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Review Article

Statistical Inference to the Parameter of the Inverse Power Ishita Distribution under Progressive Type-II Censored Data with Application to COVID-19 Data

Kariema A. Elnagar, Dina A. Ramadan, and Beih S. El-Desouky

Mathematics Department, Faculty of Science, Mansoura University, Dakahlia Governorate 35516, Egypt

Correspondence should be addressed to Dina A. Ramadan; dina_ahmed2188@yahoo.com

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The goal of the article is the inference about the parameters of the inverse power ishita distribution (IPID) using progressively type-II censored (Prog–II–C) samples. For IPID parameters, maximum likelihood and Bayesian estimates were obtained. Two bootstrap "confidence intervals" (CIs) are also proposed in addition to "approximate confidence intervals" (ACIs). In addition, Bayesian estimates for "squared error loss" (SEL) and LINEX loss functions are provided. The Gibbs within Metropolis–Hasting samplers process is used to provide Bayes estimators of unknown parameters also "credible intervals" (CRIs) of them by using the "Markov Chain Monte Carlo" (MCMC) technique. Then, an application of the suggested approaches is considered a set of real-life data this data set COVID-19 data from France of 51 days recorded from 1 January to 20 February 2021 formed of mortality rate. To evaluate the quality of the proposed estimators, a simulation study is conducted.

1. Introduction

Statistical researchers in the fields of industrial and mechanical engineering are spent so much time looking at component and unit failure, which provide the foundation of operational systems. Their research involves watching operating units until they fail, recording their lifetimes, applying statistical inference methods to analyse obtained data and then predicting reliability and risk functions for the entire system using the data. However, because some experimental units are costly and have a high level of reliability. Following a specified censoring scheme, units are excluded from the experiment, and the censored sample is the final sample obtained by Klein et al. [1]. One of the main motivations for filtering is to reduce the total test time and the accompanying test. The life test is considered to be complete when all of the items under test are monitored until failure. In most real conditions, the data available is incomplete. As a result, whether due to test time or budget limits, censoring is critical in lifetime data analysis. The most common censoring techniques are type-I and type-II censored. Following that, a progressive censoring scheme is suggested. As a result, the researchers can release the unused units of various points throughout the test, making it more flexible and realistic.

1.1. Progressive Type-II Censored. A progressive type-II censoring strategy is a useful method for removing a particular fraction of at-risk participants from an experiment at each of many ordered failure times. The following is an example of the Prog –II– C scheme: On the life test, the tester allocates independent and identical units. Let us suppose at the start of the experiment, there are n units to test. The lifetime test is ended when the mth $(m < n)$ unit fails, assuming there are n units to test when the experiment begins. After the first failure, time t_1 is recorded, and R_1 units were selected randomly as from reminder $n-1$ survival units. As a result, when the second failure occurs, time t_2 is recorded, with the remnants $n - R_1 - 2$ survival units R_2 units were chosen at random. This experiment ends when the mth failure occurs, at time t_m, and $R_m = n - m - \sum_{i=1}^{m-1} R_i$ as shown in Figure 1.

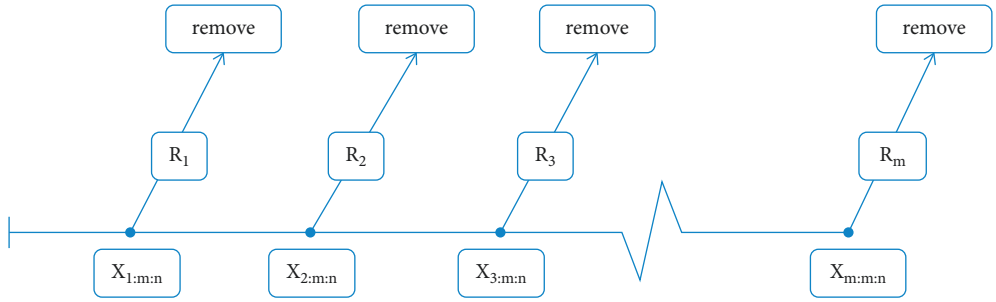


Figure 1: Type-II progressive censoring.

