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Research Article

An Efficient Numerical Scheme for Solving Multiorder Tempered Fractional Differential Equations via Operational Matrix

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In this paper, we extend the operational matrix method to solve the tempered fractional differential equation, via shifted Legendre polynomial. Although the operational matrix method is widely used in solving various fractional calculus problems, it is yet to apply in solving fractional differential equations defined in the tempered fractional derivatives. We first derive the analytical expression for tempered fractional derivative for x^p, hence, using it to derive the new operational matrix of fractional derivative. By using a few terms of shifted Legendre polynomial and via collocation scheme, we were able to obtain a good approximation for the solution of the multiorder tempered fractional differential equation. We illustrate it using some numerical examples.

1. Introduction

Tempered fractional calculus is a type of fractional derivative/integral operator which multiplies an exponential factor to its power law kernel. This type of exponential tempering had been received increasing attention from researchers as having both mathematical and practical advantages [1, 2]. Several phenomena were best described by using this tempered fractional derivative/integral operator such as tempered fractional Brownian motion [3], epidemic modelling [4], and diffusion-wave equation [5]. Besides that, the tempered fractional model also been proven superior to the standard mechanism-based models in an experiment for quantifying colloid fate and transport in complex soil-vegetation systems [6].

In this research direction, reliable numerical schemes are needed to obtain the approximation solution for tempered fractional differential equations. This is because normally there are no exact solution or the analytical solution for these tempered fractional differential equations is difficult to obtain. To date, limited researches are done to tackle this problem, which include third-order semidiscretized schemes [7], two-dimensional Gegenbauer wavelets method [8], predictor–corrector scheme [9], and finite difference iterative method [10]. However, some of the established numerical schemes which already been successfully applied to solve fractional differential equations in the Caputo sense are still not been employed to solve these tempered fractional differential equations, which include the operational matrix method. Hence, in this paper, we aim to develop a reliable numerical scheme that involves an operational matrix via shifted Legendre polynomial to tackle this tempered fractional differential equation. The tempered fractional differential equations will be transformed into a system of algebraic equations; then, solving the system of algebraic equation will solve multiorder tempered fractional differential equations as follows:

$$\sum_{r=1}^{l} q_r {}^{T}D_x^{(\alpha_r,\beta_r)} f(x) = h(x), \qquad (1)$$
$$f^{(i)}(0) = d_i, i = 0, 1, \cdots, m-1,$$

where $f(x)$ is the unknown solution, ${}^{T}D_{x}^{(\alpha_r,\beta_r)}$ is tempered fractional derivatives, $q_r \in \mathbb{R}, r = 1, \cdots, l$ are constants, and $\alpha_r, \beta_r \geq 0$ are real derivative orders which β denotes the tempered coefficient, while $h(x)$ is the unhomogeneous terms.

To date, various numerical or analytical methods were derived to find the solution for different fractional calculus problems, such as [11–13]. On top of that, the operational matrix method via different types of the polynomial is one of the common numerical schemes which had been widely used in solving various types of fractional calculus problems, such as the poly-Bernoulli operational matrix for solving fractional delay differential equation [14], poly-Genocchi operational matrix for solving fractional differential equation [15], Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation [16], and Fibonacci wavelet operational matrix of integration for solving of nonlinear Stratonovich Volterra integral equations [17]. Recently, the operational matrix method had been successfully extended to solve other fractional operator problems, such as solving Prabhakar fractional differential equation [18]. Although the operational matrix method is widely used in solving various fractional calculus problems, it is yet to apply in solving fractional differential equations defined in the tempered fractional derivatives. Hence, we hope it can fill in this research gap. The main advantages of using this operational matrix method over other existing methods are its simplicity of implementation and programmable easily in using any computer algebra system. Besides that, if the fractional differential equations are in multiorder or having variable coefficients, operational matrix method is also efficient in finding the numerical solution.

The rest of this paper is as follows. Section 2 discusses some important concepts for tempered fractional calculus. Section 3 presents the shifted Legendre operational matrix for tempered fractional derivative and is followed by the procedure of solving tempered fractional differential equation via collocation scheme using this new shifted Legendre operational matrix. Some numerical examples and conclusion are presented in Sections 4 and 5.

2. Some Concepts regarding Tempered Fractional Calculus

Definition 1 (see [19, 20]). For $\alpha \in [0,1]$ and $\alpha, \beta \in \mathbb{C}$ where $\mathrm{Re}\,(\alpha) > 0, \mathrm{Re}\,(\beta) \geq 0$, the tempered fractional integral of order (α, β) for a function $f \in L^1[a,b]$ is given by

$$\begin{aligned} {}_0^T I_x^{(\alpha,\beta)} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} e^{-\beta(x-u)} f(u)\, du \\ &= \frac{e^{-\beta x}}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} e^{\beta u} f(u)\, du, x \in [a,b]. \end{aligned} \tag{2}$$

Definition 2 (see [20]). For $\alpha, \beta \in \mathbb{C}$ where $\mathrm{Re}\,(\alpha) > 0, \mathrm{Re}\,(\beta) \geq 0$, the Riemann-Liouville tempered fractional derivative of order (α, β) for a function $f \in L^1[a,b]$ is given by

$$\begin{aligned} {}_0^{RT} D_x^{(\alpha,\beta)} f(x) &= e^{-\beta x} \left({}_0^T D_x^{(\alpha)} e^{\beta x} f(x) \right) \\ &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-u)^{n-\alpha-1} e^{\beta(u)} f(u)\, du, x \in [a,b], \end{aligned} \tag{3}$$

where $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ is integer part of α.

