$ into itself. Moreover, for any $x, y \in \overline{B}_{\delta_{2,m_2}}(z_1)$, we get from (18) that

$$\begin{aligned}\left\|h_1(x) - h_1(y)\right\| &= \left\|f^{-1}(x - g(x)) - f^{-1}(y - g(y))\right\| \\ &\le \lambda_{10}^{-1}\left[\|g(x) - g(y)\| + \|x - y\|\right] \\ &\le \lambda_{10}^{-1}(L + 1)\|x - y\|.\end{aligned} \tag{28}$$

It follows from the second relation of (15) and (22) that

$$\lambda_{10} > L + 1, \tag{29}$$

which together with (28) yields that h_1 is contractive in $\overline{B}_{\delta_{2,m_2}}(z_1)$. It follows from the Banach contractive mapping principle and (27) that there exists a unique point $z_1^* \in B_{\delta_{2,m_2}}(z_1)$ satisfying $h_1(z_1^*) = z_1^*$. Consequently, $F(z_1^*) = z_1^*$, that is, z_1^* is a fixed point of F in $B_{\delta_{2,m_2}}(z_1)$.

It should prove that z_1^* is a regular expanding fixed point of F in some neighborhood of z_1^*. To do this, take

$$r_1^* = \frac{r_1 + \left\|z_1 - x_{10}\right\|}{2}. \tag{30}$$

Then, it follows from $z_1^* \in B_{\delta_{2,m_2}}(z_1)$ and the first relation of (15) that

$$\begin{aligned}\left\|x_{10} - z_1^*\right\| &= \left\|x_{10} - z_1 + z_1 - z_1^*\right\| \\ &\le \left\|x_{10} - z_1\right\| + \left\|z_1 - z_1^*\right\| \\ &\le \left\|x_{10} - z_1\right\| + \delta_{2,m_2} \\ &< \left\|x_{10} - z_1\right\| + \frac{r_1 - \left\|z_1 - x_{10}\right\|}{2} \\ &\quad - \delta_{10} = r_1^* - \delta_{10},\end{aligned} \tag{31}$$

which implies that $\overline{B}_{\delta_{10}}(x_{10}) \subset B_{r_1^*}(z_1^*)$. For any $x \in B_{r_1^*}(z_1^*)$,

$$\begin{aligned}\left\|x - z_1\right\| &= \left\|x - z_1^* + z_1^* - z_1\right\| \le \left\|x - z_1^*\right\| + \left\|z_1^* - z_1\right\| \\ &< r_1^* + \delta_{2,m_2} < \frac{r_1 + \left\|z_1 - x_{10}\right\|}{2} + \frac{r_1 - \left\|z_1 - x_{10}\right\|}{2} - \delta_{10} \\ &= r_1 - \delta_{10},\end{aligned} \tag{32}$$

which implies that $\overline{B}_{r_1^*}(z_1^*) \subset B_{r_1}(z_1)$. Consequently, $f(B_{r_1^*}(z_1^*))$ is an open set. For any $x, y \in \overline{B}_{r_1^*}(z_1^*)$,

$$\begin{aligned}\|F(x)-F(y)\| &= \|f(x)+g(x)-f(y)-g(y)\| \\ &\geq \|f(x)-f(y)\| - \|g(x)-g(y)\| \geq (\lambda_{10}-L)\|x-y\|.\end{aligned} \tag{33}$$

Then, it follows from (29) and (33) that z_1^* is an expanding fixed point of F in $\overline{B}_{r_1^*}(z_1^*)$ with expanding coefficient $\lambda_{10}-L>1$. Since $f(B_{r_1^*}(z_1^*))$ is an open set, it follows from Lemma 1 that

$$B_{(\lambda_{10}-L)r_1^*}(z_1^*) = B_{(\lambda_{10}-L)r_1^*}(F(z_1^*)) \subset F\left(B_{r_1^*}(z_1^*)\right). \tag{34}$$

which implies that z_1^* is an interior point of $F(B_{r_1^*}(z_1^*))$. Hence, z_1^* is a regular fixed point of F in $\overline{B}_{r_1^*}(z_1^*)$.

Here, it is to show that $F(B_{r_1^*}(z_1^*))$ is an open set. For each given point $y \in F(B_{r_1^*}(z_1^*))$, there is a point $x \in B_{r_1^*}(z_1^*)$ satisfying $F(x)=y$. Then, there is a constant $\overline{r}_1>0$ satisfying $B_{\overline{r}_1}(x) \subset B_{r_1^*}(z_1^*)$. From the third paragraph of the proof, it is easy to see that $f(B_{\overline{r}_1}(x))$ is an open set because of $B_{r_1^*}(z_1^*) \subset B_{r_1}(z_1)$. It also follows from Lemma 1 again that

$$B_{(\lambda_{10}-L)\overline{r}_1}(y) = B_{(\lambda_{10}-L)\overline{r}_1}(F(x)) \subset F\left(B_{\overline{r}_1}(x)\right) \subset F\left(B_{r_1^*}(z_1^*)\right), \tag{35}$$

which implies that y is an interior point of $F(B_{r_1^*}(z_1^*))$ and then $F(B_{r_1^*}(z_1^*))$ is an open set.

With a similar argument to the existence of z_1^*, we can obtain the following positive constants

$$\begin{aligned}\delta_{1,m_1} &< \begin{cases} \min\left\{\begin{array}{c}\lambda_{10}\delta_{10}, \\ \dfrac{r_2-\|z_2-x_{20}\|}{2}-\delta_{20}\end{array}\right\}, & \text{if } m_1=1; \\ \min\left\{\begin{array}{c}\mu_{1,m_1-1}\delta_{1,m_1-1}, \\ \dfrac{r_2-\|z_2-x_{20}\|}{2}-\delta_{20}\end{array}\right\}, & \text{if } m_1>1.\end{cases} \\ \varepsilon_2 &= \frac{(\lambda_{20}-1)\delta_{1,m_1}}{1+\delta_{1,m_1}}, \\ r_2^* &= \frac{r_2+\|z_2-x_{20}\|}{2},\end{aligned} \tag{36}$$

such that when

$$\max\left\{L, \left\|g(z_2)\right\|\right\} < \varepsilon_2, \tag{37}$$

there exists a point $z_2^* \in B_{\delta_{1,m_1}}(z_2)$ satisfying that z_2^* is a regular expanding fixed point of F in $(\overline{B}_{r_2^*}(z_2^*))$ with expanding coefficient $\lambda_{20}-L>1$ and $F(B_{r_2^*}(z_2^*))$ is an open set.

A summary for this step is given as follows. When the following condition holds

$$\max\left\{L, \left\|g(z_1)\right\|, \left\|g(z_2)\right\|\right\} < \min\{\varepsilon_1, \varepsilon_2\}, \tag{38}$$

the map F will have two regular expanding fixed points $z_1^* \in B_{r_1^*}(z_1^*)$ and $z_2^* \in B_{\delta_{1,m_1}}(z_2)$, and $F(B_{r_i^*}(z_i^*))$ is an open set for $i=1,2$.

Step 2. It is to show that for each $i\,(1 \leq i \leq 2)$ there exists a point y_{i0} in $B_{r_i^*}(z_i^*)$ such that $F^{m_i}(y_{i0}) = z_{t(i)}^*$, where $t(i) = [i \bmod 2]+1$.

We first prove that there exist m_1 points $y_{1j} \in B_{\delta_{1j}}(x_{1j})$, $0 \leq j \leq m_1-1$ such that

$$\begin{aligned} F\left(y_{1j}\right) &= y_{1,j+1}, \quad \text{for } 0 \leq j \leq m_1-2, \\ F\left(y_{1,m_1-1}\right) &= z_2^*.\end{aligned} \tag{39}$$

That is, there exists a point $y_{10} \in B_{\delta_{10}}(x_{10}) \subset B_{r_1^*}(z_1^*)$ such that $F^{m_1}(y_{10}) = z_2^*$.