As special cases (if $R_i = 0$ and $i = 1, 2, \ldots, m$ with $n = m$) and the conventional type-II right censoring scheme if $R_i = 0$ and $i = 1, 2, \ldots, m-1$ with $R_m = n - m$. Many authors have devoted for a Prog–II–C method with various failure time breakdowns. For example, Balakrishnan and Aggarwala [2], Alshenawy [3], Balakrishnan [4], Ahmed [5], and Almetwally et al. [6]. Mann et al. [7] and Meeker et al. [8] looked into the characteristics of progressively censored order statistics and it offered an outline of several inferential approach expansions depend on Prog-I and Prog-II right censored samples, as well as identifying some for future research challenges. Balakrishnan and Cramer [9] give a thorough survey of the literature on progressive censoring, as well as specifics on this progressive censoring technique and its various uses. According to Tse et al. [10], the amount of patients falls out of a clinical trial at each stage is random and not possible predicted. With each failure, the pattern of removal become more random. See Balakrishnan and Sandhu [11] for further details on the gradually censored samples. As lifetime distributions are exponential, log normal, or Weibull, Aggarwala and Balakrishnan [12] has studied inference for Prog–II–C cases. Balakrishnan [11] and Aggarwala [12] have devised a method for simulating general Prog–II–C samples drawn from continuous or uniform distributions. Eryilmaz and Bairamov [13]and Montanari et al. [14] also have been researching parameter estimation from various lifetime distributions using Prog–II–C. Other authors, include Balakrishnan et al. [9], Mousa et al. [15], and Mousa et al. [16], have been researching the estimate of parameters from various lifetime distributions depending on Prog–II–C. In Prog-II right censored order statistics, Salemi et al. [17] recently investigated A-optimal and D-optimal censoring strategies, Qin et al. [18] offer a novel spacing-based test statistic for determining whether general Prog–II–C data are from an exponential distribution and parameters of a log-logistic distribution under the Prog–II–C sample, when Maiti et al. [19] given access to a managed step-wise sample of the second type, there was also a look at the generalized Rayleigh distribution. The IPID is a generalized of the Ishita distribution, which was presented by Elnagar et al. [20] is shown to be a good model for the Covid-19 and glass fiber data failure times. They also started at the maximum likelihood of the unknown parameters estimators, besides their asymptotic confidence intervals, using complete data. The random variable X of the IPI distribution, has a probability density function (PDF):

$$f(y) = \frac{\alpha\theta^3}{\theta^3 + 2}\left(\theta + y^{-2\alpha}\right)y^{-\alpha-1}e^{-\theta y^{-\alpha}}, \quad y > 0, \theta, \alpha > 0, \tag{1}$$

and cumulative distribution function (CDF) is

$$F(y) = \left[1 + \frac{\theta y^{-\alpha}\left(\theta y^{-\alpha} + 2\right)}{\theta^3 + 2}\right]e^{-\theta y^{-\alpha}} \quad y > 0, \theta, \alpha > 0. \tag{2}$$

The IPID is a generalized of the ishita distribution, which was presented by Shanker and Shukla [21] and proven to be adaptable to a variety of applications.

Modeling the COVID-19 is the driving force behind the new distribution. The COVID-19 of France was employed of 51 days that is recorded from 1 January to 20 February 2021 as real data for evaluating model techniques. COVID-19 data on new cases or deaths are discrete data (count). For France, we have been using the COVID-19 daily mortality rate. The daily mortality rate is continuous data.

Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$; $1 \le m \le n$ be a result of a Prog–II–C sample lifetime test that included n units derived from a IPI (α, θ) distribution and its censoring scheme is $R_1, R_2, \ldots, R_m$. Balakrishnan and Aggarwala [2] gives the joint PDF of a Prog-II.

$$f\left(x_{1:m:n}, x_{2:m:n}, \ldots . x_{m:m:n}\right) = C\prod_{i=1}^{m} f\left(x_{i:m:n;\alpha,\theta}\right).\left[1 - F\left(x_{i:m:n;\alpha,\theta}\right)\right]^{R_i}, m < n, \tag{3}$$

as $C = n(n - R_1 - 1)\ldots(n - \sum_{i=1}^{m-1}(R_i - 1))$.

Balakrishnan et al. [4] have excellent provides a review essay a recent report on progressive censoring schemes. This article is organized as follows: Section 2, introduced MLEs of α and θ. We survey the confidence intervals for the unknown parameters in Section 3. Two parametric bootstrap approaches are used to constructed unknown parameters in Section 4. In Section 5 different loss functions, such as SEL

and LINEX loss functions, Bayes estimates for unknown parameters were introduced. In Section 6, a real data set was studied. Section 7 introduces a simulation study to evaluate the accuracy of the various estimators generated in this study. At the end of this paper, conclusions are given in Section 8.

2. Maximum-Likelihood Estimation

The maximum likelihood estimation method (MLE) is used to study the problem of estimating IPID parameters under Prog–II–C data.

Let $x_{1:\,m:\,n}, x_{2:\,m:\,n}, \ldots, x_{m:\,m:\,n}$ be a Prog–II–C sample from the IPI distribution with PDF (1) and parameters α and θ. The likelihood function has the following:

$$L(\alpha,\theta) = C\frac{\alpha^m\theta^{3m}}{\left(\theta^3+2\right)^m}e^{-\theta\sum_{i=1}^{m}y_i^{-\alpha}}\prod_{i=1}^{m}\left[\left(\theta+y_i^{-2\alpha}\right)y_i^{-\alpha-1}\right]\prod_{i=1}^{m}\left[1-\left[1+\frac{\theta y_i^{-\alpha}\left(\theta y_i^{-\alpha}+2\right)}{\theta^3+2}\right]e^{-\theta y_i^{-\alpha}}\right]^{R_i}. \tag{4}$$

On Prog–II–C samples, the log-likelihood function of α and θ parameters as a result of

$$\begin{aligned} l &= \ln L(\alpha,\theta)\\ &= \ln c + m\ln\alpha + 3m\ln\theta - m\ln\left(\theta^3+2\right) - \theta\sum_{i=1}^{m}y_i^{-\alpha} + \sum_{i=1}^{m}\ln\left(\theta+y_i^{-2\alpha}\right)\\ &\quad -(\alpha+1)\sum_{i=1}^{m}\ln y_i + \sum_{i=1}^{m}R_i\ln\left[1-\left[1+\frac{\theta y_i^{-\alpha}\left(\theta y_i^{-\alpha}+2\right)}{\theta^3+2}\right]e^{-\theta y_i^{-\alpha}}\right]. \end{aligned} \tag{5}$$