For Caputo type of tempered fractional derivative, we take the derivative of the function under the integral (3), and we obtain Definition 3.

Definition 3 (see [20]). For $\alpha, \beta \in \mathbb{C}$ where $\mathrm{Re}\,(\alpha) > 0, \mathrm{Re}\,(\beta) \geq 0$. The Caputo tempered fractional derivative of order (α, β) for a function $f \in \mathrm{L}^1[a,b]$ is given by

$$ {}_0^{CT} D_x^{(\alpha,\beta)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} \frac{d^n e^{\beta(u)} f(u)}{du^n}\, du, x \in [a,b], \tag{4}$$

where $n = \lfloor \alpha \rfloor + 1$ and $\lfloor \alpha \rfloor$ is integer part of α.

The relationship between the tempered fractional derivative and tempered fractional integral is given by

$$ {}_0^T D_x^{(\alpha,\beta)} f(x) = \left(\frac{d}{dx} + \beta \right)^n \left({}_0^T I_x^{(n-\alpha,\beta)} f(x) \right), \tag{5}$$

where $n = \lfloor \mathrm{Re}\,(\alpha) \rfloor + 1$.

The tempered fractional derivative for function x^p, where p is integer positive, we obtain

$$\begin{aligned} {}_0^T D_x^{(\alpha,\beta)} x^p &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} \frac{d^n}{du^n} \left(e^{\beta u} u^p \right) du, x \in [a,b], \\ &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} \frac{d^n}{du^n} \sum_{k=0}^{\infty} \frac{\beta^k u^{k+p}}{k!}\, du \\ &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \frac{\Gamma(k+p+1) u^{k+p-n}}{\Gamma(k+p+1-n)}\, du \\ &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-n)} \int_0^x (x-u)^{n-\alpha-1} u^{k+p-n}\, du. \end{aligned} \tag{6}$$

By using integration $\int_0^x (x-t)^{a-1} t^{b-1} dt = B(a,b) x^{a+b-1} = (\Gamma(a)\Gamma(b)/\Gamma(a+b)) x^{a+b-1}$ where $B(a,b)$ is beta function and $a, b > 0$, we obtain

$$\begin{aligned} T_0 D_x^{(\alpha,\beta)} x^p &= \frac{e^{-\beta x}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p-n+1)} \frac{\Gamma(n-\alpha)\Gamma(k+p-n+1)}{\Gamma(n-\alpha+k+p-n+1)} x^{n-\alpha+k+p-n} \\ &= e^{-\beta x} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \frac{\Gamma(k+p+1)}{\Gamma(k+p-\alpha+1)} x^{k+p-\alpha}. \end{aligned} \tag{7}$$

For the function $f(x) = (x-a)^p$ with $\mathrm{Re}\,(p) > -1$, we have [21],

$$T_aD_x^{(\alpha,\beta)}(x-a)^k = (x-a)^{k-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}{}_1F_1(-\alpha\,; k-\alpha+1;-\beta(x-a)). \tag{8}$$

Here, we take k as positive integer number. For $a=0$, both expressions in (7) and (8) have the same result.

While using (8) to derive the operational matrix, we need the following results related to the hypergeometric function:

$${}_1F_1(a\,;c\,;x) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(c)}{\Gamma(c+k)}\frac{x^k}{k!}, \tag{9}$$

$${}_2F_2(a,b\,;c,d\,;x) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b+k)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+k)}\frac{\Gamma(d)}{\Gamma(d+k)}\frac{x^k}{k!}, \tag{10}$$

$$\int_0^1 {}_1F_1(a\,;c\,;x)dx = \frac{c-1}{a-1}{}_1F_1(a-1\,;c-1\,;x), \tag{11}$$

$$\int_0^1 x^l{}_1F_1(a\,;c\,;mx)dx = \frac{1}{l+1}{}_2F_2(a,l+1\,;c,l+2\,;m). \tag{12}$$

In another aspect, the solution for single tempered fractional differential equation, i.e., ${}^TD_x^{\alpha,\beta}y(x) = \kappa y(x) + h(x)$ for $\alpha \in (0,1]$, $\beta > 0$, and the initial condition is $y(0) = y_0$ can be expressed as follows [20]:

$$y(x) = y_0e^{-\beta x}E_{\alpha,1}(\kappa x^\alpha) + \int_0^x h(x-s)e^{-\beta s}s^{\alpha-1}E_{\alpha,\alpha}(\kappa s^\alpha)ds. \tag{13}$$

where $E_{a,b}(x) = \sum_{k=0}^{\infty}x^k/\Gamma(ax+b)$. However, this type of solution may fail when there are involving multiorder tempered fractional differential equations. Hence, we proposed to solve these multiorder tempered fractional differential equations via collocation scheme using shifted Legendre operational matrix. Other types of polynomials can be used also to derive the new operational matrix to tackle the tempered fractional derivative.