In order to do that, we first prove the existence of y_{1,m_1-1} by solving the following equation:

$$F(x) = z_2^*, \quad x \in \overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right), \tag{40}$$

which can be translated into the following:

$$f(x) = z_2^* - g(x), \quad x \in \overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right). \tag{41}$$

It follows from assumption (ii) and (13) that $f\colon \overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1}) \longrightarrow f(\overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1}))$ is homeomorphic with the inverse map $f^{-1}\colon f(\overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})) \longrightarrow \overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$ satisfying

$$\left\|f^{-1}(x) - f^{-1}(y)\right\| \leq \mu_{1,m_1-1}^{-1}\|x-y\|, \quad \forall x, y \in f\left(\overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right)\right). \tag{42}$$

Then, equation (41) can be translated into the following:

$$f^{-1}\left(z_2^* - g(x)\right) = x, \quad x \in \overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right). \tag{43}$$

Here, it needs to prove that

$$z_2^* - g(x) \in f\left(\overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right)\right), \quad x \in \overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right). \tag{44}$$

Suppose that g also satisfies

$$\max\left\{L, \left\|g\left(x_{1j}\right)\right\|, 0 \le j \le m_1 - 1\right\} < \varepsilon_3, \tag{45}$$

where

$$\varepsilon_3 = \min\left\{\varepsilon_1, \varepsilon_2, \frac{\min\left\{\lambda_{10}\delta_{10} - \delta_{11}, \mu_{1j}\delta_{1j} - \delta_{1,j+1}, \quad \text{for } 1 \le j \le m_1 - 1\right\}}{1 + \max\left\{\delta_{1j}, 0 \le j \le m_1 - 1\right\}}\right\}. \tag{46}$$

From (14) and (36), we get that $\varepsilon_3 > 0$. On the one hand, for any $x \in \overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$, it follows from (45) and (46) that

$$\begin{aligned}
\left\|z_2^* - g(x) - z_2\right\| &= \left\|g\left(x_{1,m_1-1}\right) - g(x) - g\left(x_{1,m_1-1}\right) + z_2^* - z_2\right\| \\
&\le L\left\|x - x_{1,m_1-1}\right\| + \left\|g\left(x_{1,m_1-1}\right)\right\| + \left\|z_2^* - z_2\right\| \\
&< \varepsilon_3\delta_{1,m_1-1} + \varepsilon_3 + \delta_{1,m_1} = \left(1 + \delta_{1,m_1-1}\right)\varepsilon_3 + \delta_{1,m_1} \\
&\le \left(1 + \delta_{1,m_1-1}\right)\frac{\mu_{1,m_1-1}\delta_{1,m_1-1} - \delta_{1,m_1}}{1 + \delta_{1,m_1-1}} + \delta_{1,m_1} = \mu_{1,m_1-1}\delta_{1,m_1-1}.
\end{aligned} \tag{47}$$

On the other hand, for any $x \in \partial B_{\delta_{1,m_1-1}}(x_{1,m_1-1})$, it follows from (13) that

$$\left\|f(x) - z_2\right\| = \left\|f(x) - f\left(x_{1,m_1-1}\right)\right\| \ge \mu_{1,m_1-1}\left\|x - x_{1,m_1-1}\right\| = \mu_{1,m_1-1}\delta_{1,m_1-1}. \tag{48}$$

Since $z_2 \in f(B_{\delta_{1,m_1-1}}(x_{1,m_1-1}))$ and $f(B_{\delta_{1,m_1-1}}(x_{1,m_1-1}))$ is open, it follows from (47) and (48) that (44) is true. So, we can define a map

$$h_2(x) = f^{-1}\left(z_2^* - g(x)\right), \quad x \in \overline{B}_{\delta_{1,m_1-1}}\left(x_{1,m_1-1}\right). \tag{49}$$

It follows from (45) and (46) that

$$L < \varepsilon_3 < \frac{\mu_{1,m_1-1}\delta_{1,m_1-1} - \delta_{1,m_1}}{1 + \delta_{1,m_1-1}} < \mu_{1,m_1-1}\frac{\delta_{1,m_1-1}}{1 + \delta_{1,m_1-1}} < \mu_{1,m_1-1}. \tag{50}$$

Then, for any $x \in \overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$, it follows from (42) and (47) that

$$\begin{aligned}
\left\|h_2(x) - x_{1,m_1-1}\right\| &= \left\|f^{-1}\left(z_2^* - g(x)\right) - f^{-1}\left(z_2\right)\right\| \\
&\le \mu_{1,m_1-1}^{-1}\left\|z_2^* - g(x) - z_2\right\| < \delta_{1,m_1-1}.
\end{aligned} \tag{51}$$

That is, h_2 maps $\overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$ into itself. Moreover, for any $x, y \in \overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$, it follows from (42) that

$$\begin{aligned}
\left\|h_2(x) - h_2(y)\right\| &= \left\|f^{-1}\left(z_2^* - g(x)\right) - f^{-1}\left(z_2^* - g(y)\right)\right\| \\
&\le \mu_{1,m_1-1}^{-1}\|g(x) - g(y)\| \le L\mu_{1,m_1-1}^{-1}\|x - y\|,
\end{aligned} \tag{52}$$

which together with (50) implies that h_2 is contractive in $\overline{B}_{\delta_{1,m_1-1}}(x_{1,m_1-1})$. It follows from the Banach contractive mapping principle and (51) that there exists a unique point $y_{1,m_1-1} \in B_{\delta_{1,m_1-1}}(x_{1,m_1-1})$ such that $h(y_{1,m_1-1}) = z_2^*$. Consequently, $F(y_{1,m_1-1}) = z_2^*$, that is, equation (40) has a unique solution $y_{1,m_1-1} \in B_{\delta_{1,m_1-1}}(x_{1,m_1-1})$.

Using a similar method as above, we can prove that there exist $m_1 - 1$ unique points $y_{1j} \in B_{\delta_{1j}}(x_{1j})$ for $0 \le j \le m_1 - 2$ such that $F(y_{1j}) = y_{1,j+1}$. Then, we get that $y_{10} \in B_{\delta_{10}}(x_{10}) \subset B_{r_1^*}(z_1^*)$ such that $F^{m_1}(y_{10}) = z_2^*$.

Next, set g also to satisfy

$$\max\left\{L, \left\|g\left(x_{2j}\right)\right\|, 0 \le j \le m_2 - 1\right\} < \varepsilon_4, \tag{53}$$

where

$$\varepsilon_4 = \min\left\{\varepsilon_1, \varepsilon_2, \frac{\min\left\{\lambda_{20}\delta_{20} - \delta_{21}, \mu_{2j}\delta_{2j} - \delta_{2,j+1}, \quad \text{for } 1 \le j \le m_2 - 1\right\}}{1 + \max\left\{\delta_{2j}, 0 \le j \le m_2 - 1\right\}}\right\}. \tag{54}$$

It follows from (14) and (15) that $\varepsilon_4 > 0$.

Repeating a similar discussion as above, we can get when g satisfies (53), there exist m_2 unique points $y_{2j} \in B_{\delta_{2j}}(x_{2j})$, $0 \le j \le m_2 - 1$, such that $F(y_{2j}) = y_{2,j+1}$ for $0 \le j \le m_2 - 2$, and $F(y_{2,m_2-1}) = z_1^*$. That is, there exists a point $y_{20} \in B_{\delta_{20}}(x_{20}) \subset B_{r_2^*}(z_2^*)$ such that $F^{m_2}(y_{20}) = z_1^*$.

Let $\varepsilon_0 = \min\{\varepsilon_j, 1 \le j \le 4\}$. If g satisfies the following condition

$$\max\left\{L, \|g(z_i)\|, \left\|g(x_{ij})\right\|, 1 \le i \le 2, 0 \le j \le m_i - 1\right\} < \varepsilon_0, \tag{55}$$

then the statements in Step 2 hold. Consequently, F has a heteroclinic cycle Γ' connecting repellers z_1^* and z_2^*.

Step 3. It is to show that the heteroclinic cycle Γ' connecting repellers z_1^* and z_2^* of F is regular and nondegenerate.

When the map g satisfies (55), it follows from the discussions in Step 2 that $y_{ij} \in B_{\delta_{ij}}(x_{ij})$ for $1 \le i \le 2$, $0 \le j \le m_i - 1$. Hence, for $1 \le i \le 2$, $0 \le j \le m_i - 1$, we can take positive constants $\delta_{ij}' < \delta_{ij}$ such that

$$y_{ij} \in B_{\delta_{ij}'}\left(y_{ij}\right) \subset B_{\delta_{ij}}\left(x_{ij}\right). \tag{56}$$

It follows from (11), (13), and (56) that for $1 \le i \le 2$, $1 \le j \le m_i - 1$,

$$\|F(x) - F(y)\| \ge \|f(x) - f(y)\| - \|g(x) - g(y)\| \ge (\lambda_{i0} - L)\|x - y\|, \quad \forall x, y \in \overline{B}_{r_i^*}(z_i^*), \tag{57}$$

$$\|F(x) - F(y)\| \ge \|f(x) - f(y)\| - \|g(x) - g(y)\| \ge \left(\mu_{ij} - L\right)\|x - y\|, \quad \forall x, y \in \overline{B}_{\delta_{ij}'}\left(y_{ij}\right), \tag{58}$$

where $\lambda_{i0} > L + 1$ and $\mu_{ij} > L$ can be derived from (45), (46), (53), and (54).

