Research Article

Kernel L-Ideals and L-Congruence on a Subclass of Ockham Algebras

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In this paper, we study L-congruences and their kernel in a subclass $\mathcal{K}_{n,0}$ of the variety of Ockham algebras A. We prove that the class of kernel L-ideals of an Ockham algebra forms a complete Heyting algebra. Moreover, for a given kernel L-ideal ξ on A, we obtain the least and the largest L-congruences on A having ξ as its kernel.

1. Introduction

The concept of an Ockham algebra was first introduced by Berman [1], in 1977. Next, it has been studied by Urquhart et al. [2], Goldberg et al. [3, 4], and Blyth and Varlet [5]. Blyth and Silva [5] presented the concept of kernel ideals in Ockham algebra. Wang et al. [6] presented Congruences and kernel ideals on a subclass $\mathcal{K}_{n,0}$ of the variety of Ockham Algebras (in which $h^{2n} = id_A$). The varieties of Boolean algebras, De Morgan algebras, Kleene algebras, and Stone algebras are some of the well-known subvarieties of Ockham algebra. We see [7] for the basic concepts of the class of Ockham algebras.

On the other side, for the first time, the concept of fuzzy sets was presented by Zadeh as an extension of the classical notion of set theory [8]. He defined a fuzzy subset of a nonempty set K as a function from K to $[0, 1]$. Goguen in [9] presented the notion of L-fuzzy subsets by replacing the interval $[0, 1]$ with a complete lattice L in the definition of fuzzy subsets. Swamy and Swamy [10] studied that complete lattices that fulfill the infinite meet distributive law are the most appropriate candidates to have the truth values of general fuzzy statements.

The study of fuzzy subalgebras of different algebraic structures has been begun after Rosenfeld presented his paper [11] on fuzzy subgroups. This paper has provided sufficient motivation to researchers to study the fuzzy subalgebras of different algebraic structures.

Fuzzy congruence relations on algebraic structures are fuzzy equivalence relations that are compatible (in a fuzzy sense) with all fundamental operations of the algebra. The concept of fuzzy congruence relations was presented in different algebraic structure: in semigroups (see [12, 13]), in groups, semirings, and rings (see [14–19]), in modules and vector spaces (see [20, 21]), in lattices (see [22, 23]), in universal algebras (see [24, 25]), and more recently in MS-algebras and Ockham Algebras (see [26–28]).

Initiated by the above results, we present Kernel L-ideals and L-Congruence on a subclass $\mathcal{F}\mathcal{I}_k(A)$ of $\mathcal{K}_{n,0}$ of the variety of Ockham algebras and study their characteristics. We prove that the class of kernel L-ideal $\mathcal{L}\mathcal{I}_k(A)$ of $\mathcal{K}_{n,0}$-algebra A forms Heyting algebras. Also, we get the least and the biggest L-congruences, respectively, on $\mathcal{K}_{n,0}$-algebra A having a given L-ideal as an L-kernel.

2. Preliminaries

This section contains basic definitions and important results which will be used in the sequel.

Definition 1. (see [5]). An algebra $(A; \wedge, \vee, h, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is said to be an Ockham algebra if

(1) $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice

(2) $h: A \mapsto A$ is a dual endomorphism

In this paper, for simplicity, any Ockham algebra $(A; \wedge, \vee, h, 0, 1)$ is denoted by a pair (A, h). If the dual endomorphism on A satisfies that $h^{2n} = id_A$, then this subclass is the Berman subclass $\mathcal{K}_{n,0}$.

Definition 2 (see [5]). An equivalence relation Ψ on A is a congruence on (A, h) if it is lattice congruence on A and for every $a, b \in A$,

$$(a, b) \in \Psi \Rightarrow (h(a), h(b)) \in \Psi. \tag{1}$$

Definition 3 (see [7]). An ideal I of (A, h) is said to be a kernel ideal if there is a congruence Ψ on A such that

$$I = \Psi(0) = \{a \in A: (a, 0) \in \Psi\}. \tag{2}$$

Definition 4 (see [29]). A Heyting algebra is an algebra $(K, \vee, \wedge, \longrightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ where $(K, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\longrightarrow$ is a binary operation on K such that for every $a, b \in K$, $c \leq a \longrightarrow b \Leftrightarrow c \wedge a \leq b$.

Lemma 1 (see [6]). *Let $(A, h) \in \mathcal{K}_{n,0}$ and $a, b \in A$. If $a^{\vee} = \bigvee_{k=0}^{n-1} h^{2k}(a)$ and $a^{\wedge} = \bigwedge_{k=0}^{n-1} h^{2k+1}(a)$, then*

(1) $a \leq b \Rightarrow a^{\vee} \leq b^{\vee}$

(2) $a \leq b \Rightarrow a^{\wedge} \leq b^{\wedge}$

(3) $h(a^{\vee}) = b^{\wedge}$

(4) $h(a^{\wedge}) = a^{\vee}$

(5) $a^{\vee\vee} = a^{\wedge\wedge} = a^{\vee}$

(6) $a^{\vee\wedge} = a^{\wedge\vee} = a^{\wedge}$

(7) $(a \vee b)^{\vee} = a^{\vee} \vee b^{\vee}$

(8) $(a \vee b)^{\wedge} = a^{\wedge} \wedge b^{\wedge}$

(9) $(a \wedge b^{\wedge})^{\wedge} = a^{\wedge} \vee b^{\vee}$

(10) $(a \wedge b^{\wedge})^{\vee} = a^{\vee} \wedge b^{\wedge}$

Throughout this article, L is a none trivial complete lattice satisfying infinite meet distributive law: $a \wedge \wedge K = \vee \{a \wedge x: x \in\}, \forall a \in A$, and $K \subseteq L$. An L-fuzzy subset λ of a nonempty K is a mapping from K into L.

In this work, for simplicity, we say L-subsets instead of L-fuzzy subsets and write $\zeta \in L^K$ to say that ζ is an L-subset of K.