3. Shifted Legendre Operational Matrix for Tempered Fractional Derivative

3.1. Shifted Legendre Polynomials. The Legendre polynomials is an orthogonal polynomial on the interval $[-1,1]$. One of the ways to obtain Legendre polynomials is via recurrence relation as follows:

$$L_{i+1}(t) = \frac{2i+1}{i+1}tL_i(t) - \frac{i}{i+1}L_{i-1}(t), i = 1,2,\cdots, \tag{14}$$

where $L_0(t) = 1$ and $L_1(t) = t$. The Legendre polynomials in domain $[-1,1]$ can be transformed into the domain of $[0,1]$ by using $t = 2x-1$, which we get shifted Legendre polynomials, $\tilde{L}_i(x)$ as follows:

$$\tilde{L}_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1}\tilde{L}_i(x) - \frac{i}{i+1}\tilde{L}_{i-1}(x), i = 1,2,\cdots, \tag{15}$$

where $\tilde{L}_0(x) = 1$ and $\tilde{L}_1(x) = 2x-1$. Besides that, the shifted Legendre polynomials $\tilde{L}_i(x)$ of degree i can be obtained via the analytical form:

$$\tilde{L}_i(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!x^k}{(i-k)!(k!)^2}, i = 0,1,2,\cdots, \tag{16}$$

where $\tilde{L}_i(0) = (-1)^i$ and $\tilde{L}_i(1) = 1$. The orthogonality condition is

$$\int_0^1 \tilde{L}_i(x)\tilde{L}_j(x)dx = \begin{cases}\dfrac{1}{2i+1}, & \text{for } i = j,\\ 0, & \text{for } i \neq j.\end{cases} \tag{17}$$

The shifted Legendre polynomials have a nice property that is useful for function approximation. In this case, a square integrable function $f(x) \in L^2[0,1]$ can be expressed in terms of shifted Legendre polynomials as follows:

$$f(x) = \sum_{j=0}^{\infty}c_j\tilde{L}_j(x), \tag{18}$$

where the coefficients c_j are given by

$$c_j = (2j+1)\int_0^1 f(x)\tilde{L}_j(x)dx, j = 0,1,2,\cdots. \tag{19}$$

In order to use equation (18) to approximate the function, normally, we truncated after $(N+1)$ terms shifted Legendre polynomials as follows:

$$f_N^*(x) = \sum_{j=0}^{N}c_j\tilde{L}_j(x) = C^T\mathbf{\Phi}_L(x), \tag{20}$$

where the shifted Legendre coefficient vector, C is given by $C^T = [c_0, c_1, \cdots, c_N]$ and the shifted Legendre vector, $\mathbf{\Phi}_L(x)$ can be expressed as

$$\mathbf{\Phi}_L(x) = \left[\tilde{L}_0(x), \tilde{L}_1(x), \cdots, \tilde{L}_N(x)\right]^T. \tag{21}$$

3.2. Shifted Legendre Polynomial Operational Matrix. In this subsection, we derive the new shifted Legendre operational matrix for tackling tempered fractional derivative. We have the following theorem.

Theorem 4. *Let $\Phi_{\tilde{L}}(x)$ be the shifted Legendre vector as shown in (20). For $n-1<\alpha<n, n \in \mathbb{N}$, then,*

$$ {}_0^T D_x^{(\alpha,\beta)} \Phi_{\tilde{L}}(x) = \mathbf{P}_{x;\tilde{L}}^{\alpha,\beta} \Phi(x), \tag{22}$$

where $\mathbf{P}_x^{\alpha,\beta}$ is the $N \times N$ operational matrix of tempered fractional derivative of order α, β defined as

$$\mathbf{P}_x^{\alpha,\beta} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \xi_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \xi_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \xi_{\lceil\alpha\rceil,N-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \xi_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \xi_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \xi_{i,N-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{N-1} \xi_{N-1,0,k} & \sum_{k=\lceil\alpha\rceil}^{N-1} \xi_{N-1,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{N-1} \xi_{N-1,N-1,k} \end{bmatrix}, \tag{23}$$

where $\xi_{i,j,k}$ is given by

$$\xi_{i,j,k} = (2j+1)\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)} \times {}_2F_2(-\alpha, 1+k-\alpha+l; k-\alpha+1, 2+k-\alpha+l; -\beta), \tag{24}$$

where $i=\lceil\alpha\rceil, \cdots, N-1, j=0,1,2,\cdots,N-1$.