For each $i\,(1 \le i \le 2)$, since z_i^* is a regular expanding fixed point of F and $F(B_{r_i^*}(z_i^*))$ is an open set, the backward orbit of y_{i0} lies in $B_{r_i^*}(z_i^*)$ and (57) holds in some neighborhood of each point on the backward orbit. The forward orbit of y_{i0} consists of y_{ij} for $1 \le j \le m_i - 1$ and (58) holds for each y_{ij} in $\overline{B}_{\delta_{ij}'}(y_{ij})$. Therefore, the heteroclinic cycle Γ' connecting repellers z_1^* and z_2^* is nondegenerate. In addition, it is clear that F is continuous in $B_{r_i^*}(z_i^*)$ and $B_{\delta_{ij}'}(y_{ij})$ for $1 \le i \le 2$, $1 \le j \le m_i - 1$. It follows from (3) of Remark 2.2 in [25] that if we prove that for each point y_0 on the cycle Γ', there exists a positive constant r_0 such that $F(y_0)$ is an interior point of $F(B_{r_0}(y_0))$, then this cycle Γ' is regular.

Firstly, for each point y_0 on Γ' lying in $B_{r_i^*}(z_i^*)$, there exists a constant r_0 such that $\overline{B}_{r_0}(y_0) \subset B_{r_i^*}(z_i^*)$. It follows from (57) and Lemma 1, by using F to replace f and making $g = 0$, that

$$B_{(\lambda_{i0}-L)r_0}(F(y_0)) \subset F\left(B_{r_0}(y_0)\right), \tag{59}$$

which implies that $F(y_0)$ is an interior point of $F(B_{r_0}(y_0))$.

Secondly, for each point y_{ij}, $1 \le j \le m_i - 1$, on Γ' lying out $B_{r_i^*}(z_i^*)$, it follows from (58) and Lemma 1, by using F to replace f and making $g = 0$ again, that

$$B_{(\mu_{ij}-L)\delta_{ij}'}\left(F\left(y_{ij}\right)\right) \subset F\left(B_{\delta_{ij}'}\left(y_{ij}\right)\right), \tag{60}$$

which implies that $F(y_{ij})$ is an interior point of $F(B_{\delta_{ij}'}(y_{ij}))$.

Hence, the cycle Γ' is regular. That is, the heteroclinic cycle Γ' connecting repellers z_1^* and z_2^* of F is regular and nondegenerate. Consequently, it follows form Lemma 2 that there exists an uncountable, perfect, bounded, and closed set V such that system (9) is chaotic on V in the sense of both Devaney and Li-Yorke. This completes the proof.

Remark 4. Theorem 1 gives a relatively explicit range of the Lipschitz perturbation g which is characterized by a constant ε_0 determined by the properties of the original map f. From (10), we see that it only needs L and the values of g at z_i, x_{ij} are less than ε_0, and it does not need to compute all the values of g in some domains. Hence, the conditions about g in Theorem 1 are relatively easy to check out in practice. In addition, it only needs the original map f to be continuous near some points of interest without having to be continuous in the whole space.

Remark 5. From the proof of Theorem 1, it is easy to see that the perturbed map F will have a regular and nondegenerate heteroclinic cycle Γ' connecting repellers if the unperturbed map f with a regular and nondegenerate heteroclinic cycle Γ connecting repellers undergoes a small perturbation, and the cycle Γ' is near to Γ. The perturbed range of g is characterized by ε_0 determined in Theorem 1. Thus, this result can be viewed as persistence of regular and nondegenerate heteroclinic cycles connecting repellers in Banach spaces.

When the original map f is continuously differentiable in some domains of interest, using a similar method to Theorem 1, we can get the following result.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space and $f: X \longrightarrow X$ be a map with $k\,(\ge 2)$ different fixed points $z_1, \ldots, z_k \in X$. Assume that*

(i) *For each $i\,(1 \le i \le k)$, f is continuously differentiable in $B_{r_i'}(z_i)$ for some constant $r_i' > 0$ and $Df(z_i)$ is an invertible linear map satisfying $\|Df(z_i)\|^0 > 1$, which is equivalent to that there exists a positive constant $r_i \le r_i'$ such that z_i is a regular expanding fixed point of f in $\overline{B}_{r_i}(z_i)$.*

(ii) *f has a heteroclinic cycle Γ connecting repellers $z_1, \ldots, z_k$.*

(iii) *f is continuously differentiable in some neighborhood U_{x_0} of each point x_0 on the cycle Γ, and $Df(x_0)$ is an invertible linear map.*

Then, there exists a constant $\varepsilon_0 > 0$ such that for any Lipschitz map g in each set of $\overline{B}_{r_i}(z_i)$ and U_{x_0} for $x_0 \in \Gamma$, with the Lipschitz constant L satisfying

$$\max\{L, \|g(x_0)\| \text{ for } x_0 \in \Gamma\} < \varepsilon_0, \tag{61}$$

the results of Theorem 1 hold.

Proof. It follows from the assumptions in Theorem 2 and Lemma 3 that f has a regular and nondegenerate heteroclinic cycle Γ connecting repellers $z_1, \ldots, z_k$. For each $i(1 \le i \le k)$, since z_i is a regular expanding fixed point of f, there exist a point $x_{i0} \in B_{r_i}(z_i)$ and a positive integer $m_i \ge 1$ such that $f(x_{i0}) \notin B_{r_i}(z_i)$ and $f^{m_i}(x_{i0}) = z_{t(i)}$, where $t(i) = [i \bmod k] + 1$. The rest of the proof is similar to that of Theorem 1, so it is omitted.

For a function $f \in C^1(U, X)$, the following norm is often used

$$\|f\|_{C^1,U} := \sup\{\|f(x)\|, \|Df(x)\|, x \in U \subset X\}. \tag{62}$$

Therefore, if the conditions in Theorems 1 and 2 about g are replaced by those based on the above norm, then we can obtain two consequences of Theorems 1 and 2. For convenience, we list them as the following theorems. □

Theorem 3. *Suppose that $(X, \|\cdot\|)$ is a Banach space, $f: X \longrightarrow X$ is a map with $k(\ge 2)$ different fixed points $z_1, \ldots, z_k \in X$ and satisfies the conditions (i) and (ii) in Theorem 1. Then, there exists a constant $\varepsilon_0 > 0$ such that for any $g \in C^1(U, X)$ with $\|g\|_{C^1,U} < \varepsilon_0$, the results of Theorem 1 hold, where $U = B_{r_i}(z_i) \cup (\cup_{j=1}^{m_i-1} U_{ij}), 1 \le i \le k$.*

Theorem 4. *Suppose that $(X, \|\cdot\|)$ is a Banach space, $f: X \longrightarrow X$ is a map with $k(\ge 2)$ different fixed points $z_1, \ldots, z_k \in X$ and satisfies the conditions (i)–(iii) in Theorem 2. Then, there exists a constant $\varepsilon_0 > 0$ such that for any $g \in C^1(U, X)$ with $\|g\|_{C^1,U} < \varepsilon_0$, the results of Theorem 1 hold, where $U = \cup_{x_0 \in \Gamma} U_{x_0}$.*

At the last of this section, we discuss a usually used Banach space $\mathbf{R}^n$, which is the Euclidean space. As is well known, there are many different norms in $\mathbf{R}^n$. A map in $\mathbf{R}^n$ can expand in different norms, see [11, 26] and references therein. It is natural to ask whether there is the persistence of heteroclinic cycles connecting repellers in $\mathbf{R}^n$, where the repellers expand in different norms. The following Theorem 5 will answer this question.

The usually used Euclidean norm is denoted by

$$\|x\| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}, \quad x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n. \tag{63}$$

In the following, we will use the neighborhood of a point $x \in \mathbf{R}^n$ in different norms. For convenience, let $\overline{B}_r(x)$ and $B_r(x)$ denote the closed and open balls of x with radius r in the Euclidean norm $\|\cdot\|$, let $\overline{N}_r(x)$ and $N_r(z)$ denote the closed and open balls of x with radius r in any other norm $\|\cdot\|'$.