The union and intersection of any class $\{\lambda_i\}_{i \in \Delta}$ of L-subsets of K, respectively, represented by $\bigcup_{i \in \Delta} \lambda_i$ and $\bigcap_{i \in \Delta} \lambda_i$, are defined as follows:

$$\left(\bigcup_{i \in \Delta} \lambda_i\right)(a) = \bigvee_{i \in \Delta} \lambda_i(a) \text{ and } \left(\bigcap_{i \in \Delta} \lambda_i\right)(a) = \bigwedge_{i \in \Delta} \lambda_i(a), \tag{3}$$

for all $a \in A$, respectively.

Definition 5 (see [9]). For every λ and σ in L^K, define a binary relation "$\subseteq$" on L^K by

$$\lambda \subseteq \sigma \Leftrightarrow \lambda(a) \leq \sigma(a), \quad \forall a \in K. \tag{4}$$

It can be easily proved that "$\subseteq$" is a partial ordering on the set L^K of L-subsets of K and the poset $(L^K, \subseteq)$ forms a complete lattice in which for any $\{\lambda_i\}_{i \in \Delta} \subseteq L^K$,

$$\bigvee_{i \in \Delta} \lambda_i = \bigcup_{i \in \Delta} \lambda_i \text{ and } \bigwedge_{i \in \Delta} \lambda_i = \bigcap_{i \in \Delta} \lambda_i. \tag{5}$$

The partial ordering "$\subseteq$" is called the point wise ordering.

For $\lambda \in L^K$ and $\alpha \in L$, the set

$$\lambda_{\alpha} = \{x \in K: \lambda(a) \geq \alpha\}, \tag{6}$$

is called the α-level subset of λ and for each $a \in K$, we have

$$\lambda(a) = \vee \{\alpha \in L: a \in \lambda_{\alpha}\}. \tag{7}$$

For any $\alpha \in L$, we write $\overline{\alpha}$ to denote the constant L-subset of K which maps every element of K onto $\{\alpha\}$.

Definition 6 (see [11]). Suppose h is a function from T into R, and suppose λ is an L-subset of T and σ is an L-subset of R. Then, the image of λ under h, $h(\lambda)$, is an L-subset of R defined as for each $b \in R$,

$$h(\lambda)(b) = \begin{cases} \sup\{\lambda(a): a \in h^{-1}(b)\}, & \text{if } h^{-1}(b) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The preimage of σ under h, $h^{-1}(\sigma)$, is an L-subset of T and $h^{-1}(\sigma)(a) = \sigma(h(a))$ for each $a \in T$.

Definition 7 (see [22]). An L-fuzzy subset ξ of a lattice K is called an L-fuzzy ideals of K if $\xi(0) = 1, \xi(x \vee b) \geq \xi(x) \wedge \xi(b)$, and $\xi(x \wedge b) \geq \xi(x) \vee \xi(b)$.

An L-fuzzy subset ξ of a lattice K is called an L-fuzzy filters of K, if $\xi(1) = 1, \xi(x \wedge b) \geq \xi(x) \vee \xi(b)$, and $\xi(x \vee b) \geq \xi(x) \wedge \xi(b)$, for each $x, y \in K$.

It was also proved in [22] that an L-subset ξ of a lattice K with 0 is called an L-ideal of K if $\xi(0) = 1$ and $\xi(x \vee y) = \xi(x) \wedge \xi(y)$ for each $x, y \in K$. Dually, an L-subset ξ of a lattice K with 1 is called an L-filter of K if $\xi(1) = 1$ and $\xi(x \wedge y) = \xi(x) \wedge \xi(y)$, for each $x, y \in K$.

An L-ideal (respectively, L-filter) λ of K is called proper if it is not a constant map $\overline{1}$.

Definition 8 (see [28]). An L-subset λ of K is said to be an L-down set (respectively, L-up set) if $a \leq b$, then $\lambda(a) \geq \lambda(b)$ (respectively, $\lambda(a) \leq \lambda(b)$) for every $a, b \in K$.

Lemma 2 (see [28]). *Let λ be an L-subset of K. Then, the L-subset $\lambda^{\downarrow}$ of K defined by*

$$\lambda^{\downarrow}(a) = \vee\{\lambda(b): a \le b\} \text{ for every } a \in K, \qquad (9)$$

is the least L-down set containing λ.

Dually, we have the next result.

Lemma 3 (see [28]). *Let λ be an L-subset of A. Then, the L-fuzzy subset $\lambda^{\uparrow}$ of A defined by*

$$\lambda^{\uparrow}(a) = \vee\{\lambda(b): b \le a\} \text{ for each } a \in A, \qquad (10)$$

is the least L-up set including λ.

In what follows, for an Ockham algebra A, we shall denote by $\mathscr{LI}_{\mathscr{K}}(A)$ the set of all kernel L-ideals of A and by $\mathscr{LI}(A)$ the lattice of L-ideals of A in which the lattice operations $\wedge$ and $\vee$ are given by

$$\lambda \wedge \sigma = \lambda \cap \sigma; \ (\lambda \vee \sigma)(a) = \vee\{\mu(x) \wedge \sigma(y): a = x \vee y\}, \text{ for all } a \in A. \quad (11)$$

By an L-binary relation on a nonempty set K, we mean an L-subset of $K \times K$. For an L-binary relation Ψ on K and each $\alpha \in L$, the set

$$\Psi_{\alpha} = \{(a,b) \in K \times K: \Psi(a,b) \ge \alpha\}, \qquad (12)$$

is called the α–level binary relation of Ψ on K.

Definition 9 (see [30]). An L-relation Ψ on a nonempty set K is said to be

(1) Reflexive if $\Psi(a,a) = 1$, for all $a \in k$
(2) Symmetric if $\Psi(a,b) = \Psi(b,a)$, for each $a,b \in X$
(3) Transitive if for each $a,b \in K$: $\Psi(a,b) \ge \Psi(a,c) \wedge \Psi(c,b)$ for all $c \in K$

An L-equivalence relation on K is a reflexive, symmetric, and transitive L-relation on K.

Definition 10 (see [28]). Ψ is an L-equivalence relation on (A,h) and is called an L-congruence relation on (A,h) if it compatible with $\wedge, \vee$ and a unary operation h.