Proof. By using equation (8) and letting $a=0$, the tempered fractional derivative for x^k is given as follows:

$$ {}_0^T D_x^{(\alpha,\beta)} x^k = x^{k-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}{}_1F_1(-\alpha; k-\alpha+1; -\beta x). \tag{25}$$

□

Using the expression as in (25), the explicit expression of tempered fractional derivative of the i-th degree shifted Legendre polynomials which is the $(i+1)$-th element of $\Phi_L(x)$ is computed:

$$\begin{aligned} {}_0^T D_x^{(\alpha,\beta)} \tilde{L}_i(x) &= \sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!^2}\left({}_0^T D_x^{(\alpha,\beta)} x^k\right), i=0,1,2,\cdots \\ &= \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!^2}\left[x^{k-\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}{}_1F_1(-\alpha; k-\alpha+1; -\beta x)\right] \\ &= \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!}\left[\frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)}{}_1F_1(-\alpha; k-\alpha+1; -\beta x)\right]. \end{aligned} \tag{26}$$

The elements $\rho_{i,j}$ of the operational matrix $\mathbf{P}_{x;\tilde{L}}^{\alpha,\beta}$ are computed by taking inner product for the tempered fractional derivative of shifted Legendre polynomials, ${}_0^T D_x^{(\alpha,\beta)} \tilde{L}_i(x)$ with shifted Legendre polynomials, $\tilde{L}_j(x), j=0,1,\cdots,N-1$:

$$ {}_0^T D_x^{(\alpha,\beta)} \tilde{L}_i(x) = \sum_{j=0}^{N-1}\rho_{i,j}\tilde{L}_j(x), \tag{27}$$

$$\rho_{i,j} = \left\langle {}_0^T D_x^{(\alpha,\beta)} \tilde{L}_i(x), \tilde{L}_j(x)\right\rangle = \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!} \left[\frac{1}{\Gamma(k-\alpha+1)}\left\langle x^{k-\alpha}{}_1F_1(-\alpha; k-\alpha+1; -\beta x), \tilde{L}_j(x)\right\rangle\right], \tag{28}$$

where the inner product can be computed as follows

$$\begin{aligned} &\left\langle x^{k-\alpha}{}_1F_1(-\alpha; k-\alpha+1; -\beta x), \tilde{L}_j(x)\right\rangle \\ &= (2j+1)\int_0^1 x^{k-\alpha}{}_1F_1(-\alpha; k-\alpha+1; -\beta x)\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2}x^l dx \\ &= (2j+1)\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2}\int_0^1 x^{k-\alpha}{}_1F_1(-\alpha; k-\alpha+1; -\beta x)x^l dx \\ &= (2j+1)\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2}\left(\frac{{}_2F_2(-\alpha, 1+k-\alpha+l; k-\alpha+1, 2+k-\alpha+l; -\beta)}{1+k-\alpha+l}\right). \end{aligned} \tag{29}$$

The integration in (29) can be obtained via the formula in (10). Putting equation (29) into (28), we obtain

$$\begin{aligned} \rho_{i,j} &= \left\langle {}_0^T D_x^{(\alpha,\beta)} \tilde{L}_i(x), \tilde{L}_j(x)\right\rangle \\ &= \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!}\left[\frac{(2j+1)}{\Gamma(k-\alpha+1)} \times \sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2}\left(\frac{{}_2F_2(-\alpha, 1+k-\alpha+l; k-\alpha+1, 2+k-\alpha+l; -\beta)}{1+k-\alpha+l}\right)\right] \\ &= (2j+1)\sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)} \\ &\quad \times {}_2F_2(-\alpha, 1+k-\alpha+l; k-\alpha+1, 2+k-\alpha+l; -\beta). \end{aligned} \tag{30}$$

Setting $\rho_{i,j} = \sum_{k=\lceil\alpha\rceil}^{i}\xi_{i,j,k}$

$$\xi_{i,j,k} = (2j+1)\frac{(-1)^{i+k}(i+k)!}{(i-k)!k!(k-\alpha)!}\sum_{l=0}^{j}\frac{(-1)^{j+l}(j+l)!}{(j-l)!l!^2(1+k-\alpha+l)} \times {}_2F_2(-\alpha, 1+k-\alpha+l; k-\alpha+1, 2+k-\alpha+l; -\beta). \tag{31}$$

Hence, each element of shifted Legendre operational matrix for tempered fractional derivative is obtained.

To test the accuracy of the operational matrix derived in Theorem 4, we use it to approximate tempered fractional derivative for $\tilde{L}_2(x) = 6x^2 - 6x + 1$ and $\tilde{L}_3(x) = 20x^3 - 30x^2$

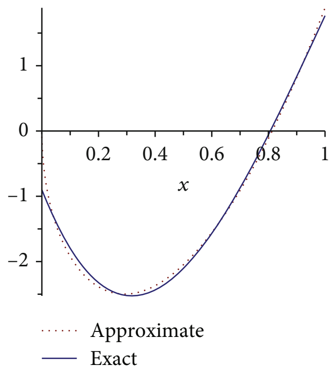

Figure 1: Comparison of the exact solution and approximation for $\tilde{L}_2(x)$.