Now, we establish a result on persistence of heteroclinic cycles connecting repellers in $\mathbf{R}^n$, where the repellers expand in different norms.

Theorem 5. *Suppose that a map $f: \mathbf{R}^n \longrightarrow \mathbf{R}^n$ has $k(\ge 2)$ different fixed points $z_1, \ldots, z_k \in \mathbf{R}^n$ and satisfies the following conditions:*

(i) for each $i(1 \le i \le k)$, f is continuously differentiable in some neighborhood of z_i and all the eigenvalues of $Df(z_i)$ have absolute values larger than 1, which implies that there exist a constant $r_i > 0$ and a norm $\|\cdot\|_i$ in $\mathbf{R}^n$ such that f is continuously differentiable in $\overline{N}_{r_i}(z_i)$, and z_i is a regular expanding fixed point of f in $\overline{N}_{r_i}(z_i)$.

(ii) for each $i(1 \le i \le k)$, there exist a point $x_{i0} \in N_{r_i}(z_i)$, $x_{i0} \ne z_i$, and a positive integer $m_i \ge 1$ such that $f^{m_i}(x_{i0}) = z_{t(i)}$, where $t(i) = [i \mod k] + 1$. Furthermore, f is continuously differentiable in some neighborhood U_{ij} of x_{ij} and satisfies that $\det Df(x_{ij}) \ne 0$, where $x_{ij} = f^j(x_{i0})$ for $1 \le j \le m_i - 1$.

Then, for any Lipschitz map g with Lipschitz constant L in the Euclidean norm $\|\cdot\|$ in each set of $\overline{N}_{r_i}(z_i)$ and U_{ij}, $1 \le i \le k, 1 \le j \le m_i - 1$, there exists a constant $\varepsilon_0 > 0$ satisfying

$$\max\left\{L, \|g(z_i)\|, \left\|g(x_{ij})\right\|, \quad 1 \le i \le k, 0 \le j \le m_i - 1\right\} < \varepsilon_0, \tag{64}$$

such that the perturbed system (9) is chaotic in the sense of both Devaney and Li-Yorke on a compact and perfect set which contains a Cantor set.

Proof. Without loss of generality and for simplicity, we also only show that Theorem 5 holds for $k = 2$.

For convenience, let $i = 1$ or 2 in the rest of proof. As pointed in the second paragraph of the proof in Theorem 1, we can also suppose that $N_{r_1}(z_1) \cap N_{r_2}(z_2) = \varnothing$ and $f(x_{i0}) \notin N_{r_i}(z_i)$.

Since all the norms on $\mathbf{R}^n$ are equivalent by Corollary 3.14 of Chapter II in [36], there exist positive constants b_{11}, b_{12}, c_{i1} and c_{i2} such that

$$\begin{aligned} b_{11}\|\cdot\|_1 &\le \|\cdot\|_2 \le b_{12}\|\cdot\|_1, \\ c_{i1}\|\cdot\|_i &\le \|\cdot\| \le c_{i2}\|\cdot\|_i. \end{aligned} \tag{65}$$

Since g is a Lipschitz map with Lipschitz constant L in the Euclidean norm $\|\cdot\|$ in $\overline{N}_{r_i}(z_i)$ and U_{ij}, for any $x, y \in \overline{N}_{r_i}(z_i)$ and any $x, y \in U_{ij}$, $1 \le j \le m_i - 1$, it follows from (65) that

$$\|g(x) - g(y)\|_i \le c_{i1}^{-1}\|g(x) - g(y)\| \le c_{i1}^{-1}L\|x - y\| \le c_{i1}^{-1}c_{i2}L\|x - y\|_i \le L'\|x - y\|_i, \tag{66}$$

where

$$L' = \max\left\{c_{i1}^{-1}c_{i2}L, \quad i = 1, 2\right\}. \tag{67}$$

Then it follows from (66) that g is also a Lipschitz map with Lipschitz constant L' in the norm $\|\cdot\|_i$ in $\overline{N}_{r_i}(z_i)$ and U_{ij}.

It follows from assumption (i) that there exists a constant $\lambda_{i0} > 1$ such that

$$\|f(x) - f(y)\|_i \geq \lambda_{i0}\|x - y\|_i, \quad \forall x, y \in \overline{N}_{r_i}(z_i), \tag{68}$$

$f\colon \overline{N}_{r_i}(z_i) \longrightarrow f(\overline{N}_{r_i}(z_i))$ is a homeomorphism and $f(N_{r_i}(z_i))$ is open, $f(D)$ is open for any open set $D \subset N_{r_i}(z_i)$. Take a constant

$$\delta_{i0} < \frac{r_i - \|z_i - x_{i0}\|_i}{2}, \tag{69}$$

such that $\overline{N}_{\delta_{i0}}(x_{i0}) \subset N_{r_i}(z_i)$. Then, it follows form (68) that $f\colon N_{\delta_{i0}}(x_{i0}) \longrightarrow f(N_{\delta_{i0}}(x_{i0}))$ is also a homeomorphism.

In addition, it follows from $\det Df(x_{ij}) \neq 0$, $1 \leq j \leq m_i - 1$, that none of the eigenvalues of $Df(x_{ij})$ is 0. Therefore, $(Df(x_{ij}))^T Df(x_{ij})$ is positive definite. Then,

$$\|Df(x_{ij})\|^0 = \left(\inf_{\|x\|=1}\left(x^T\left(Df(x_{ij})\right)^T Df(x_{ij})x\right)\right)^{\frac{1}{2}} > 0, \tag{70}$$

where $x \in \mathbf{R}^n$. It follows from (65) and (70) that

$$\begin{aligned}\left\|Df(x_{ij})\right\|_i^0 &= \inf_{x\neq 0}\frac{\left\|Df(x_{ij})x\right\|_i}{\|x\|_i} \geq c_{i1}c_{i2}^{-1}\inf_{x\neq 0}\frac{\left\|Df(x_{ij})x\right\|}{\|x\|} \\ &= c_{i1}c_{i2}^{-1}\left\|Df(x_{ij})\right\|^0 > 0,\end{aligned} \tag{71}$$

Hence, it follows from (71) and Lemma 4 that there exist positive constants μ_{ij} and δ_{ij} such that

$$\|f(x) - f(y)\|_i \geq \mu_{ij}\|x - y\|_i, \quad \forall x, y \in \overline{N}_{\delta_{ij}}(x_{ij}), \tag{72}$$

which implies that $f\colon N_{\delta_{ij}}(x_{ij}) \longrightarrow f(N_{\delta_{ij}}(x_{ij}))$ is homeomorphic, and $f(N_{\delta_{ij}}(x_{ij}))$ is open for $1 \leq j \leq m_i - 1$, where δ_{ij} satisfies the following conditions:

$$\begin{aligned}&\delta_{i1} < \lambda_{i0}\delta_{i0}, \\ &\delta_{i,j+1} < \mu_{ij}\delta_{ij}, \quad \text{for } 1 \leq j \leq m_i - 2,\end{aligned} \tag{73}$$

$\overline{N}_{\delta_{ij}}(x_{ij})$ are disjoint subsets of U_{ij} and $\overline{N}_{\delta_{ij}}(x_{ij}) \cap N_{r_i}(z_i) = \varnothing$ for fixed i and $1 \leq j \leq m_i - 1$.