For any $a \in L$ and Ψ is L-congruence relation, L-subset Ψ_t of (A,h) is defined as follows:

$$\Psi_t(a) = \Psi(t,a), \text{ for each } a \in A. \qquad (13)$$

We call Ψ_t an L-congruence class of Ψ determined by t, and in particular, Ψ_0 is called the kernel of Ψ and Ψ_1 is called the cokernel of Ψ. One can easily observe that the kernel Ψ_0 of Ψ is an L-ideal of A and the cokernel Ψ_1 of Ψ is an L-filter of A.

Put $A/\Psi = \{\Psi(a): a \in A\}$ and $\wedge, \vee$ are binary operations and h is a unary operation on A/Ψ expressed as follows:

$$\Psi(a) \wedge \Psi(b) = \Psi(a \wedge b), \Psi(a) \vee \Psi(b) = \Psi(a \vee b) \text{ and } h(\Psi(a)) = \Psi(h(a)). \qquad (14)$$

After routine work, it can be proved that $(A/\Psi, \wedge, \vee, h, \Psi(0), \Psi(1))$ is an Ockham algebra and it is said to be the quotient Ockham algebra of A modulo Ψ.

Definition 11 (see [28]). An L-ideal λ of (A,h) is called a kernel L-ideal if $\lambda = \Psi_0$ for some L-congruence Ψ of A.

Lemma 4 (see [28]). *An L-ideal λ of (A,h) is a kernel L-ideal if and only if it holds the following conditions:*

(1) $h^2(\lambda) \subseteq \lambda$
(2) $(h(\lambda))^{\uparrow}(s) \wedge \lambda(a \wedge s) \le \lambda(a)$, for each $a,s \in A$

Lemma 5 (see [28]). *ie intersection of a class of kernel L-ideals of (A,h) is a kernel L-ideal.*

3. Kernel L-Ideals in a Subclass of Ockham Algebra

In the present topic, we present the structure of the set of kernels L-ideals in a subclass $\mathscr{K}_{n,0}$ of the class O of Ockham algebra.

Lemma 6 *Let $(A,h) \in \mathscr{K}_{n,0}$. Then, any kernel L-ideal of A is determined by an L-filter of A.*

Proof. Suppose that λ be a kernel L-ideal of A. This implies that there exists an L-congruence Ψ on A such that $\lambda = \ker \Psi$. Put $\gamma = \text{coker } \Psi = \Psi_1$, which is an L-filter of A.
Consider an L-subset of A defined as follows:

$$\gamma_0(a) = \vee\left\{\gamma(x): a \le \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), x \in A\right\}, \text{ for all } a \in A. \qquad (15)$$

To determine λ by the L-filter γ of A, we want to show that $\lambda = \gamma_0$. Now, for any $a \in A$,

$$\lambda(a) = \ker \Psi(a) = \Psi(a,0) \le \Psi(h(a),1) = \gamma(h(a)). \qquad (16)$$

Since $a \le \overset{n-1}{\underset{k=0}{\vee}} h^{2k}(a) = \overset{n-1}{\underset{k=0}{\vee}} h^{2k+2}(a) = \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(h(a))$.
Now, we get that

$$\lambda(a) = \gamma(h(a)) \le \vee\left\{\gamma(h(a)): a \le \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(h(a))\right\} = \gamma_0(a). \qquad (17)$$

Therefore, $\lambda \subseteq \gamma_0$.

$$\text{Also for every } a \in A, \gamma_0(a) = \vee\left\{\gamma(x): a \leq \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), x \in A\right\}$$

$$= \vee\left\{\Psi(a, 1): a \leq \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), x \in A\right\}$$

$$\leq \vee\left\{\Psi\left(\overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), 0\right): x \leq \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), x \in A\right\} \tag{18}$$

$$\leq \Psi\left(a \wedge \overset{n-1}{\underset{k=0}{\vee}} h^{2k+1}(x), a \wedge 0\right)$$

$$= \Psi(a, 0)$$

$$= \lambda(a).$$

That is, $\gamma_0 \subseteq \lambda$ and hence $\gamma_0 = \lambda$. Thus, the result holds. $\qquad\square$

Next, we see an equivalent characterization of an L-fuzzy ideal of a $K_{n,0}$-algebra to get a kernel L-ideal.

Lemma 7. *Let $(A, h) \in \mathcal{K}_{n,0}$. Then, an L-ideal λ of (A, h) is a kernel L-ideal if and only if the following conditions are satisfied:*

(1) $\lambda(h^2(a)) \geq \lambda(a)$, *for each $a \in A$*
(2) $\lambda(x) \wedge \lambda(a \wedge \overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)) \leq \lambda(a)$, *for each $a, x \in A$*

Proof. Assume that λ is a kernel L-ideal of A. Let $a \in A$. Then, by Lemma 4 (1), we have $\lambda(a) \leq h^2(\lambda)(h^2(a)) \leq \lambda(h^2(a))$. Hence, (1) holds.
For each $x, a \in A$,

$$\lambda(x) \wedge \lambda\left(a \wedge \overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right) \leq \lambda\left(\overset{n-1}{\underset{k=0}{\vee}} h^{2k}(x)\right) \wedge \lambda\left(a \wedge \overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right)$$

$$\leq h(\lambda)\left(\overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right) \wedge \lambda\left(a \wedge \left(\overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right)\right) \tag{19}$$

$$\leq [h(\lambda))\left(\overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right) \wedge \lambda\left(a \wedge \overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(x)\right)$$

$$\leq \lambda(a) \text{ (by Lemma 2.15 (2))}.$$

Thus, (2) is proved. Conversely, suppose that the given conditions hold. Let $a \in A$,

$$h^2(\lambda)(a) = \vee\left\{\lambda(x): a = h^2(x)\right\}$$

$$\leq \vee\left\{\lambda\left(h^2(x)\right): a = h^2(x)\right\} \tag{20}$$

$$= \lambda(a).$$

Hence, $h^2(\lambda) \subseteq \lambda$. Again for every $a, b \in A$,

$$(h(\lambda))^{\uparrow}(b) \wedge \lambda(a \wedge b) = \vee\{h(\lambda)(x): x \leq b\} \wedge \lambda(a \wedge b)$$