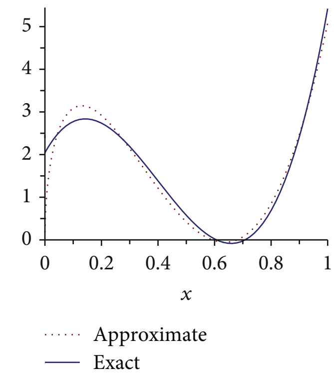

Figure 2: Comparison of the exact solution and approximation for $\tilde{L}_3(x)$.

$+\,12x - 1$. For $N = 4$, we obtain the following operational matrix $\mathbf{P}_x^{\alpha,\beta}$ when $\alpha = 1/2, \beta = 1$,

$$\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0, \\ 1.786057010727 & 1.257808290370 & -0.157033776306 & 0.090092226619, \\ -1.257808290370 & 1.678966426855 & 1.690298665024 & -0.345920769399, \\ 1.629023234421 & -0.432490374653 & 2.103160984111 & 2.122192953652. \end{bmatrix} \tag{32}$$

Figures 1 and 2 show the comparison between the exact solution for tempered fractional derivative for $\tilde{L}_2(x)$ and $\tilde{L}_3(x)$ with the approximation using operational matrix as in (32). Accuracy of the approximate can be increased with increasing the N.

4. Solving Tempered Fractional Differential Equation Using Shifted Legendre Operational Matrix Method and Error Analysis

This section consists of an explanation for the proposed method that combines collocation scheme with shifted Legendre polynomials operational matrix of tempered fractional derivative to solve multiorder tempered fractional differential equations.

$$\begin{aligned} &\sum_{r=1}^{l} q_r\, {}^{T}D_x^{(\alpha_r,\beta_r)} f(x) = h(x), \\ &f^{(i)}(0) = d_i, i = 0, 1, \cdots, m-1. \end{aligned} \tag{33}$$

Here, we present the general procedure for solving multiorder tempered fractional differential equations as in equation (33) via shifted Legendre operational matrix.

Step 1. The unknown function, i.e., the solution, $f(x)$ is approximated by truncated shifted Legendre polynomials,

$$f(x) \approx f_N(x) \approx \mathbf{C}^T \tilde{\mathbf{L}}(x), \tag{34}$$

where $\mathbf{C}^T = [c_0, c_1, c_2, \cdots, c_N]$. The tempered fractional derivative of equation (33) is approximated using the shifted Legendre operational matrix of tempered fractional derivative as in equations (22)–(24).

$${}^{T}D_x^{(\alpha,\beta)} f(x) \approx \mathbf{C}^T \mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)} \mathbf{\Phi}_{\tilde{L}}(x). \tag{35}$$

Remark 5. The function, $h(x)$ in the RHS of equation (33) can also approximated in term of truncated shifted Legendre polynomials as follows:

$$h(x) \approx \mathbf{H}^T \mathbf{\Phi}_{\tilde{L}}(x), \tag{36}$$

where $\mathbf{H} = [h_i]^T$ and the coefficients h_i are computed using equation (19). However, to increase the accuracy of the method and also increase speed of its computer implementation, collocation scheme can be applied directly to these known functions.

Step 2. From Step 1, the following is obtained:

$$\sum_{r=1}^{l} q_r \mathbf{\Phi}_{\tilde{L}}^T(x) \left(\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)}\right)^T \mathbf{C} = h(x). \tag{37}$$

After some algebraic manipulations, we obtain $\mathbf{\Phi}_{\tilde{L}}^T(x)$ $(\sum_{r=1}^{l} q_r (\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)})^T \mathbf{C}) - h(x) = 0$. Thus, the residual is

$$\mathscr{R}(x) = \mathbf{\Phi}_{\tilde{L}}^T(x) \left(\sum_{r=1}^{l} q_r \left(\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)}\right)^T \mathbf{C} \right) - h(x) = 0. \tag{38}$$

Step 3. Due to the set of shifted Legendre polynomials basis, $\mathbf{\Phi}_{\tilde{L}}^T(x) = [\tilde{L}_0(x)\tilde{L}_1(x) \cdots \tilde{L}_N(x)]$ is linearly independent, we obtain

$$\mathbf{\Phi}_{\tilde{L}}^T(x) \sum_{r=1}^{l} q_r \left(\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)}\right)^T \mathbf{C} = h(x). \tag{39}$$

We obtain a system of $N+1$ algebraic equations from equation (39). The initial condition in equation (33) is also approximated in term of shifted Legendre polynomials:

$$f^{(i)}(0) = d_i, i = 0, \cdots, m-1,$$
$$\mathbf{\Phi}_{\tilde{L}}^{T}(0)\left(\mathbf{P}_{0+;\tilde{L}}^{i}\right)^{T}\mathbf{C} = d_i. \tag{40}$$

Step 4. Select $N-m$ equations from equation (39) and combine with initial conditions from equation (40), we obtain a system of N linear algebraic equations in term of $\mathbf{C}$. Hence, solve the system with using any suitable numerical methods. Then, the approximate solution can be computed by following equation:

$$f^{*}(x) = \mathbf{C}^{T}\mathbf{\Phi}_{\tilde{L}}(x). \tag{41}$$

4.1. Error Analysis. To discuss the error analysis for our method, we follow the approach done in [22] where the alternating Legendre polynomials are applied to derive the operational matrix to solve the fractional differential equations problem defined in the classical Caputo sense.