The rest of the proof is almost exactly the same to Steps 1–3 in the proof of Theorem 1 except for three aspects. One is that L is replaced by L' and the domains in the norm $\|\cdot\|$ are replaced by those in the norms $\|\cdot\|_1$ or $\|\cdot\|_2$, respectively. In brief, in the representations of the domains, the alphabet B is replaced by the alphabet N through the proof of Theorem 1. The second is that some values in the norm $\|\cdot\|$ are replaced by those in the norms $\|\cdot\|_1$ or $\|\cdot\|_2$, respectively. It is pointed out that (18) and (42) take the values in the norm $\|\cdot\|_1$, the remainders follow the following rule: if the independent variables of functions are taken from $N_{r_i}(z_i)$ or $N_{\delta_{ij}}(x_{ij})$, then the values in the norm $\|\cdot\|$ are replaced by those in the norm $\|\cdot\|_i$. The third is that some related constants used in the proof are slightly modified since the norm $\|\cdot\|$ is replaced by the norms $\|\cdot\|_1$ or $\|\cdot\|_2$. For convenience, we list them as follows.

$$\delta_{1,m_1} < \begin{cases} \min\left\{\begin{array}{c} b_{11}\lambda_{10}\delta_{10}, \\ \dfrac{r_2 - \|z_2 - x_{20}\|_2}{2} - \delta_{20} \end{array}\right\}, & \text{if } m_1 = 1; \\ \min\left\{\begin{array}{c} b_{11}\mu_{1,m_1-1}\delta_{1,m_1-1}, \\ \dfrac{r_2 - \|z_2 - x_{20}\|_2}{2} - \delta_{20} \end{array}\right\}, & \text{if } m_1 > 1, \end{cases}$$

$$\delta_{2,m_2} < \begin{cases} \min\left\{\begin{array}{c} b_{12}^{-1}\lambda_{20}\delta_{20}, \\ \dfrac{r_1 - \|z_1 - x_{10}\|_1}{2} - \delta_{10} \end{array}\right\}, & \text{if } m_2 = 1; \\ \min\left\{\begin{array}{c} b_{12}^{-1}\mu_{2,m_2-1}\delta_{2,m_2-1}, \\ \dfrac{r_1 - \|z_1 - x_{10}\|_1}{2} - \delta_{10} \end{array}\right\}, & \text{if } m_2 > 1, \end{cases}$$

$$
\begin{aligned}
r_1^* &= \frac{r_1 + \|z_1 - x_{10}\|_1}{2},\\
r_2^* &= \frac{r_1 + \|z_2 - x_{20}\|_2}{2},\\
\varepsilon_1 &= \frac{(\lambda_{10} - 1)\delta_{2,m_2}}{1 + \delta_{2,m_2}},\\
\varepsilon_2 &= \frac{(\lambda_{20} - 1)\delta_{1,m_1}}{1 + \delta_{1,m_1}},\\
\varepsilon_3 &= \min\left\{ \varepsilon_1, \varepsilon_2, \frac{\min\left\{\lambda_{10}\delta_{10} - \delta_{11}, \mu_{1,m_1-1}\delta_{1,m_1-1} - b_{11}^{-1}\delta_{1,m_1}, \mu_{1j}\delta_{1j} - \delta_{1,j+1}, \quad \text{for } 1 \le j \le m_1 - 2\right\}}{1 + \max\left\{\delta_{1j}, 0 \le j \le m_1 - 1\right\}} \right\},\\
\varepsilon_4 &= \min\left\{ \varepsilon_1, \varepsilon_2, \frac{\min\left\{\lambda_{20}\delta_{20} - \delta_{21}, \mu_{2,m_2-1}\delta_{2,m_2-1} - b_{12}\delta_{2,m_2}, \mu_{2j}\delta_{2j} - \delta_{2,j+1}, \quad \text{for } 1 \le j \le m_2 - 2\right\}}{1 + \max\left\{\delta_{2j}, 0 \le j \le m_2 - 1\right\}} \right\}.
\end{aligned}
\tag{74}
$$

It follows from (73) and the above conditions that $\varepsilon_j > 0$ for $1 \le j \le 4$. Here, it should explain how to take the values of the numerators in terms of m_1 and m_2 in ε_3 and ε_4. We only explain the values in terms of m_1, while that for m_2 is similar. For the numerator of the fraction at the right side of ε_3, when $m_1 = 1$, it only takes the first term; when $m_1 = 2$, it only takes the first two terms; when $m_1 \ge 3$, it takes all of the terms.

Set

$$
\varepsilon_0' = \min\left\{\varepsilon_j,\ 1 \le j \le 4\right\}, \quad \varepsilon_0 = \min\left\{ \frac{\varepsilon_0'}{\max\left\{c_{11}^{-1}c_{12}, c_{21}^{-1}c_{22}\right\}}, c_{11}\varepsilon_0', c_{21}\varepsilon_0' \right\}. \tag{75}
$$

If g satisfies condition (64) in the Euclidean norm $\|\cdot\|$, that is the following:

$$
\max\left\{L, \|g(z_i)\|, \left\|g(x_{ij})\right\|, 1 \le i \le 2, 0 \le j \le m_i - 1\right\} < \varepsilon_0, \tag{76}
$$

then it follows from (65), (75) and (76) that g also satisfies the following condition in the norms $\|\cdot\|_1$ and $\|\cdot\|_2$

$$
\max\left\{L', \|g(z_i)\|_i, \left\|g(x_{ij})\right\|_i, 1 \le i \le 2, 0 \le j \le m_i - 1\right\} < \varepsilon_0', \tag{77}
$$

Therefore, repeating the Steps 1–3 in the proof of Theorem 1, we can get that if g satisfies (76), consequently (77), then F has a regular and nondegenerate heteroclinic cycle Γ' connecting repellers z_1^* and z_2^* in different norms $\|\cdot\|_1$ and $\|\cdot\|_2$. It follows from the proof of Theorem 4.1 in [26] that there exists a positive integer p such that F^p has a heteroclinic cycle Γ^* connecting repellers z_1^* and z_2^* in the unified Euclidean norm $\|\cdot\|$. Since all the points on the cycle Γ^* of F^p also lie on the cycle Γ' of F, it is easy to prove that the cycle Γ^* of F^p is also regular and nondegenerate. Consequently, F is chaotic on a compact and perfect set which contains a Cantor set in the sense of both Devaney and Li-Yorke. This completes the proof. □

Remark 6. From the proof of Theorem 5, we obtain that F^p has a regular and nondegenerate heteroclinic cycle connecting repellers in the unified Euclidean norm $\|\cdot\|$ for some positive integer p. Hence, Theorem 5 can also be regarded as the persistence of a regular and nondegenerate heteroclinic cycle connecting repellers in $\mathbf{R}^n$. In the special case that all the norms $\|\cdot\|_i$, $1 \le i \le k$, in assumption (i) become a unified norm, such as the Euclidean norm $\|\cdot\|$, then the positive integer p becomes 1. Hence, this special case of Theorem 5 is consistent with Theorem 1.

The following result is a direct consequence of Theorem 5.

Theorem 6. *Suppose that a map $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ has $k(\ge 2)$ different fixed points $z_1, \ldots, z_k \in \mathbf{R}^n$ and satisfies the following conditions*

(i) For each $i\,(1 \le i \le k)$, z_i is an expanding fixed point of f in some norm $\|\cdot\|_i$;

(ii) f has a k-heteroclinic cycle Γ connecting fixed points $z_1, \ldots, z_k$ and is continuously differentiable in some neighborhood U_{x_0} of each point $x_0 \in \Gamma$ satisfying $\det Df(x_0) \ne 0$.

Then, there exists a constant $\varepsilon_0 > 0$ such that for any Lipschitz map g in each set of $\cup_{x_0 \in \Gamma} U_{x_0}$ with Lipchitz constant L in the Euclidean norm $\|\cdot\|$ satisfying

$$
\max\left\{L, \|g(x_0)\|, \quad \text{for } x_0 \in \Gamma\right\} < \varepsilon_0, \tag{78}
$$

the results in Theorem 5 hold.

Remark 7. [28] studied the persistence of heteroclinic repellers in $\mathbf{R}^n$ for C^1 maps with C^1 perturbations, where the maps needed to be continuously differentiable in the whole space. Here, it only needs the maps to be continuously differentiable in some neighborhoods of points. The main differences between the above two theorems and the result in [28] are as follows. One is that Theorems 5 and 6 studied the Lipschitz perturbations, while the latter considered the C^1 perturbations. The second is that Theorems 5 and 6 give an explicit expression for the range of perturbations, which are determined by the properties of the original maps, while the latter did not give such a range for perturbations. The third is that Theorems 5 and 6 use different norms for expansions of fixed points which are more general in practice, while the latter only used a single norm for expansions of fixed points.

Remark 8. Just as Theorems 3 and 4, if the perturbed term g is continuously differentiable, then the conditions about g in Theorems 5 or 6 can be replaced by that $g \in C^1(U, X)$ with $\|g\|_{C^1,U} < \varepsilon_0$, where U is taken the corresponding domains used in Theorems 5 or 6, respectively, then the results in Theorems 5 or 6 hold.

4. Examples

In this section, three examples are given to illustrate the validity of the theoretical results.