$$= \vee\{\vee\{\lambda(y): h(y) = x\}: x \leq b\} \wedge \lambda(a \wedge b)$$

$$\leq \vee\{\lambda(y): h(y) \leq b\} \wedge \lambda(a \wedge b)$$

$$= \vee\{\lambda(y) \wedge \lambda(a \wedge b): h(y) \leq b\}$$

$$\leq \vee\{\lambda(y) \wedge \lambda(a \wedge h(y)): h(y) \leq b\}$$

$$\leq \vee\left\{\lambda(y) \wedge \lambda\left(a \wedge \left(\overset{n-1}{\underset{k=0}{\wedge}} h^{2k+1}(y)\right)\right): h(y) \leq b\right\}$$

$$\leq \lambda(a) \text{ (by (2))}.$$

$$\tag{21}$$

Then, it follows from Lemma 4 that λ is a kernel L-ideal. $\qquad\square$

In the following result, we characterized the least kernel L-ideal of (A, h).

Theorem 1. *Let $(A, h) \in \mathcal{K}_{n,0}$ and λ be an L-ideal of A. Let λ° be an L-subset of A defined by*

$$\lambda^{\circ}(a) = \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}, \text{ forall } a \in A, \tag{22}$$

where $x^{\vee}$ and $x^{\wedge}$ are as stated in Lemma 1. Then, λ° is the least kernel L-ideal of A containing λ.

Proof. First we prove that λ° is an L-ideal of A.

$$\lambda^{\circ}(0) = \vee\{\lambda(x): 0 \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$

$$= \vee\{\lambda(x): 0 \leq x^{\vee} \wedge x^{\wedge}\} \tag{23}$$

$$\geq \lambda(0) = 1. \left(\text{as } 0 \leq 0^{\vee} \wedge 0^{\wedge}\right).$$

Hence, $\lambda^{\circ}(0) = 1$.

Now, for each $a, b \in A$,

$$\lambda^{\circ}(a) \wedge \lambda^{\circ}(b) = \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\} \wedge \vee\{\lambda(y): b \wedge y^{\wedge} \leq y^{\vee} \wedge y^{\wedge}\}$$
$$= \vee\{\lambda(x) \wedge \lambda(y): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}, b \wedge y^{\wedge} \leq y^{\vee} \wedge y^{\wedge}\}$$
$$\leq \vee\{\lambda(x \vee y): (a \vee b) \wedge (x \vee y)^{\wedge} \leq (x \vee y)^{\vee} \wedge (x \vee y)^{\wedge}\}$$
$$= \lambda^{\circ}(a \vee b).$$

$$(24)$$

This implies that $\lambda^{\circ}(a) \wedge \lambda^{\circ}(b) \leq \lambda^{\circ}(a \vee b)$ for each $a, b \in A$.

On the other side,

$$\lambda^{\circ}(a \vee b) = \vee\{\lambda(x): (a \vee b) \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\} \qquad (25)$$
$$= \lambda^{\circ}(a).$$

With similarly approach, we can prove that $\lambda^{\circ}(a \vee b) \leq \lambda^{\circ}(b)$ and so $\lambda^{\circ}(a \vee b) \leq \lambda^{\circ}(a) \wedge \lambda^{\circ}(b)$. Therefore, $\lambda^{\circ}(a \vee b) = \lambda^{\circ}(a) \wedge \lambda^{\circ}(b)$ and thus λ° is an L-ideal of A.

Next, we prove that λ° is a kernel L-fuzzy ideal of A.

$$\lambda^{\circ}(a) = \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \vee\{\lambda(h^2(x)): h^2(a \wedge x^{\wedge}) \leq h^2(x^{\vee} \wedge x^{\wedge})\}$$
$$= \vee\{\lambda(h^2(x)): h^2(a) \wedge h^2(x^{\wedge}) \leq h^2(x^{\vee}) \wedge h^2(x^{\wedge})\}$$
$$\vee\{\lambda(h^2(x)): h^2(a) \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$= \lambda^{\circ}(h^2(a)).$$

$$(26)$$

Thus, $\lambda^{\circ}(a) \leq \lambda^{\circ}(h^2(a))$, for each $a \in A$, and the property (1) of Lemma 7 holds.

To see the property (2) of Lemma 7 holds, let $a, s \in A$. Then,

$$\lambda^{\circ}(s) \wedge \lambda^{\circ}\left(a \wedge \left(\bigwedge_{k=0}^{n-1} h^{2k+1}(t)\right)\right) = \lambda^{\circ}(s) \wedge \lambda^{\circ}(a \wedge s^{\wedge})$$

$$= \vee\{\lambda(x): s \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\} \wedge$$
$$\vee\{\lambda(y): (a \wedge s^{\wedge}) \wedge y^{\wedge} \leq y^{\vee} \wedge y^{\wedge}\}$$
$$= \vee\{\lambda(x) \wedge \lambda(y): s \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}, (a \wedge s^{\wedge}) \wedge y^{\wedge} \leq y^{\vee} \wedge y^{\wedge}\}$$
$$\leq \vee\{\lambda(x \vee y): a \wedge (x \vee y)^{\wedge} \leq (x \vee y)^{\wedge} \wedge (x \vee y)^{\vee}\}$$
$$\leq \lambda^{\circ}(y).$$

$$(27)$$

Hence, by Lemma 7, λ° is a kernel L-ideal of A. Suppose λ is a kernel ideal of A such that $\lambda \subseteq \gamma$.

This implies there exists an L-congruence Ψ on A such that $\gamma = \ker \Psi$. Let $x \in A$. As $a \wedge a^{\wedge} \leq a^{\vee} \wedge a^{\wedge}$, we clearly get

$$\lambda^{\circ}(a) = \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge},\} \geq \lambda(a). \qquad (28)$$

And hence $\lambda \subseteq \lambda^{\circ}$. Again for each $a \in A$,

$$\lambda^{\circ}(a) = \vee\{\lambda(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \vee\{\gamma(x): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$= \vee\{\Theta(x, 0): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \vee\{\Psi(h^{2k+1}(x), 1): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \vee\left\{\Psi\left(\bigwedge_{k=0}^{n-1} h^{2k+1}(x), 1\right): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\right\} \qquad (29)$$
$$= \vee\{\Psi(x^{\wedge}, 1): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \Psi(a^{\wedge}, 1).$$

Similarly, we can show that $\lambda^{\circ}(a) \leq \vee\{\Theta(x^{\vee}, 0): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$.