Lemma 6. *Suppose that $f(x) \in C^{n+1}(\Theta)$ and $f(x) \approx f_N(x) \approx \mathbf{C}^{T}\tilde{\mathbf{L}}(x)$ be its expansion in terms of shifted Legendre polynomials, as described by equation (34). Then*

$$\|f(x) - f_N(x)\|_2 \le \frac{M}{(N+1)!2^{2N+1}}, \tag{42}$$

where M is a constant such that $|f^{(N+1)}(x)| \le M$.

Proof. Assume that $s_n(x)$ is the interpolating polynomials to $f(x)$ at points x_j, where x_j be the roots of the shifted Chebyshev polynomials of degree $n+1$, then

$$f(x) - s_n(x) = \frac{f^{(n+1)}(\varsigma_x)}{(n+1)!}\prod_{j=0}^{n}\left(x - x_j\right), \varsigma_x \in \Theta = [0,1]. \tag{43}$$

As proof in [22], the above approximation has the bound as follows:

$$|f(x) - s_n(x)| \le \frac{M}{(n+1)!2^{2n+1}}, \forall x \in \Theta. \tag{44}$$

Then, we have

$$\begin{aligned}\|f(x) - f_N(x)\|_2^2 &\le \|f(x) - s_N(x)\|_2^2 \\ &= \int_0^1 \|f(x) - s_N(x)\|^2 dx = \int_0^1 \left(\frac{M}{(N+1)!2^{2N+1}}\right)^2 dx \\ &= \left(\frac{M}{(N+1)!2^{2N+1}}\right)^2.\end{aligned} \tag{45}$$

Table 1: Absolute errors obtained by proposed method with $N = 4, 6$ for Example 1.

x	Exact solution	Abs. error ($N=4$)	Abs. error ($N=6$)
0.1	0.402	1.61661E-02	2.72390E-03
0.2	0.816	1.87810E-02	6.74832E-03
0.3	1.254	1.25442E-02	6.06411E-03
0.4	1.728	2.15555E-02	1.63810E-03
0.5	2.250	7.68546E-02	2.20223E-03
0.6	2.832	1.22791E-02	1.38416E-03
0.7	3.486	6.92580E-03	4.29494E-03
0.8	4.224	1.30742E-02	7.55223E-03
0.9	5.058	5.24204E-02	9.99599E-03
1.0	6.000	1.15813E-02	8.14496E-02

Hence, take the square root of both sides, we obtain

$$\|f(x) - f_N(x)\|_2 \le \frac{M}{(N+1)!2^{2N+1}}. \tag{46}$$

This completes the proof. □

In order to perform the error estimation for this new numerical scheme, we apply residual correction procedure. From equation (39), i.e., $\mathscr{R}(x) = \mathbf{\Phi}_{\tilde{L}}^{T}(x)(\sum_{r=1}^{l} q_r (\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)})^{T}\mathbf{C}) - h(x) = 0$, hence

$$\mathbf{\Phi}_{\tilde{L}}^{T}(x)\left(\sum_{r=1}^{l} q_r\left(\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)}\right)^{T}\mathbf{C}\right) - h(x) = 0. \tag{47}$$

If $N \longrightarrow \infty$, using the operational matrix via shifted Legendre polynomials to approximate multiorder tempered fractional derivative and approximate $h(x)$ via shifted Legendre polynomials, we obtain

$$\left|\mathbf{\Phi}_{\tilde{L}}^{T}(x)\sum_{r=1}^{l} q_r\left(\mathbf{P}_{x;\tilde{L}}^{(\alpha_r,\beta_r)}\right)_{(N\times N)}^{T} - \sum_{r=1}^{l} q_r{}^{T}D_x^{(\alpha_r,\beta_r)}f(x)\right| \approx 0, N \longrightarrow \infty. \tag{48}$$

For our proposed method, N is finite. Hence, suppose m term of shifted Legendre polynomials had been used, then, small error, e_m, is inevitable.

$$\sum_{r=1}^{l}\left|D_x^{(\alpha_r,\beta_r)}f(x) - \mathbf{P}_{x;L}^{(\alpha_r,\beta_r)}\right|_2 = e_m, N = m. \tag{49}$$

Let e_m^{*} is the approximation solution of equation (1) obtained by the shifted Legendre operational matrix method, if $\|e_m - e_m^{*}\| < \varepsilon$ is sufficiently small; then, the absolute errors e_m can be estimated by e_m^{*}. Hence, we obtain the optimal value m (i.e., N).