Computing the first partial derivatives of (5) in relation to α and θ and solving this nonlinear system of equations $\partial/\partial\alpha \ln L(\alpha,\theta) = 0$ and $\partial/\partial\theta \ln L(\alpha,\theta) = 0$, the maximum likelihood estimates ($\widehat{\alpha}$ and $\widehat{\theta}$) are obtained. Accordingly, results in

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\left(\theta y_i^{-\alpha}-1\right)\ln y_i + \sum_{i=1}^{m}R_i\left[\frac{\theta y_i^{-\alpha}\ln y_i e^{-\theta y_i^{-\alpha}}}{1-\left[1+\theta y_i^{-\alpha}\left(\theta y_i^{-\alpha}+2\right)/\theta^3+2\right]e^{-\theta y_i^{-\alpha}}}\right] = 0, \tag{6}$$

and

$$\begin{aligned} &\frac{3m}{\theta} - m\frac{3\theta^2}{\theta^3+2} - \sum_{i=1}^{m}y_i^{-\alpha} + \sum_{i=1}^{m}\frac{1}{\theta+y_i^{-2\alpha}},\\ &+\sum_{i=1}^{m}R_i e^{-\theta y_i^{-\alpha}}\left[y_i^{-\alpha}\left[1+\frac{\theta y_i^{-\alpha}\left(\theta y_i^{-\alpha}+2\right)}{\theta^3+2}\right] - y_i^{-\alpha}\left[\frac{2\left(\theta^3+2\right)\left(\theta y_i^{-\alpha}+1\right)-3\theta^2\left(\theta y_i^{-\alpha}+2\right)}{\left(\theta^3+2\right)^2}\right]1\right.\\ &\left.-\left[1+\theta y_i^{-\alpha}\left(\theta y_i^{-\alpha}+2\right)/\theta^3+2\right]e^{-\theta y_i^{-\alpha}}\right] = 0, \end{aligned} \tag{7}$$

Since Equations (6) and (7) cannot be solved analytically, so to obtain the estimations, a numerical method like Newton–Raphson method should be employed. In Ahmed [5], the algorithm is given in detail.

3. Asymptotic Confidence Intervals

The elements of the Fisher information matrix's inverse give us the approximate variances and covariances of the MLEs, $\widehat{\alpha}$

and $\widehat{\theta}$; $I_{ij} = E[-[\partial^2 l(\Psi)/\partial\psi_i\partial\psi_j]]$, As $i, j = 1, 2$ also $\Psi = (\psi_1, \psi_2) = (\alpha, \theta)$. So, precise asymptotic forms for these equations are tough to comprehend get. The Fisher information matrix $\widehat{I_{ij}} = -[\partial^2 l(\Psi)/\partial\psi_i\partial\psi_j]_{\Psi=\widehat{\Psi}}$ is then used, we will use the expectation to establish confidence intervals (CIs) for the parameters, which is obtained through inference. Consequently, the information matrix that has been seen is

$$\widehat{I}_{(\widehat{\alpha},\widehat{\theta})} = \begin{bmatrix} \dfrac{-\partial^2 l}{\partial\alpha^2} & \dfrac{-\partial^2 l}{\partial\alpha\,\partial\theta} \\ \dfrac{-\partial^2 l}{\partial\theta\,\partial\alpha} & \dfrac{-\partial^2 l}{\partial\theta^2} \end{bmatrix}_{(\alpha=\widehat{\alpha},\theta=\widehat{\theta})}. \tag{8}$$

As a result, the asymptotic variance-covariance matrix $\widehat{V}$ for MLEs derived through converting observed information matrix $\widehat{I}(\alpha, \theta)$ or similar to the asymptotic variance-covariance matrix

$$\widehat{V} = \widehat{I}^{-1}_{(\widehat{\alpha},\widehat{\theta})} = \begin{bmatrix} \widehat{\mathrm{var}(\alpha)} & \mathrm{cov}(\alpha,\theta) \\ \mathrm{cov}(\alpha,\theta) & \widehat{var(\theta)} \end{bmatrix}_{(\widehat{\alpha},\widehat{\theta})}. \tag{9}$$

It is evident that $(\widehat{\alpha}, \widehat{\theta})$ is approximately distributed as multivariate normal with mean (α, θ) and covariance matrix $I^{-1}(\alpha, \theta)$ given specific symmetry requirements, see Bebbington et al. [22]. As a result. The $(1-\gamma)100\%$ convergent confidence intervals (ACIs) for α and θ may be calculated as follows:

$$\begin{aligned} \widehat{\alpha} &\pm z_{\gamma/2}\sqrt{\widehat{\mathrm{var}(\alpha)}}, \\ \widehat{\theta} &\pm z_{\gamma/2}\sqrt{\widehat{\mathrm{var}(\theta)}}, \end{aligned} \tag{10}$$

as $z_{\gamma/2}$ is the right-tail probability percentile standard normal distribution for $\gamma/2$ See Lawless [23].