This implies that

$$\lambda^{\circ}(a) \leq \vee\{\Psi(x^{\wedge}, 1): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\} \wedge \vee\{\Psi(x^{\vee}, 0): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$= \vee\{\Psi(x^{\wedge}, 1) \wedge \Psi(x^{\vee}, 0): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$= \vee\{\Psi(a \wedge x^{\wedge}, a) \wedge \Psi(x^{\vee} \wedge x^{\wedge}, 0): a \wedge x^{\wedge} \leq x^{\vee} \wedge x^{\wedge}\}$$
$$\leq \Psi(x^{\vee} \wedge x^{\wedge}, a) \wedge \Psi(x^{\vee} \wedge x^{\wedge}, 0)$$
$$\leq \Psi(a, 0) = \gamma(a).$$

$$(30)$$

And so $\lambda^{\circ} \subseteq \gamma$. Therefore, from these observations, λ° is the least kernel L-ideal of A including λ. □

The following corollaries follows Theorem 1.

Corollary 1. *Suppose $(A, h) \in \mathcal{K}_{n,0}$ and λ is an L-ideal of A. Then, λ is a kernel L-ideal if and only if $\lambda = \lambda^{\circ}$.*

Corollary 2. *If $(A, h) \in \mathcal{K}_{n,0}$, then the following properties hold, for all $\lambda, \gamma \in \mathcal{LF}(A)$:*

(1) $\lambda \subseteq \gamma \Rightarrow \lambda^{\circ} \subseteq \gamma^{\circ}$

(2) $\lambda \subseteq \lambda^{\circ} = \lambda^{\circ \circ}$

(3) $\lambda^{\circ} \vee \gamma^{\circ} \subseteq (\lambda \vee \gamma)^{\circ}$

(4) $(\lambda \cap \gamma)^{\circ} \subseteq \lambda^{\circ} \cap \gamma^{\circ}$

Given a $\mathscr{K}_{n,0}$- algebra A and L-ideals λ, γ of A, we shall define

$$\lambda^{\circ} \sqcap \gamma^{\circ} = (\lambda \cap \gamma)^{\circ} \text{ and } \lambda^{\circ} \sqcup \gamma^{\circ} = (\lambda \vee \gamma)^{\circ}. \tag{31}$$

Suppose that λ and γ kernel L-ideals of A, then $\lambda \sqcap \gamma = \lambda \cap \gamma$. Indeed, since λ and γ are kernel L-ideals of A, we have $\lambda = \lambda^{\circ}$ and $\gamma = \nu^{\circ}$, and $\lambda \cap \gamma$ is also a kernel L-ideal. Thus,

$$\lambda \sqcap \gamma = \lambda^{\circ} \sqcap \gamma^{\circ} = (\lambda \cap \gamma)^{\circ} = \lambda \cap \gamma. \tag{32}$$

Then, we have the following.

Theorem 2. *Let $(A, h) \in \mathscr{K}_{n,0}$ and $\mathscr{LI}_{k}(A)$ denote the set of all kernel L-ideals. Then, $(\mathscr{LI}_{k}(A), \sqcap, \sqcup)$ is a complete bounded distributive lattice.*

Proof. Suppose that $\lambda, \gamma \in \mathscr{LI}_{k}(L)$. Then, $\lambda \sqcap \gamma = \lambda \cap \gamma$ is a kernel L-ideal and the infimum of λ and γ. It follows by Theorem 1 that $\lambda \sqcup \gamma = \lambda^{\circ} \sqcup \gamma^{\circ} = (\lambda \vee \gamma)^{\circ}$ is a kernel L-ideal and $\lambda, \gamma \subseteq \lambda \vee \gamma \subseteq (\lambda \vee \gamma)^{\circ}$. Suppose that ξ is a kernel L-ideal of A such that ξ is an upper bounded of λ and γ, then $\lambda \vee \gamma \subseteq \xi$ and there follows $(\lambda \vee \gamma)^{\circ} \subseteq \xi^{\circ} = \xi$. Hence, $(\lambda \vee \gamma)^{\circ}$ is the supremum of both λ and γ in $\mathscr{LI}_{K}(A)$. Thus, $(\mathscr{FI}_{K}(A), \sqcap, \cap)$ is a lattice. Obviously, $\chi_{\{0\}}$ and χ_{L} are the least and biggest kernel L-ideals in $\mathscr{LI}_{k}(A)$, respectively. This implies that $\mathscr{LI}_{k}(A)$ is bounded. The completeness is clear since the intersection of a family of kernel L-ideals is also a kernel L-ideal of A (Lemma 4). As far as the distributivity, let $\lambda, \gamma, \xi \in \mathscr{LI}_{k}(A)$. Since

$$\begin{aligned}
\lambda \sqcap (\gamma \sqcup \xi) &= \lambda^{\circ} \sqcap (\gamma^{\circ} \sqcup \xi^{\circ}) \\
&= \lambda^{\circ} \sqcap (\gamma \vee \xi)^{\circ} \\
&= [\lambda \cap (\gamma \vee \xi)]^{\circ} \\
&= [(\lambda \cap \xi) \vee (\gamma \vee \lambda)]^{\circ} \\
&= (\lambda \cap \xi)^{\circ} \vee (\gamma \wedge \xi)^{\circ} \\
&= (\lambda^{\circ} \sqcap \xi^{\circ}) \vee (\gamma \wedge \xi)^{\circ} \\
&= (\xi \cap \xi)^{\circ} \sqcup (\gamma^{\circ} \sqcap \xi^{\circ}) \\
&= (\lambda \sqcap \xi) \sqcup (\gamma \sqcap \xi).
\end{aligned} \tag{33}$$