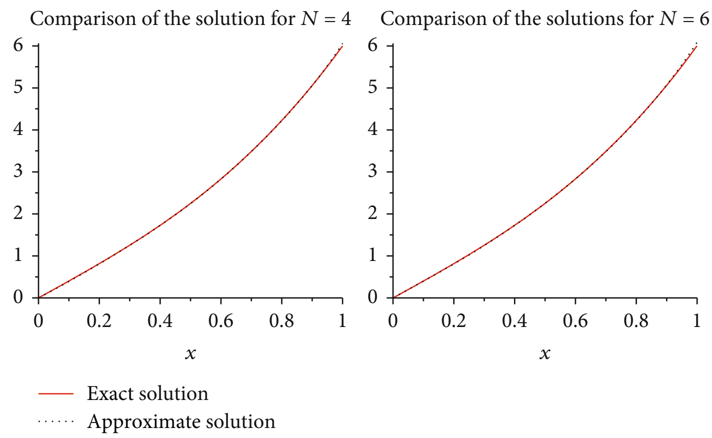

Figure 3: The comparison result of the approximate and exact solutions for Example 1 ($N = 4, 6$).

Table 2: Absolute errors obtained by proposed method with $N = 6, 8$ for Example 2.

x	Exact solution	Abs. error ($N = 6$)	Abs. error ($N = 8$)
0.1	0.00002	6.44679E-05	2.57668E-07
0.2	0.00064	2.17039E-05	4.92156E-07
0.3	0.00486	8.06877E-06	6.44773E-07
0.4	0.02048	3.64403E-06	5.63021E-07
0.5	0.06250	3.18313E-05	5.17865E-07
0.6	0.15552	3.62011E-05	3.12227E-07
0.7	0.33614	5.94866E-07	9.34225E-07
0.8	0.65536	3.47375E-05	2.46696E-07
0.9	1.18098	7.69841E-05	8.54648E-08
1.0	2.00000	6.15397E-04	4.12816E-05

5. Numerical Examples

In this section, we will apply the new operational matrix for tempered fractional derivative to solve the tempered fractional differential equations.

Example 1. Consider the multiorder tempered fractional differential equation as follows:

$$\begin{aligned}
&{}_O^T D_x^{(1/2,1/2)} y(x) + {}_0^T D_x^{(1/2,1/4)} y(x) \\
&\quad = x^2 y(x) - 2x^5 - 4x^3 + \sqrt{2}\big(x^3 + 3x^2 - x \\
&\qquad + 5\big) \operatorname{erf}\left(\sqrt{\frac{x}{2}}\right) + \frac{2x}{\sqrt{\pi x}} e^{-x/2}\left(x^2 + 2x - 1\right) \\
&\qquad + \left(x^3 + 6x^2 - 10x + 28\right) \operatorname{erf}\left(\frac{\sqrt{x}}{2}\right) \\
&\qquad + \frac{2x}{\sqrt{\pi x}} e^{-x/4}\left(x^2 + 4x - 10\right),
\end{aligned} \tag{50}$$

with initial condition $y(0) = 0$. The exact solution is known as $y(x) = 2x^3 + 4x$.

Solution: by using $N = 4, 6$, we have the following numerical result as shown in Table 1 and Figure 3. Only using few terms of shifted Legendre polynomials, we were able to obtain good result for this multiorder tempered fractional differential equation.

Example 2. Consider a simple tempered fractional differential equation as follows:

$$\begin{aligned}
&{}_O^T D_x^{(1/2,3/4)} y(x) \\
&\quad = -2.0\, x^{15/2} + 1.128379167\, \mathrm{e}^{-0.75\,x} x^5 + 3.009011112\, \mathrm{e}^{-0.75\,x} x^4 \\
&\qquad + 1.732050807\, x^{11/2} \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad + 5.773502692\, x^{9/2} \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad - 6.018022225\, \mathrm{e}^{-0.75\,x} x^3 \\
&\qquad - 7.698003593\, x^{7/2} \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad + 15.39600718\, x^{5/2} \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad + 13.37338272\, \mathrm{e}^{-0.75\,x} x^2 \\
&\qquad - 25.66001197\, x^{3/2} \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad + 23.94934450 \operatorname{erf}\left(0.8660254038\sqrt{x}\right) \\
&\qquad \cdot \sqrt{x} - 23.40341976\, \mathrm{e}^{-0.75\,x} x,
\end{aligned} \tag{51}$$

with initial condition $y(0) = 0$. The exact solution is known as $y(x) = 2x^5$.

Solution: by using $N = 6, 8$, we have the following numerical result as shown in Table 2 and Figure 4. Again, by only using few terms of shifted Legendre polynomials, we were able to obtain good result for this tempered fractional differential equation.