Example 3. The original map f is taken as the following map on $\mathbf{R}$:

$$f(x) = \begin{cases} 2x, & \text{if } x \in [-2, 2] \\ 5x - 9, & \text{if } x \in (2, 2.5) \\ 0.1x - 0.2875, & \text{else.} \end{cases} \tag{79}$$

The perturbed map g is taken as $g(x) = \gamma|x|$, where $x \in \mathbf{R}$ and γ is a positive real number. It is obvious that f is piecewise continuous on $\mathbf{R}$, g is a Lipschitz map with a Lipschitz constant $L = \gamma$ and it is not differentiable on $\mathbf{R}$.

It is easy to see that $z_1 = 0$ and $z_2 = 2.25$ are two regular expanding fixed points of f. Set $x_{10} = 1.125 \in (-2, 2)$, then $f(x_{10}) = z_2$. Set $x_{20} = 2.375 \in (2, 2.5)$, then $x_{21} = f(x_{20}) = 2.875$ and $f(x_{21}) = z_1$, that is, $f^2(x_{20}) = z_1$. So, f has a 2-heteroclinic cycle Γ connecting repellers z_1 and z_2. It is clear that the cycle is regular and nondegenerate, and assumptions (i) and (ii) in Theorem 1 holds with $k = 2$, $r_1 = 2$, $r_2 = 0.25$, $\lambda_{10} = 2$, $\lambda_{20} = 5$, $m_1 = 1$, $m_2 = 2$, x_{10} and x_{20} as the above.

Some constants that appear in the proof of Theorem 1 are taken as follows: $\mu_{21} = 0.1$, $\delta_{10} = 0.43 < (r_1 - |z_1 - x_{10}|)/2) = 0.4375$, $\delta_{20} = 0.06 < ((r_2 - |z_2 - x_{20}|/2) = 0.0625$, $\delta_{11} = 0.002 < \min\{\lambda_{10}\delta_{10}, ((r_2 - |z_2 - x_{20}|)/2) - \delta_{20}\} = 0.0025$, $\delta_{21} = 0.28 < \lambda_{20}\delta_{20} = 0.3$, $\delta_{22} = 0.007 < \min\{\mu_{21}\delta_{21}, ((r_1 - |z_1 - x_{10}|)/2) - \delta_{10}\} = 0.0075$, $\varepsilon_1 = ((\lambda_{10} - 1)\delta_{22}/1 + \delta_{22}) \approx 0.006951$, $\varepsilon_2 = (((\lambda_{20} - 1)\delta_{11})/1 + \delta_{11}) \approx 0.007984$, $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2, ((\delta_{10}\lambda_{10} - \delta_{11})/\ 1 + \delta_{10})\} = 0.006951$, $\varepsilon_4 = \min\ \{\varepsilon_1, \varepsilon_2, ((\min\ \{\lambda_{20}\delta_{20} - \delta_{21}, \mu_{21}\delta_{21} - \delta_{22}\})/1 + \max\{\delta_{20}, \delta_{21}\})\} = 0.006951$, $\varepsilon_0 = \min\left\{\varepsilon_j, 1 \le j \le 4\right\} = 0.006951$. It is easy to check that the perturbation g satisfies condition (10) for $\gamma \le 0.0024$. Then, it follows from the result of Theorem 1 that the perturbed system $F = f + g$ also has a regular and nondegenerate heteroclinic cycle Γ' connecting repellers which is near to Γ. Consequently, F and f are chaotic in the sense of both Devaney and Li-Yorke. For illustrating the persistence of a heteroclinic cycle connecting repellers, we take $\gamma = 0.002$ for example. It is easy to calculate the following results. The perturbed map F has two regular expanding fixed pints $z_1^* = 0$ and $z_2^* = (1500/667) \approx 2.248876$. There exist two points $x_{10}^* = (750000/667667) \approx 1.123314$ and $x_{20}^* = (301375/127551) \approx 2.362780$ such that $F(x_{10}^*) = z_2^*$, $x_{21}^* = F(x_{20}^*) = (575/204) \approx 2.818627$ and $F(x_{21}^*) = z_1^*$. Then, F has a 2-heteroclinic cycle Γ' connecting repellers $z_1^* = 0$ and $z_2^* = 0$. It is clear that Γ' is near to Γ. With the increase of γ, the heteroclinic cycle Γ' will gradually run away from Γ until it breaks or disappears. Since f and F are chaotic on some intervals of $\mathbf{R}$ and the computer simulations of them are on intervals, we omit the computer simulations.

Example 4. The original map f is taken as the following map on $\mathbf{R}^2$

$$f(x, y) = \begin{cases} 8(x, y), & \text{if } (x, y) \in \overline{B}_1(0, 0), \\ (2x - 2, 2y - 2), & \text{if } (x, y) \in \overline{B}_4(0, 0)/\overline{B}_1(0, 0), \\ \left(\sin\left[x - 2 - \dfrac{\pi}{2} + \left(y - 2 - \dfrac{\pi}{2}\right)^2\right],\right. & \\ \left.\sin\left[\left(x - 2 - \dfrac{\pi}{2}\right)^2 + y - 2 - \dfrac{\pi}{2}\right]\right), & \text{if } (x, y) \notin \overline{B}_4(0, 0). \end{cases} \tag{80}$$

This map is used as an example in [25] for illustrating chaos induced by a heteroclinic cycle connecting repellers. The perturbed map g is taken as $g(x, y) = \gamma(x, y)$, where $(x, y) \in \mathbf{R}^2$ and γ is a real number. It is obvious that f is only continuously differentiable in some domains of $\mathbf{R}^2$, g is continuously differentiable in $\mathbf{R}^2$ and has a Lipschitz constant $L = |\gamma|$.

On the one hand, it is clear that $z_1 = (0,0)$ and $z_2 = (2,2)$ are two fixed points of f, f is continuously differentiable in $\overline{B}_1(z_1)$, $\overline{B}_{1.171}(z_2)$, and satisfies that

$$\begin{aligned} Df(z_1) &= 8I_2, \\ Df(z_2) &= 2I_2, \end{aligned} \tag{81}$$

where I_2 is the identity matrix. So, the eigenvalues of $Df(z_1)$ and $Df(z_2)$ have absolute values larger than 1, which implies that z_1 and z_2 are two regular expanding expanding fixed points of f in $\overline{B}(z_1)$ and $\overline{B}_{1.171}(z_2)$ in the Euclidean norm $\|\cdot\|$ with $\lambda_{10} = 8$, $\lambda_{20} = 2$, respectively. It is obvious that $\overline{B}_1(z_1) \cap \overline{B}_{1.171}(z_2) = \varnothing$ and both lie in $B_4(z_1)$. Set $x_{10} = ((1/4),(1/4)) \in B_1(z_1)$, then $f(x_{10}) = z_2$. Set $x_{20} = (2 + (\pi/8), 2 + (\pi/8)) \in B_{1.171}(z_2)$, then $x_{21} = f(x_{20}) = (2 + (\pi/4), 2 + (\pi/4)) \in B_4(z_1)\backslash B_1(z_1)$, $x_{22} = f(x_{21}) = (2 + (\pi/2), 2 + (\pi/2)) \notin \overline{B}_4(z_1)$ and $f(x_{22}) = z_1$, that is, $f^3(x_{20}) = z_1$.

On the other hand, it is also obvious that f is continuously differentiable in some neighborhoods of x_{10}, x_{20}, x_{21}, and x_{22} and satisfies

$$\begin{aligned} Df(x_{10}) &= 8I_2, \\ Df(x_{20}) &= Df(x_{21}) \\ &= 2I_2, \\ Df(x_{22}) &= I_2. \end{aligned} \tag{82}$$

Then, it follows from (82) that

$$\begin{aligned} \|Df(x_{10})\|^0 &= 8, \\ \|Df(x_{20})\|^0 &= \|Df(x_{21})\|^0 \\ &= 2, \\ \|Df(x_{22})\|^0 &= 1, \end{aligned} \tag{83}$$

which together with Lemma 4 imply that the cycle Γ is nondegenerate. Furthermore, it follows from (4) of Remark 2.2 in [25] that the cycle Γ is also regular. Consequently, f has a regular and nondegenerate 2-heteroclinic cycle Γ connecting the repellers z_1 and z_2.