It follows that $\mathscr{LI}_{k}(A)$ is a distributive lattice. $\square$

In order to further characterize the structure of the lattice of kernel L-ideals of a $\mathscr{K}_{n,0}$-algebra, we require the following:

Theorem 3. *Let $(A, h) \in \mathscr{K}_{n,0}$. Then, $(\mathscr{LI}_{k}(A), \sqcap, \sqcup, *)$ is a Heyting algebra in which for $\lambda, \gamma \in \mathscr{LI}_{k}(A)$, the relative pseudocomplement of λ and γ is defined as for every $a \in A$,*

$$(\lambda * \gamma)(a) = \vee\{\xi(a^{\vee}): \xi \in \mathscr{LI}(A), \xi^{\circ} \sqcap \lambda \subseteq \gamma\}. \tag{34}$$

Proof. Since $\gamma^{\circ} \sqcap \lambda \subseteq \gamma$ and

$$\gamma(0^{\vee}) = \gamma\left(\bigvee_{k=0}^{n-1} h^{2k}(0)\right) = \bigwedge_{k=0}^{n-1} \gamma\left(h^{2k}(0)\right) \geq \gamma(0) = 1, \tag{35}$$

we have $(\lambda * \gamma)(0) \geq \gamma(0^{\vee}) = 1$.

$$\begin{aligned}
(\lambda * \gamma)(a) &= \vee\{\xi(a^{\vee}): \xi \in \mathscr{FI}(A), \xi^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{\xi((a \vee b)^{\vee}): \xi \in \mathscr{FI}(A)\xi^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= (\lambda * \gamma)(a \vee b).
\end{aligned} \tag{36}$$

Similarly, we can prove that $(\lambda * \gamma)(b) \leq (\lambda * \gamma)(a \vee b)$ and hence

$$(\lambda * \gamma)(a) \vee (\lambda * \gamma)(b) \leq (\lambda * \gamma)(a \vee b). \tag{37}$$

On the other side, for each $a, b \in A$,

$$\begin{aligned}
(\lambda * \gamma)(a) \wedge (\lambda * \gamma)(b) &= \vee\{\xi(a^{\vee}): \xi \in \mathscr{FI}(A), \xi^{\circ} \sqcap \lambda \subseteq \gamma\} \wedge \\
&\quad \vee\{\sigma(y^{\vee}): \sigma \in \mathscr{FI}(A), \sigma^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= \vee\{\xi(a^{\vee}) \wedge \sigma(b^{\vee}): \xi^{\circ} \sqcap \lambda \subseteq \gamma, \sigma^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{(\xi \sqcup \sigma)(a^{\vee}) \wedge (\xi \sqcup \sigma)(b^{\vee}): (\xi \sqcup \sigma)^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{(\xi \sqcup \sigma)(a \vee b)^{\vee}: (\xi \sqcup \sigma)^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= (\lambda * \gamma)(a \vee b).
\end{aligned} \tag{38}$$

Thus, $(\lambda * \gamma)(a \vee b) = (\lambda * \gamma)(a) \wedge (\lambda * \gamma)(b)$, for each $a, b \in A$, and hence $\lambda * \gamma$ is an L-ideal of A.

Next, we show that $\lambda * \gamma$ is a kernel L-ideal of A.

$$\begin{aligned}
(\lambda * \gamma)(a) &= \vee\{\xi(a^{\vee}): \xi \in \mathscr{FI}(A), \xi^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= \vee\{\gamma(f^{2}(a^{\vee})): \gamma \in \mathscr{FI}(A), \gamma^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= (\lambda * \gamma)f^{2}(a^{\vee}).
\end{aligned} \tag{39}$$

Also, for any $a, s \in A$,

$$\begin{aligned}
(\lambda * \gamma)(s) \wedge (\lambda * \gamma)(a \wedge t^{\wedge}) &= \vee\{\xi(s^{\vee}): \xi \in \mathscr{FI}(A), \xi^{\circ} \sqcap \lambda \subseteq \gamma\} \wedge \\
&\quad \vee\{\delta((a \wedge s^{\wedge})^{\vee}): \delta \in \mathscr{FI}(A), \delta^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&= \vee\{\xi(s^{\vee}) \wedge \delta(a^{\vee} \wedge s^{\wedge}): \xi^{\circ} \sqcap \lambda \subseteq \gamma, \delta^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{(\xi \vee \delta)(s^{\vee}) \wedge (\xi \vee \delta)(a^{\vee} \wedge s^{\wedge}): \xi^{\circ} \sqcap \lambda \subseteq \gamma, \delta^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{(\xi \vee \delta)(s^{\vee} \vee (a^{\vee} \wedge s^{\wedge})): (\xi \vee \delta)^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq \vee\{(\xi \vee \delta)(a^{\vee}): (\xi \vee \delta)^{\circ} \sqcap \lambda \subseteq \gamma\} \\
&\leq (\lambda * \gamma)(a).
\end{aligned} \tag{40}$$

Hence, by Lemma 4, $\lambda * \gamma$ is kernel L-ideals of A.

Next, we prove that $\mu * \lambda$ is the biggest kernel L-ideals of A such that $\lambda \sqcap \lambda * \gamma \subseteq \xi$. Suppose that $\sigma \in \mathscr{LI}_K(A)$ such that $\lambda \sqcap \sigma \subseteq \xi$. Now, since for each $a \in A$,

$$\sigma(a) \leq \sigma(a^\vee) \text{ and } \sigma^\circ \sqcap \lambda = \sigma^\circ \sqcap \lambda^\circ = (\sigma \cap \lambda)^\circ = \sigma \sqcap \lambda \subseteq \gamma. \quad (41)$$

We have clearly

$$(\lambda * \gamma)(a) = \vee\{\xi(a^\vee): \xi \in \mathscr{LI}(A), \xi^\circ \sqcap \lambda \subseteq \gamma\} \geq \sigma(a^\vee) \geq \sigma(a). \quad (42)$$

And hence $\sigma \subseteq \lambda * \gamma$. Therefore, $\lambda * \gamma$ is the pseudo-complement of λ relative to γ. □