4. Bootstrap Confidence Intervals

Different bootstrap confidence intervals are suggested here. The percentile bootstrap (Bp) and bootstrap-t (Bt) confidence intervals are used in this study. Efron [24] and Hall [25] have more information on bootstrap confidence intervals.

4.1. Parametric Percentile Bootstrap (Boot-P). The formula for calculating the parametric percentile bootstrap confidence interval for model parameters is as follows:

(1) Calculate the MLE of α and θ by maximizing Equations (6) and (7).

(2) Create bootstrap samples with α and θ to get the bootstrap estimate of α and θ, say $\widehat{\psi}^*$ where ψ be α or θ, from the bootstrap sample.

(3) To get $\widehat{\psi}_1{}^*, \widehat{\psi}_2{}^*, \ldots, \widehat{\psi_{N\,\mathrm{boot}}}{}^*$, repeat step 2 N times, where $\widehat{\psi}_i{}^* = (\widehat{\alpha}_i{}^*, \widehat{\theta}_i^*)$, $i = 1, 2, 3, ..N$ boot.

(4) Assign $\widehat{\psi}_i^* = 1, 2, 3, .., N\,boot$ to $\widehat{\psi}^*_{(1)}, \widehat{\psi}^*_{(1)}, \ldots, \widehat{\psi}^*_{(N\mathrm{boot})}$ in ascending order.

$$\left[\widehat{\psi}_{\mathrm{boot}-p}\left(\frac{\eta}{2}\right), \widehat{\psi}_{\mathrm{boot}-p}\left(1-\frac{\eta}{2}\right)\right], \tag{11}$$

Gives a two-sided $100(1-\eta)\%$ bootstrap-p for the unknown parameters α and θ.

4.2. Parametric Bootstrap-t Boot-t. The percentile bootstrap-t confidence interval for model parameters can be calculated using the following algorithm:

(1) Steps 1 and 2 in boot-p are the same

(2) Using the asymptotic variance-covariance matrix (9), the variance-covariance matrix $I^{-1^*}(\partial^2 l/\partial\psi_i\partial\psi_j)$, $i, j = 1, 2$ and get the t-statistic of ψ as $T^{*\psi} = (\widehat{\psi}^* - \widehat{\psi})/\sqrt{\widehat{\mathrm{var}(\widehat{\psi}^*)}}$

(3) NBoot times repeat steps 2 and 3 to get $T_1^{*\psi}, T_2^{*\psi}, \ldots, T_{N\mathrm{boot}}^{*\psi}$

(4) $T_1^{*\psi}, T_2^{*\psi}, \ldots, T_{N\mathrm{boot}}^{*\psi}$ the ordered sequences are sorted in ascending order as $T_1^{*\psi}, T_2^{*\psi}, \ldots, T_{N\mathrm{boot}}^{*\psi}$

$$\left[\widehat{\psi}_{\mathrm{boot}-t}\left(\frac{\eta}{2}\right), \widehat{\psi}_{\mathrm{boot}-t}\left(1-\frac{\eta}{2}\right)\right], \tag{12}$$

Calculates a two-sided bootstrap-t $100(1-\eta)\%$ CI for the unknown parameters α and θ.

5. Bayes Estimation

The Bayesian study of the IPID under Prog–II–C is discussed in this section. Under the SEL function and LINEX loss functions, we obtain Bayes estimators. When the parameters α and θ are independent and follow the following gamma prior distributions according to Zellner [26].

$$\begin{aligned} \pi_1(\alpha) &\propto \alpha^{k_1-1}\exp(-h_1\alpha), \alpha > 0, \\ \pi_2(\theta) &\propto \theta^{k_2-1}\exp(-h_2\theta), \theta > 0, \end{aligned} \tag{13}$$

where the hyper-parameters k_i and h_i, $i = 1, 2$ are assumed to be known and chosen to reflect the prior belief about the unknown parameters. The posterior distribution is derived from the likelihood function equation (4) and the prior distribution (13), and with the outcomes posterior distribution of α and θ denoted as $\pi^*(\alpha, \theta|\,\underline{x})$.

$$\pi^*(\alpha,\theta|\,\underline{x}) = \frac{L(\alpha,\theta;x)\pi_1(\alpha)\pi_2(\theta)}{\int_0^\infty\int_0^\infty L(\alpha,\theta;x)\pi_1(\alpha)\pi_2(\theta)\mathrm{d}\alpha\mathrm{d}\theta}. \tag{14}$$

5.1. Loss Function. We need to design an asymmetric loss function to make statistical Bayesian inference more practical and relevant. The loss function, as defined by Press and James [27], is a real-valued function that satisfies all feasible estimates and parameters. We must adopt an asymmetric loss function as well as create statistical Bayesian inference extra sensible and relevant. According to Press and James [27], the loss function is a real-valued function that satisfy all possible estimations and parameters.