Therefore, assumptions (i) and (ii) in Theorem 5 hold with $k = 2$, $r_1 = 1$, $r_2 = 1.171$, $\lambda_{10} = 8$, $\lambda_{20} = 2$, $m_1 = 1$, $m_2 = 3$, x_{10} and x_{20} as the above. Consequently, f has a 2-heteroclinic cycle Γ connecting repellers z_1 and z_2. As is pointed out in Remark 6, when the norms used in Theorem 5 become a unified norm, the special case of Theorem 5 is consistent with Theorem 1. So, we can take some constants that appear in the proof of Theorem 1 as follows:

$$\begin{aligned}
\mu_{21} &= 2, \\
\mu_{22} &= 1, \\
\delta_{10} &= 0.2 < \frac{r_1 - \|z_1 - x_{10}\|}{2} \approx 0.323223, \\
\delta_{20} &= 0.175 < \frac{r_2 - \|z_2 - x_{20}\|}{2} \approx 0.307820, \\
\delta_{11} &= 0.1328 < \min\left\{\lambda_{10}\delta_{10}, \frac{r_2 - \|z_2 - x_{20}\|}{2} - \delta_{20}\right\} \\
&= 0.132820, \\
\delta_{21} &= 0.2 < \lambda_{20}\delta_{20} \\
&= 0.35, \\
\delta_{22} &= 0.2 < \mu_{21}\delta_{21} \\
&= 0.4, \\
\delta_{23} &= 0.025 < \min\left\{\mu_{22}\delta_{22}, \frac{r_1 - \|z_1 - x_{10}\|}{2} - \delta_{10}\right\} \\
&= 0.123223, \\
\varepsilon_1 &= \frac{(\lambda_{10} - 1)\delta_{23}}{1 + \delta_{23}} \approx 0.170732, \\
\varepsilon_2 &= \frac{(\lambda_{20} - 1)\delta_{11}}{1 + \delta_{11}} \approx 0.117232, \\
\varepsilon_3 &= \min\left\{\varepsilon_1, \varepsilon_2, \frac{\delta_{10}\lambda_{10} - \delta_{11}}{1 + \delta_{10}}\right\} \\
&= 0.117232, \\
\varepsilon_4 &= \min\left\{\varepsilon_1, \varepsilon_2, \frac{\min\{\lambda_{20}\delta_{20} - \delta_{21}, \mu_{21}\delta_{21} - \delta_{22}, \mu_{22}\delta_{22} - \delta_{23}\}}{1 + \max\{\delta_{20}, \delta_{21}, \delta_{22}\}}\right\} \\
&= 0.117232,
\end{aligned} \tag{84}$$

$\varepsilon_0 = \min\{\varepsilon_j, 1 \le j \le 4\} = 0.117232$. It is easy to check that the perturbation g satisfies condition (10) for $|\gamma| \le 0.0297$. Then, it follows from the result of Theorem 5 that the perturbed system $F = f + g$ also has a regular and nondegenerate heteroclinic cycle Γ' connecting repellers which is near to Γ. Consequently, F and f are chaotic in the sense of both Devaney and Li-Yorke.

As is done in Example 3, for a given γ, one can also directly calculate the heteroclinic cycle Γ' of F to check whether it is near to Γ of f. However, it is not easy to directly calculate such a cycle for high-dimensional maps. If there is the persistence of a heteroclinic cycle connecting repellers, then the computer simulations of them will not change very much. The behaviors of the unperturbed map f with an initial point $(x, y) = (0.1, 0.1)$ are illustrated in Figure 1. We do some computers simulations of F as γ increases from -0.0297 to 0 or from 0 to 0.0297, and find that all the simulations are similar with that of the original map f in Figure 1. Here, we give one simulation of F with an initial point $(x, y) = (0.1, 0.1)$ for $\gamma = 0.0297$, see Figure 2. We can see that Figure 2 is a small change to Figure 1, which shows that the heteroclinic cycle Γ' of F is near to Γ of f. When we let $|\gamma|$ continuous to increase, we find that the computer simulations gradually change until there is a big difference from that of the original map. This shows that the heteroclinic cycle Γ breaks or disappears. Are there new heteroclinic cycles connecting repellers not near Γ or new snap-back repellers to make the perturbed system still chaotic? It is an interesting question, while it is out of the scope of this paper and will be our further study.

Example 5. The original map f is taken as the following map on $\mathbf{R}^3$

$$f(x,y,z) = \begin{cases} (6x, 6y, 6z), & \text{if } (x,y,z) \in \overline{B}_1(O), \\ (4x-9, 4y-9, 4z-9), & \text{if } (x,y,z) \in \overline{B}_8(O)/\overline{B}_1(O), \\ \left(\sin\left[x-5+(y-5)^2\right], \right. & \sin\left[y-5+(z-5)^2\right], \\ \left. \sin\left[(x-5)^2+z-5\right]\right), & \text{if } (x,y,z) \notin \overline{B}_8(O), \end{cases} \tag{85}$$

where $O = (0,0,0)$ is the origin. The perturbed map g is taken as $g(x,y,z) = \gamma(x,y,z)$, where $(x,y,z) \in \mathbf{R}^3$ and γ is a real number. It is obvious that f is only continuously differentiable in some domains of $\mathbf{R}^3$, g is continuously differentiable in $\mathbf{R}^3$ and has a Lipschitz constant $L = |\gamma|$.

Theorem 5 is also used to verify the persistence of a heteroclinic cycle connecting repellers, and the process is similar to that of Example 4. So, we omit some details and only give some main results as follows. Assumptions (i) and (ii) in Theorem 5 hold with $k = 2$, $z_1 = (0,0,0)$, $z_2 = (3,3,3)$, $r_1 = 1$, $r_2 = 2.8$, $\lambda_{10} = 6$, $\lambda_{20} = 4$, $m_1 = 1$, $m_2 = 2$, $x_{10} = (0.5, 0.5, 0.5) \in B_1(z_1)$ and $x_{20} = (3.5, 3.5, 3.5) \in B_{2.8}(z_2) \subset B_8(O)$. The points z_1 and z_2 are two regular expanding fixed points of f in the Euclidean norm $\|\cdot\|$. In addition, $f(x_{10}) = z_2$, $x_{21} = f(x_{20}) = (5,5,5) \notin \overline{B}_8(O)$ and $f(x_{21}) = z_1$, that is, $f^2(x_{20}) = z_1$. Then f has a regular and nondegenerate 2-heteroclinic cycle

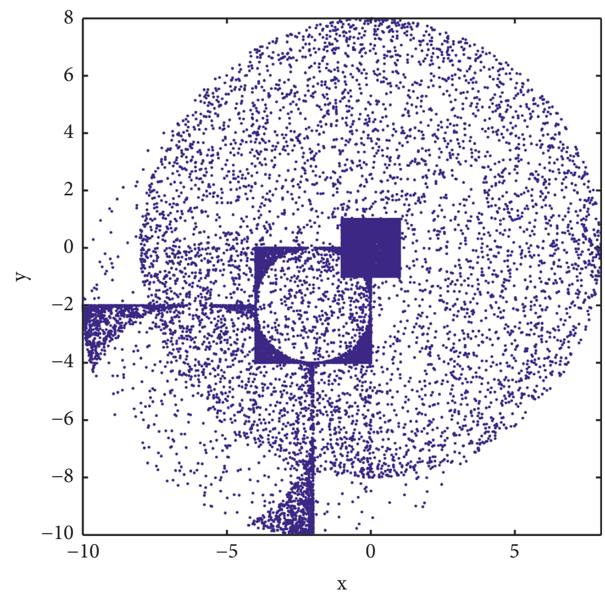

Figure 1: Complex behaviors of the original map f in the (x, y) space, where the initial point is take as $(0.1, 0.1)$ and $n = 0, 1, 2, \ldots, 20000$.

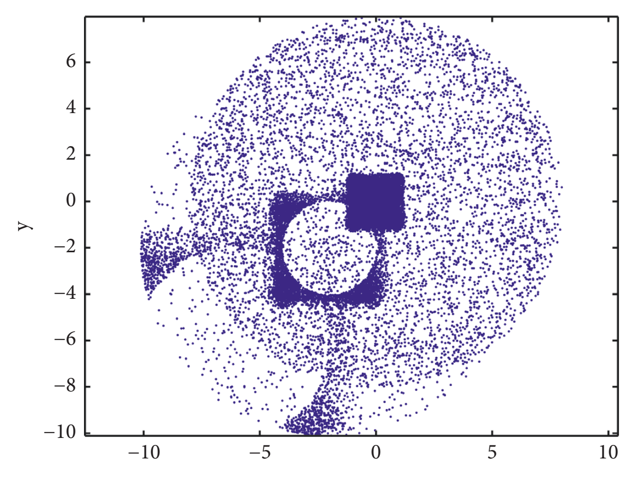

Figure 2: Complex behaviors of the perturbed map F in the (x, y) space, where $\gamma = 0.0297$, the initial point is taken as $(0.1, 0.1)$ and $n = 0, 1, 2, \ldots, 20000$.