Theorem 4. *Let* $(A, h) \in \mathscr{K}_{n,0}$ *and* λ *be a proper kernel L-ideal of A and $L = [0, 1]$. Then, λ is the intersection of a family of prime kernel L-ideals including it.*

Proof. Assume that λ is a proper L-ideal of A. Then, there exists $x \in A$ such that $\lambda(x) \neq 1$. Put $\lambda(x) = \alpha$ and consider the set

$$\Delta = \{\sigma \in \mathscr{LI}_K(A): \sigma \subseteq \lambda \text{ and } \sigma(x) \leq \alpha\}. \quad (43)$$

Clearly, $\lambda \in \Delta$ and so Δ is nonempty and hence it forms a poset under the point wise ordering "$\subseteq$". By using Zorn's lemma, we can choose a maximal element, say ξ in Δ; we prove that ξ is a prime kernel L-ideal. Let γ and σ be L-ideals of A such that $\gamma \cap \sigma \subseteq \xi$. Suppose on contrary that $\gamma \nsubseteq \xi$ and $\sigma \nsubseteq \xi$. Then,

$$\xi \subset \xi \vee \gamma \subseteq (\xi \vee \gamma)^\circ \text{ and } \xi \subset \xi \vee \sigma \subseteq (\sigma \vee \xi)^\circ. \quad (44)$$

By the maximality of ξ in Δ, it follows that $(\gamma \vee \xi)^\circ \notin \Delta$ and $(\sigma \vee \xi)^\circ \notin \Delta$. So that $(\nu \vee \xi)^\circ(a) \nleq \alpha$ and $(\sigma \vee \xi)^\circ(a) \nleq \alpha$. Now,

$$(\xi \vee \gamma)^\circ \sqcap (\xi \vee \sigma)^\circ = ((\xi \vee \gamma) \cap (\xi \vee \sigma))^\circ$$
$$= (\xi \vee (\gamma \cap \sigma))^\circ \quad (45)$$
$$\subseteq \xi^\circ = \xi,$$

which is a contradiction.

Let $\{\xi_i\}_{i \in \Delta}$ be the set of prime kernel L-ideals that contain λ. Then, $\lambda \subseteq \cap_{i \in \Delta}\xi_i$. Suppose that $\cap_{i \in \Delta}\xi_i \nsubseteq \lambda$. Then, there is $x \in A$ such that $(\cap_{i \in \Delta}\xi_i)(x) \nleq \lambda(x)$. This implies that $\lambda(x) \neq 1$ and hence λ is proper. Thus, from the above, there exists a prime kernel L-ideal ξ which contains λ and $\xi \leq \alpha = \lambda(x)$ and hence $(\cap_{i \in \Delta}\xi_i)(x) \leq \xi(x) \leq \alpha = \lambda(x)$, which is a contradictions. Hence, $\underset{i \in \Delta}{\cap} \xi_i \subseteq \lambda$ and so $\lambda = \cap_{i \in \Delta}\xi_i$. □

4. L-Congruences of a $\mathscr{K}_{n,0}$-Algebra

In this topic, we characterize an interesting property of the least L-congruence on the $\mathscr{K}_{n,0}$-algebra A such that the given kernel L-ideal ξ of A as its L-congruence class.

Let ξ be a kernel L-ideal of (A, h). As shown in [28], the smallest L-congruence $\Omega_L(\xi)$ on A with kernel ξ which is given by: for $x, y \in A$,

$$\Omega_L(\xi)(x, y) = \vee\{\xi(a) \wedge [h(\xi)](b): (x \vee a) \wedge b = (y \vee a) \wedge b\}. \quad (46)$$

Theorem 5. *If* $(L, h) \in \mathscr{K}_{n,0}$ *and* $\xi, \lambda \in \mathscr{LI}_K(A)$, *then*

$$\Omega_L(\xi \sqcup \lambda) = \Omega_L(\xi) \vee \Omega_L(\lambda). \quad (47)$$

Proof. Let $\xi, \lambda \in \mathscr{LI}_k(A)$. It is easily proved that

$$\xi \subseteq \lambda \Leftrightarrow \Omega_L(\xi) \subseteq \Omega_L(\lambda). \quad (48)$$

Then, we have $\Omega_L(\xi) \vee \Omega_L(\lambda) \leq \Omega_L((\xi \vee \lambda)^0) = \Omega_L(\xi \sqcup \lambda)$. On the other side, let $x, y \in A$. Then, by (46), we have

$$\Omega_L(\xi \sqcup \lambda)(x, y) = \vee\{(\mu \sqcup \lambda)(r) \wedge [h(\xi \sqcup \lambda)](t): (x \vee r) \wedge t = (y \vee r) \wedge t\}. \quad (49)$$

Consider the following equation:

$$
\begin{aligned}
(\xi \sqcup \lambda)(r) &= (\xi \vee \lambda)^0(r) \\
&= \vee\{(\xi \vee \lambda)(a): r \wedge a^\wedge \leq a^\wedge \wedge a^\vee\} \\
&= \vee\{(\xi \vee \lambda)(a): (r \vee a^\vee) \wedge a^\wedge = a^\wedge \wedge a^\vee\} \\
&= \vee\{\vee\{\xi(r_1) \wedge \lambda(t_1): a = r_1 \vee t_1\}: (r \vee a^\vee) \wedge a^\wedge = a^\wedge \wedge a^\vee\}.
\end{aligned}
\quad (50)
$$