5.1.1. Squared Error Loss Function (SEL). The SEL function is

$$L(\psi, \widehat{\psi}) = (\widehat{\psi} - \psi)^2, \tag{15}$$

Then, for any function of α and θ, the Bayes estimate is $g(\alpha, \theta)$ under the SEL function is

$$\widehat{g}_{BS}(\alpha, \theta|x) = E_{\alpha,\theta|x}(g(\alpha, \theta)), \tag{16}$$

where

$$E_{\alpha,\theta|x}(g(\alpha, \theta)) = \frac{\int_0^\infty \int_0^\infty g(\alpha, \theta)\pi_1(\alpha)\pi_2(\theta)L(\alpha, \theta|x)\mathrm{d}\alpha\mathrm{d}\theta}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\pi_2(\theta)L(\alpha, \theta|x)\mathrm{d}\alpha\mathrm{d}\theta}. \tag{17}$$

5.1.2. Linear Exponential (LINEX) Loss Function. The LINEX loss function $L(\Delta)$ for a parameter ψ is proposed according to Varian and Hall [28] given by the following equation:

$$\begin{aligned} L(\Delta) &= \left(e^{c\Delta} - c\Delta - 1\right), \\ c &\neq 0, \\ \Delta &= \widehat{\psi} - \psi, \end{aligned} \tag{18}$$

Hence, the bayes estimate of a function $g(\alpha, \theta)$ under LINEX loss function illustrated by the following equation:

$$\widehat{g}_{BL}(\alpha, \theta|x) = -\frac{1}{c}\log\left[E\left(e^{-g(\alpha,\theta)}|x\right)\right], c \neq 0,$$

$$E\left(e^{-g(\alpha,\theta)}\right) = \frac{\int_0^\infty \int_0^\infty e^{-g(\alpha,\theta)}\pi_1(\alpha)\pi_2(\theta)L(\alpha, \theta|x)\mathrm{d}\alpha\mathrm{d}\theta}{\int_0^\infty \int_0^\infty \pi_1(\alpha)\pi_2(\theta)L(\alpha, \theta|x)\mathrm{d}\alpha\mathrm{d}\theta}. \tag{19}$$

It should be noted that the ratio of multiple integrals in (17) and (19) cannot be expressed explicitly. To produce samples from the joint posterior density function in (14), the MCMC approach is used. We use the Gibbs within Metropolis–Hasting samplers process to add the MCMC approach. The joint posterior distribution expressed as follows:

$$\pi^*(\alpha, \theta|x) \propto m\frac{\alpha^{m+k_1-1}\theta^{3m+k_2-1}}{\left(\theta^3+2\right)^m}e^{-\left(h_1\alpha+h_2\theta\sum_{i=1}^m y_i^{-\alpha}\right)}$$

$$\prod_{i=1}^m\left[\left(\theta + y_i^{-2\alpha}\right)y_i^{-\alpha-1}\right]\prod_{i=1}^m\left[1-\left[1+\frac{\theta y_i^{-\alpha}(\theta y_i^{-\alpha}+2)}{\theta^3+2}\right]e^{-\theta y_i^{-\alpha}}\right]^{R_i}. \tag{20}$$

The conditional posterior densities of α and θ represented as follows:

$$\pi_1^*(\alpha\theta, x) \propto \alpha^{m+k_1-1}e^{-\left(h_1\alpha+\theta\sum_{i=1}^m y_i^{-\alpha}\right)}\prod_{i=1}^m\left[\left(\theta + y_i^{-2\alpha}\right)y_i^{-\alpha-1}\right]\prod_{i=1}^m\left[1-\left[1+\frac{\theta y_i^{-\alpha}(\theta y_i^{-\alpha}+2)}{\theta^3+2}\right]e^{-\theta y_i^{-\alpha}}\right]^{R_i}, \tag{21}$$

and

$$\pi_2^*(\theta|\alpha, x) \propto \frac{\theta^{3m+k_2-1}}{\left(\theta^3+2\right)^m}e^{-\theta\left(h_2+\sum_{i=1}^m y_i^{-\alpha}\right)}\prod_{i=1}^m\left[\left(\theta + y_i^{-2\alpha}\right)\right]\prod_{i=1}^m\left[1-\left[1+\frac{\theta(\theta y_i^{-\alpha}+2)}{\theta^3+2}\right]e^{-\theta y_i^{-\alpha}}\right]^{R_i}. \tag{22}$$

Because the conditional posteriors of α and θ in the previous equations do not qualify for a specified distribution, the Metropolis–Hasting sampler must be used to apply the MCMC approach. Tierney and Luke [29] proposed the Metropolis–Hastings algorithm within Gibbs sampling, which generates the posterior samples as follows:

(1) Begin with the first suggestion $(\alpha^{(0)}, \theta^{(0)})$

(2) Specify $j = 1$

(3) Produce $\alpha^{(j)}$ and $\theta^{(j)}$ by using M-H algorithm below using Equations (23) and (24) with the normal indicated distribution $N(\alpha^{j-1}, \mathrm{var}(\alpha))$ and the inverse Fisher information matrix is used to calculate $N(\theta^{j-1}, \mathrm{var}(\theta))$, where $\mathrm{var}(\alpha)$ and $\mathrm{var}(\theta)$

(i) Create a $N(\alpha^{j-1}, \mathrm{var}(\alpha))$ is suggestion to α^* and θ^* by $N(\theta^{j-1}, \mathrm{var}(\theta))$