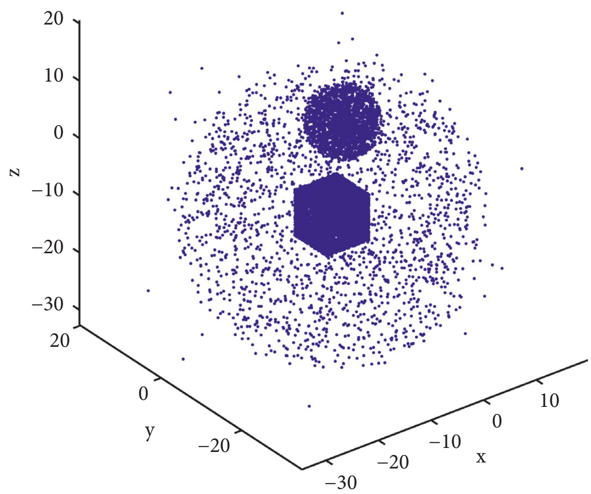

Figure 3: Complex behaviors of the original map f in the (x, y, z) space, where the initial point is take as $(0.1, 0.1, 0.1)$ and $n = 0, 1, 2, \ldots, 20000$.

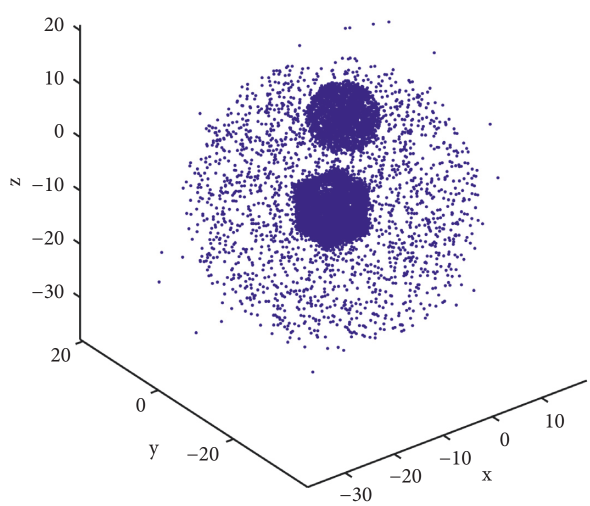

Figure 4: Complex behaviors of the perturbed map F in the (x, y, z) space, where $\gamma = 0.0133$, the initial point is taken as $(0.1, 0.1, 0.1)$ and $n = 0, 1, 2, \ldots, 20000$.

Γ connecting the repellers z_1 and z_2. Some constants that used to determine the range of perturbations are taken as follows:

$$\begin{aligned}
\mu_{21} &= 1,\\
\delta_{10} &= 0.03 < \frac{r_1 - \|z_1 - x_{10}\|}{2} \approx 0.066987,\\
\delta_{20} &= 0.2 < \frac{r_2 - \|z_2 - x_{20}\|}{2} \approx 0.966987,\\
\delta_{11} &= 0.04 < \min\left\{\lambda_{10}\delta_{10}, \frac{r_2 - \|z_2 - x_{20}\|}{2} - \delta_{20}\right\}\\
&= 0.18,\\
\delta_{21} &= 0.2 < \lambda_{20}\delta_{20}\\
&= 0.8,\\
\delta_{22} &= 0.03 < \min\left\{\mu_{21}\delta_{21}, \frac{r_1 - \|z_1 - x_{10}\|}{2} - \delta_{10}\right\}\\
&= 0.036987,\\
\varepsilon_1 &= \frac{(\lambda_{10} - 1)\delta_{22}}{1 + \delta_{22}} \approx 0.145631,\\
\varepsilon_2 &= \frac{(\lambda_{20} - 1)\delta_{11}}{1 + \delta_{11}} \approx 0.115385,\\
\varepsilon_3 &= \min\left\{\varepsilon_1, \varepsilon_2, \frac{\delta_{10}\lambda_{10} - \delta_{11}}{1 + \delta_{10}}\right\}\\
&= 0.115385,\\
\varepsilon_4 &= \min\left\{\varepsilon_1, \varepsilon_2, \frac{\min\{\lambda_{20}\delta_{20} - \delta_{21}, \mu_{21}\delta_{21} - \delta_{22}\}}{1 + \max\{\delta_{20}, \delta_{21}\}}\right\}\\
&= 0.115385,
\end{aligned} \tag{86}$$

$\varepsilon_0 = \min\{\varepsilon_j, 1 \le j \le 4\} = 0.115385$. It is also easy to check that the perturbation g satisfies condition (10) for $|\gamma| \le 0.0133$. Then, it follows from the result of Theorem 5 that the perturbed system $F = f + g$ also has a regular and nondegenerate heteroclinic cycle Γ' connecting repellers which is near to Γ. Consequently, F and f are chaotic in the sense of both Devaney and Li-Yorke.

The behaviors of the unperturbed map f with an initial point $(x, y, z) = (0.1, 0.1, 0.1)$ are illustrated in Figure 3. We also do some computers simulations of F as γ increases from -0.0133 to 0 or from 0 to 0.0133, and find that all the simulations are also similar with that of the original map f in Figure 3. Here, we give one simulation of F with an initial point $(x, y, z) = (0.1, 0.1, 0.1)$ for $\gamma = 0.0133$, see Figure 4. We can see that Figure 4 is also a small change to Figure 3, which shows that the heteroclinic cycle Γ' of F is near to Γ of f. When we let $|\gamma|$ continuous to increase, we also find that the computer simulations gradually change until there is a big difference from that of the original map. This shows that the heteroclinic cycle Γ breaks or disappears.

Remark 9. In the above examples, it only needs the Lipschitz constant L and the values of g at z_i, x_{ij} for $1 \le i \le 2$, $0 \le j \le m_i - 1$ to satisfy condition (10), and does not need to compute the values of g at any other points. This is very easy to check out and is very convenient in applications. Since there are few literature giving concrete methods to identify an exact expanding area of a fixed point, it is very hard to get the largest perturbation range. But we think that the results obtained in this paper are also useful in practice. Because when a perturbation range ε_0 is determined as in the above examples, it can ensure that the persistence is maintained for a large range of parameters. The perturbation range obtained in these examples may not be the largest one for the persistence to be maintained. A more precise perturbed range is needed in practice and this will also be our further research.

5. Conclusions

In this paper, we studied persistence of heteroclinic cycles connecting repellers in Banach spaces. We proved that if a map with a regular and nondegenerate heteroclinic cycle connecting repellers undergoes a small Lipschitz perturbation, then the perturbed map still has a regular and nondegenerate heteroclinic cycle connecting repellers. Consequently, the perturbed map and the original map are simultaneously chaotic in the sense of both Devaney and Li-Yorke. We believe that the results obtained in the paper will be useful for studying the existence of chaos and will provide certain theoretical basis for practical applications of heteroclinic cycles of connecting repellers. Compared with some related papers, three major achievements on the persistence are summarized as follows. One is that the maps discussed in the paper only need to be continuous or continuously differentiable in some domains instead of the whole space. Since a lot of maps may not be continuous or continuously differentiable in the whole space, our results are more general in practice than those in some related papers. The second is that an explicit expression for the range of perturbations is given, while most related papers did not give such an expression. The expression is determined by some properties of the original maps. It only needs to check out some values of the perturbation map at certain points in practice. This is very convenient and has great potential in applications. The third is that different repellers are allowed to expand in different norms in $\mathbf{R}^n$, while some related papers only used the single Euclidean norm to do that. This is very meaningful since it is more general in practice for some fixed points to expand in different norms. To show the validity of the theoretical results, we give some illustrative examples. However, the range of perturbations obtained in this paper is only a sufficient condition for the persistence to be maintained, and it may not be the largest one. Since it is hard to determine the exact area of a fixed point and few researches have given concrete methods to do this, it is not easy to find the largest range of perturbations and this will be our further research.

Data Availability

The data in this paper are acquired from the corresponding author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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