$$
\begin{aligned}
[h(\xi \sqcup \lambda)](t) &= \vee\{(\xi \vee \lambda)^0(b): h(b) \leq t\} \\
&= \vee\{\vee\{(\xi \vee \lambda)(c): b \wedge c^\wedge \leq c^\wedge \wedge c^\vee\}: h(b) \leq t\} \\
&= \vee\{\vee\{(\xi \vee \lambda)(c): (b \vee c^\vee) \wedge c^\wedge = c^\wedge \wedge c^\vee\}: h(b) \leq t\} \\
&= \vee\{\vee\{\vee\{\xi(r_2) \wedge \lambda(t_2): c = r_2 \vee t_2\}: (b \vee c^\vee) \wedge c^\wedge = c^\wedge \wedge c^\vee\}: h(b) \leq t\} \\
&= \vee\{\xi(r_2) \wedge \lambda(t_2): c = r_2 \vee t_2, (b \vee c^\vee) \wedge c^\wedge = c^\wedge \wedge c^\vee, h(b) \leq t\} \\
&= \vee\{\xi(r_2) \wedge \lambda(t_2): (b \vee r_2^\vee \vee t_2^\vee) \wedge r_2^\wedge \wedge t_2^\wedge = (r_2^\vee \vee t_2^\vee) \wedge r_2^\wedge \wedge t_2^\wedge, h(b) \leq t\} \\
&\leq \vee\{\xi(r_2) \wedge \lambda(t_2): (h(b) \wedge r_2^\wedge \wedge t_2^\wedge) \vee r_2^\vee \vee t_2^\vee = (r_2^\wedge \wedge t_2^\wedge) \vee r_2^\vee \vee t_2^\vee, h(b) \leq t\}.
\end{aligned}
\quad (51)
$$

By (49) and (50), we have

$$\Omega_L(\xi \sqcup \lambda)(x, y) \leq \vee\{(\xi \sqcup \lambda)(r): (x \vee r) \wedge t = (y \vee r) \wedge t\}$$

$$= \vee\{\xi(r_1) \wedge \lambda(t_1): (r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge} \wedge t_1^{\vee} \quad (52)$$

$$= (r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge} \wedge t_1^{\vee}(x \vee r) \wedge t = (y \vee r) \wedge t,\}.$$

From (51), we have

$$\xi(r_1) \leq \Omega_L(\xi)(0, r_1)$$

$$\leq \Omega_L(\xi)(0, r_1^{\vee}) \wedge \Omega_L(\xi)(1, r_1^{\wedge}) \quad (53)$$

$$\leq \Omega_L(\xi)(r, r \vee r_1^{\vee}) \wedge \Omega_L(\xi)(r \vee r_1^{\vee} \vee t_1^{\vee}, (r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}),$$

$$\Omega_L(\xi)(1, r_1^{\wedge}) = \Omega_L(\xi)(1, r_1^{\wedge}) \wedge \Omega_L(\xi)(r_1^{\vee} \vee t_1^{\vee}, r_1^{\vee} \vee t_1^{\vee})$$

$$\leq \Omega_L(\xi)(r_1^{\vee} \vee t_1^{\vee}, (r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\vee}). \quad (54)$$

Also, from (51), we have

$$\lambda(t_1) \leq \Omega_L(\lambda)(0, t_1)$$

$$\leq \Omega_L(\lambda)(0, t_1^{\vee}) \wedge \Omega_L(\lambda)(1, t_1^{\wedge})$$

$$= \Omega_L(\lambda)(0, t_1^{\vee}) \wedge \Omega_L(\lambda)(r \vee r_1^{\vee}, r \vee r_1^{\vee}) \wedge \Omega_L(\lambda)(1, t_1^{\wedge}) \wedge$$

$$\Omega_L(\lambda)((r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}, (r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}) \quad (55)$$

$$\leq \Omega_L(\lambda)(r \vee r_1^{\vee}, r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge \Omega_L(\lambda)(1, t_1^{\wedge}) \wedge$$

$$\Omega_L(\lambda)((r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}, (r \vee r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge} \wedge t_1^{\wedge}),$$

$$\Omega_L(\lambda)(t_1^{\wedge}, 1) = \Omega_L(\lambda)(t_1^{\wedge}, 1) \wedge \Omega_L(\lambda)(r_1^{\vee} \vee t_1^{\vee}, r_1^{\vee} \vee t_1^{\vee})$$

$$\leq \Omega_L(\lambda)((r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}, (r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}) \quad (56)$$

$$\leq \Omega_L(\lambda)((r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge} \wedge t_1^{\wedge}, (r_1^{\vee} \vee t_1^{\vee}) \wedge r_1^{\wedge}).$$

From (52), (53), (54), (55), and (56), we have

$$\Omega_L(\xi \sqcup \lambda)(x, y) \leq \vee\{(\Omega_L(\xi) \vee \Omega_L(\lambda))(r, 0)\}. \quad (57)$$

Similarly,

$$\Omega_L(\xi \sqcup \lambda)(x, y) \leq \vee\{(\Omega_L(\xi) \vee \Omega_L(\lambda))(t, 1)\}. \quad (58)$$

From (57) and (58), we get

$$\Omega_L(\xi \sqcup \lambda)(x, y) \leq (\Omega_L(\xi) \vee \Omega_L(\lambda))(x, y). \quad (59)$$

Hence, $\Omega_L(\xi \sqcup \lambda) = \Omega_L(\xi) \vee \Omega_L(\lambda)$. $\qquad \square$

We now give a description on the biggest L-congruence on a $\mathscr{K}_{n,0}$-algebra A such that the given kernel L-ideal of A as its L-congruence class.

Theorem 6. *Let $(A, h) \in \mathscr{K}_{n,0}$ and ξ be a kernel L-ideal of A. A binary relation $\delta(\xi)$ on A is defined as follows: $\delta(\xi)(a, b) = \vee\{\xi(h^K(a) \wedge x) \wedge \xi(h^K(b) \wedge x)\}$ for all $x \in A$ and $(k = 0, 1, 2, 2n - 1)$. Then, $\delta(\xi)$ is the biggest congruence on A with $\delta(\xi) = \xi$.*

5. Conclusion

In this work, we studied Kernel L-ideals and L-congruence on a subclass of Ockham algebras and investigate their properties. We proved that the set of kernel L-ideal $\mathscr{F}\mathscr{I}_k(A)$ of $\mathscr{K}_{n,0}$-algebra A forms Heyting algebras. Also, we obtain the least, respectively, the biggest L-congruences on $K_{n,0}$-algebra A having a given L-ideal as a kernel and describe it using algebraic operations in an L setting.

Data Availability

No data were used to support the results of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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