(ii) Calculate the probabilities of approval

$$\rho_\alpha = \min\left[1, \frac{\pi_1^*\left(\alpha^*|\theta^{j-1}, x\right)}{\pi_1^*\left(\alpha^{j-1}|\theta^{j-1}, x\right)}\right], \tag{23}$$

and

$$\rho_\theta = \min\left[1, \frac{\pi_2^*\left(\theta^*|\alpha^j, x\right)}{\pi_2^*\left(\theta^{j-1} \mid \alpha^j, x\right)}\right]. \tag{24}$$

(iii) Create a u_1 and u_2 from a Uniform $(0, 1)$ distribution

(iv) If $u_1 = \rho_\alpha$, confess the proposition and Specify $\alpha^j = \alpha^*$ else Specify $\alpha^j = \alpha^{j-1}$

(v) If $u_1 = \rho_\theta$, confess the proposition and Specify $\theta^j = \theta^*$ else Specify $\theta^j = \theta^{j-1}$

(4) Calculate $\alpha^{(j)}$ and $\theta^{(j)}$

(5) Specify $j = j + 1$

(6) Steps from 3 to 6 should be reiterated N times

(7) To evaluate the CRIs of α and θ of ψ_k^i, $i = 1, 2, .., N$, $k = 1, 2$ and $(\psi_1, \psi_2) = (\alpha, \theta)$ as $\psi_k^1 < \psi_k^2 < \ldots < \psi_k^N$ then the $(1 - \gamma)100\%$ CRIs of ψ_k is

$$\left(\psi_k\left(\frac{\gamma}{2}(N-M)\right), \psi_k\left(\left(1-\frac{\gamma}{2}\right)(N-M)\right)\right). \tag{25}$$

For sufficiently large N, the samples chosen are $\psi_k^{(i)}$, $j = M + 1, \ldots, N$. SEL function is accustomed to get the approximate Bayes estimates of $\widehat{\alpha}$ and $\widehat{\theta}$ as follows:

$$\widehat{\alpha}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N}\alpha^{(j)}, \tag{26}$$

and

$$\widehat{\theta}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N}\theta^{(j)}. \tag{27}$$

And, the estimates for the mentioned parameters under LINEX loss function are

$$\alpha_{BL} = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\alpha^{(i)}}\right], \tag{28}$$

and

$$\theta_{BL} = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-c\theta^{(i)}}\right]. \tag{29}$$

6. Real-Life Data Application

In this section, the proposed estimation methods are applied to the parameters of the IPI model under a Prog–II–C scheme using a real data set from the COVID-19 data from France, which was analyses by Almetwally [6]. The considered COVID-19 data belong to France of 51 days that is recorded from 1 January to 20 February 2021. This data formed of mortality rate. A Prog–II–C sample of size $m = 10$ simulated randomly from the sample of size $n = 20$ with censoring scheme $(5, 5, 0, 0, 0, 0, 0, 0, 0, 0)$. For the data, we calculated the Kolmogorov–Smirnov (K–S) distance (D) between the fitted and empirical distribution functions, with K–S = 0.0999069 and p-value = 0.688698.The MLEs of parameters depend on Prog-II failure data in Table 1 are $\widehat{\alpha}$ and $\widehat{\theta}$ respectively, and are shown in Table 2. The mean of 1000 Boot-p and Boot-t samples of the lifetime parameters also given in Table 2 through using the algorithm defined in Section 4 of the bootstrap methods. For various values of the LINEX loss function's shape parameter c, the parameters α and θ, Table 3 calculated and shewed the Bayes estimates related to both SEL and LINEX functions; the 95% ACIs and CRIs of the parameters α and θ have calculated. The LINEX loss function becomes symmetric as c going to zero, as is well known. In Table 2, the outputs of the SEL and LINEX loss functions are identical at $c = 0.0001$, confirming accuracy of presented s. Figure 2 indicate simulation numbers of the parameters α and θ for COVID-19 data.

7. Simulated Data

Simulation experiments are carried out utilizing 1000 (Prog –II–C) samples per one simulation in order to compare parameter estimators for the IPI distribution. IPID is used to create (Prog –II– C) samples, with initial values $\alpha = 1$ and $\theta = 0.6$. The comparison of the different approaches of the producing α and θ estimators, has been taken into account in the computation of their mean square error (MSE), for $k = 1, 2\,(\psi_1 = \alpha, \psi_2 = \theta)$, as MSE $(\psi_k) = 1/M\,(\widehat{\psi}_k - \psi_k)^2$, as $M = 1000$ simulated samples numbers. The 95% CIs generated through utilizing asymptotic distributions of the MLEs and CRIs are compared using another criterion. The average confidence interval lengths (ACLs) and coverage probability (CP) are used to make comparisons between them. In this study the progressive schemes that have considered:

Scheme I: $R_1 = n - m$, $R_i = 0$ for $i \neq 1$

Scheme II: $R_{m/2} = R_{m/2+1} = n - m/2$, $R_i = 0$ for $i \neq m/2$ and $i \neq m/2 + 1$

Scheme III: $R_m = n - m$, $R_i = 0$ for $i \neq m$

Tables 4 and 5 illustrate the estimated parameters' results and their MSE and Table 6 shows outcomes of ACL and CP of 95% CIs. From the results, the following notices can be observed:

(1) Tables 4 and 5 show that as sample size increase, MSEs decrease, with Bayes estimates having the smallest MSEs for α and θ. As a result, Bayes estimates performance MLE approaches in all situations studied.