Example 3. Consider the tempered fractional differential equation taken from [23] as follows:

$${}_0^T D_x^{(\alpha,\beta)} y(x) = e^{-\beta x}\left(\frac{\Gamma(6)}{\Gamma(6-\alpha)} x^{5-\alpha} - e^{-\beta x} x^{10} + e^{\beta x} y^2(x)\right), 0 < \alpha < 1, \tag{52}$$

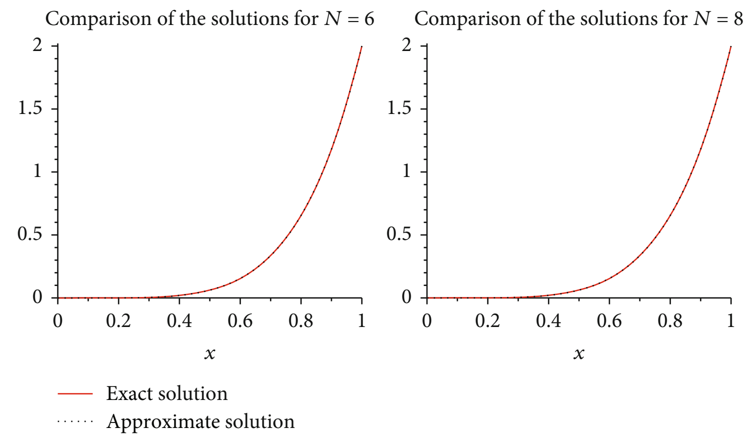

FIGURE 4: The comparison result of the approximate and exact solutions for Example 2 ($N = 6, 8$).

TABLE 3: Comparison of errors for proposed method with Jacobi-predictor-corrector algorithm [23] for Example 3 with $\alpha = 0.4$.

β	N	Max abs. error	Stepsize (iteration)	Error
0	6	1.06818E-04	1/10	2.4208E-04
0	8	4.10270E-06	1/20	4.9371E-06
0	10	3.37294E-07	1/40	1.0390E-07
3	6	2.43410E-04	1/10	7.4482E-07
3	8	8.27866E-06	1/20	6.8759E-08
3	10	4.77932E-07	1/40	3.1715E-09

TABLE 4: Relative absolute errors obtained by proposed method with $N = 6, 8, 10$ for Example 4.

x	Exact solution	RAE ($N = 6$)	RAE ($N = 8$)	RAE ($N = 10$)
0.2	0.0725173321	1.80058E-03	9.61065E-04	2.41946E-04
0.4	0.2620226201	1.09846E-04	6.15192E-05	9.17638E-06
0.6	0.5332522504	2.56602E-04	4.80993E-05	2.88812E-05
0.8	0.8586273304	7.19889E-04	7.26678E-05	2.39416E-06

with initial condition $dy(0)/dx = 0$. The exact solution is given as $y(x) = e^{-\beta x}x^5$. This can be verified by using the procedure as in equations (6) and (7), where ones should get

$$ {}_0^TD_x^{(\alpha,\beta)} e^{-\beta x}x^p = \frac{p+1}{\Gamma(p-\alpha+1)} e^{-\beta x}x^{p-\alpha}, \alpha > -1. \tag{53}$$

Solution: by using $N = 6, 8, 10$ and $\alpha = 0.4$, we have the following numerical result for $\beta = 0, 3$, as shown in Table 3. We compare our solution with Jacobi-predictor-corrector algorithm [23]. Again, only using few terms of shifted Legendre polynomials, we were able to obtain good results compare with Jacobi-predictor-corrector algorithm. More specifically, say if $N = 10$ has been used, the size of each subinterval is 0.1, which is equivalent to the iteration Jacobi-predictor-corrector algorithm with stepsize 1/10. For $\beta = 3$, our absolute error is 4.779320E-07, $(N = 10)$, which is comparable with 7.4482E-07, stepsize = 1/10.

Example 4. Consider the multiorder tempered fractional differential equation as follows:

$$ {}_0^TD_x^{(0.5,0.5)}y(x) + {}_0^TD_x^{(0.25,0.5)}y(x) = \frac{16}{3}\frac{e^{-x/2}x^{3/2}}{\sqrt{\pi}} + \frac{64}{21}\frac{e^{-x/2}x^{7/4}}{\sqrt{3/4}}, \tag{54}$$

with initial condition $y(0) = 0$. The exact solution is given as $y(x) = 2e^{-x/2}x^2$.

Solution: by using $N = 6, 8, 10$, we have the following numerical result as shown in Table 4. Again, for the relative absolute errors presented, it is obvious that by only using few terms of shifted Legendre polynomials via its operational matrix, we able to obtain good results for this multiorder tempered fractional differential equations.

6. Conclusion

In this paper, we manage to derive a new operational matrix for tempered fractional derivatives. Hence, we use it to solve tempered fractional differential equations. The proposed method is easy to apply and yet able to give accurate results. The accuracy of the method can be improved by increasing the number of the term of shifted Legendre polynomials. The proposed method should extend to solve some other tempered fractional calculus problems, such as tempered fractional partial differential equations. Besides that, this operational matrix also can be extended to solve other kinds of fractional calculus problems such as those in [24–27].

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no conflicts of interest.

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