(2) In the sense that MSEs are smaller, Bayes estimates under LINEX with $c = -2$ better results estimates.

(3) Scheme I outperforms schemes II and III in terms of MSEs when the sample size is fixed at n and the effective size is fixed at m.

(4) Adapted from Table 6. It can be seen that the CRIs produce more accurate results. Then, the ACIs for different sample sizes, observed failures, and schemes.

Table 1: The Progressively Type-II failure COVID-19 data.

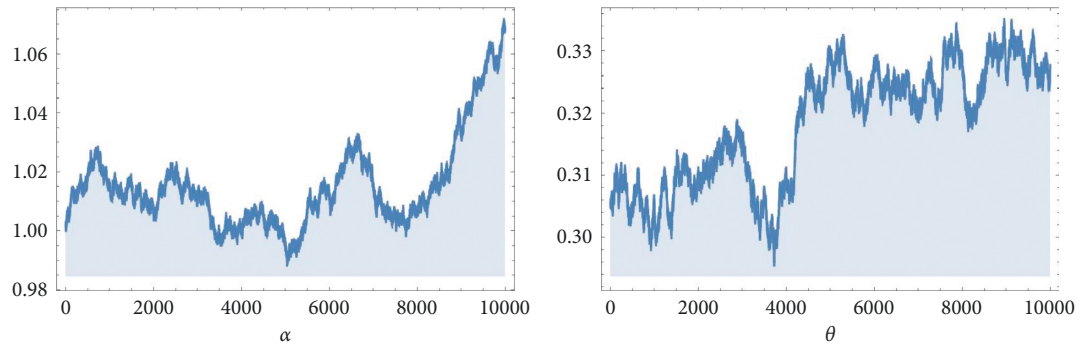
x_i:	0.0995	0.0525	0.0615	0.0455	0.1474
R_i	5	5	0	0	0
x_i:	0.3373	0.1087	0.1055	0.2235	0.0633
R_i	0	0	0	0	0
x_i:	0.0565	0.2577	0.1345	0.0843	0.1023
R_i	0	0	0	0	0
x_i:	0.2296	0.0691	0.0505	0.1434	0.2326
R_i	0	0	0	0	0

Table 2: Point estimates for the parameters α, θ, for COVID-19 data.

Parameter	MLE	Boot-p	Boot-t	SEL	LINEX		
					$c_1 = -2$	$c_2 = 2$	$c_3 = 0.0001$
α	1.00871	1.3251	1.1135	0.982731	0.983072	0.982389	0.982731
θ	0.307831	0.5218	0.4231	0.27371	0.273741	0.27371	0.27371

Table 3: 95% ACIs and CRI α and θ COVID-19 data.

Parameter	MLE	Boot-p	Boot-t	MCMC
α	(0.603973, 1.41345)	(0.8751, 1.6324)	(0.5891, 1.3271)	(0.951173,1.01104)
θ	(-0.0186961, 0.634358)	(0.6782, 0.8792)	(0.4351, 0.7452)	(0.264976,0.285227)



(a) (b)

Figure 2: Simulation numbers of the parameters for COVID-19 data.

Table 4: MSE of MLE and BE parameters estimators α with $\alpha_0 = 1$.

(n, m)	Scheme	**MLE**	**SEL**	LINEX		
				$C = -2$	$C = 2$	$C = 0.0001$
(15, 10)	I	0.3703 (40.24)	0.6855 (9.0013)	1.878 (1.24439)	0.5904 (10.3129)	0.6854 (9.001)
	II	0.3784 (39.37)	0.8011 (23.2342)	2.738 (6.55607)	1.1106 (22.5079)	0.8007 (23.2216)
	III	0.4204 (35.29)	0.7239 (20.2274)	2.8229 (3.03739)	1.5548 (42.8603)	0.7236 (20.2287)
(20, 15)	I	0.2742 (5.294)	0.4937 (0.80339)	1.159 (0.60741)	0.1183 (1.18222)	0.4936 (0.80343)
	II	0.2774 (5.268)	0.2852 (3.8652)	0.897 (3.8086)	0.3366 (5.715)	0.2851 (3.8653)
	III	0.3221 (4.651)	0.8523 (1.068194)	2.5606 (1.07274)	0.7491 (1.20591)	0.8519 (1.68009)

Table 4: Continued.

(n, m)	Scheme	MLE	SEL	LINEX		
				$C = -2$	$C = 2$	$C = 0.0001$
(30, 20)	I	0.2088 (0.6282)	0.097 (0.13922)	0.2514 (0.11393)	0.0584 (0.17185)	0.097 (0.13923)
	II	0.202 (0.6389)	0.2392 (0.14607)	0.4757 (0.13187)	0.0126 (0.1949)	0.2392 (0.14608)
	III	0.2124 (0.6242)	0.2377 (0.15856)	0.4532 (0.14083)	0.02 (0.20695)	0.2377 (0.15856)
(40, 20)	I	0.1596 (0.7071)	0.2231 (0.10083)	0.3151 (0.009619)	0.1302 (0.01739)	0.2231 (0.010083)
	II	0.1871 (0.06631)	0.3382 (0.16286)	0.0891 (0.007436)	0.0891 (0.017436)	0.3382 (0.016284)
	III	0.2888 (0.5133)	0.3395 (0.096754)	1.5987 (0.02266)	0.969 (0.0758)	0.3394 (0.096828)
(50, 20)	I	0.158 (0.007097)	0.1879 (0.0011341)	0.2888 (0.001328)	0.0828 (0.01085)	0.1879 (0.001879)
	II	0.2254 (0.06032)	0.4393 (0.0018168)	0.8991 (0.0026371)	0.0106 (0.023192)	0.4393 (0.0018168)
	III	0.3913 (0.379)	2.9493 (0.0823423)	7.8626 (0.0212)	1.9799 (0.05568)	2.9482 (0.0823)

Table 5: MSE of MLE and BE estimates for the parameter β with $\beta_0 = 0.6$.

(n, m)	Scheme	MLE	SEL	LINEX		
				$C = -2$	$C = 2$	$C = 0.0001$
(15, 10)	I	0.9585 (14.85)	0.6084 (12.6991)	2.44 (1.80526)	2.44 (14.96)	0.6082 (12.6988)
	II	0.9306 (12.8)	0.6808 (11.3128)	2.8284 (1.5829)	2.8284 (2.31639.9981)	0.6806 (11.3137)
	III	0.9845 (17.54)	0.9995 (2.3791)	3.2166 (0.31721)	3.2166 (4.35105)	0.9992 (2.379)
(20, 15)	I	0.8452 (0.812)	0.6675 (0.38233)	2.3146 (0.2418)	0.1183 (0.8796)	0.6672 (0.38221)
	II	0.8261 (6.98)	0.6913 (5.751)	2.0049 (0.83298)	2.0049 (0.99981)	2.0049 (0.99981)
	III	0.9156 (12.71)	0.8301 (0.0325622)	3.103 (0.0356730)	3.103 (0.9974)	0.8294 (0.325915)
(30, 20)	I	0.8798 (0.1006)	1.1282 (0.030879)	1.7577 (0.020327)	1.1281 (0.02195)	1.2181 (0.0309)
	II	0.8307 (0.725)	0.8266 (0.39378)	1.6663 (0.047928)	1.663 (0.047928)	0.8265 (0.39377)
	III	1.0391 (2.074)	1.0708 (0.058934)	1 .9744 (0.0087964)	1.9744 (0.049418)	1.0707 (0.058932)
(40, 20)	I	1.1567 (0.03138)	0.8393 (0.00437)	1.739 (0.00571)	1.739 (0.00571)	0.8372 (0.0636)
	II	1.0661 (0.2308)	0.9043 (0.04472)	2.9307 (0.05844)	1.7436 (0.059018)	1.1098 (0.06479)
	III	0.9937 (1.643)	1.11 (0.0080091)	2.5991 (0.001266)	2.5991 (0.01403)	1.1098 (0.0901)
(50, 20)	I	1.1712 (0.03312)	1.2884 (0.00392)	2.035 (0.000067)	2.035 (0.00319)	1.2884 (0.003914)
	II	1.0273 (0.1902)	1.094 (0.00744)	2.1157 (0.00894)	2.1157 (0.00813)	1.094 (0.0074286)
	III	0.9647 (0.1407)	0.534 (0.002475)	3.4552 (0.000545)	3.4552 (0.00472)	0.5336 (0.02475)

Table 6: ACL and CB of 95% CIs for the parameters α and β.

(n, m)	Scheme	α MLE	α MCMC	β MLE	β MCMC
(15, 10)	I	1.9729	3.7715	2.0206	4.868
		(0.953)	(0.9706)	(0.9278)	(0.9447)
	II	2.46234	4.48107	3.0967	5.1826
		(0.9405)	(0.9421)	(0.9353)	(0.9615)
	III	3.1753	5.1875	3.849	5.6644
		(0.9491)	(0.968)	(0.9297)	(0.9499)
(20, 15)	I	1.28025	2.3428	1.91155	3.8025
		(0.9459)	(0.9342)	(0.926)	(0.9305)
	II	1.28637	2.244	1.33569	3.7107
		(0.9384)	(0.9354)	(0.9711)	(0.9261)
	III	2.38825	4.0415	3.25721	5.7014
		(0.9644)	(0.9652)	(0.9296)	(0.9371)
(30, 20)	I	0.636235	1.0541	0.592525	2.5754
		(0.9403)	(0.9408)	(0.9565)	(0.9527)
	II	0.685303	1.2593	0.738147	2.8534
		(0.9506)	(0.9431)	(0.9636)	(0.9464)
	III	0.712548	1.2123	0.781155	3.1536
		(0.9444)	(0.9611)	(0.9387)	(0.9536)
(40, 20)	I	0.428134	0.834	0.634894	3.0317
		(0.9737)	(0.9743)	(0.9337)	(0.9586)
	II	0.624713	1.1338	0.726405	2.8997
		(0.9254)	(0.9715)	(0.935)	(0.9481)
	III	1.97717	3.4896	2.10026	4.0023
		(0.964)	(0.947)	(0.9397)	(0.9454)
(50, 20)	I	0.478148	0.8091	0.72361	2.7625
		(0.9329)	(0.9553)	(0.9526)	(0.9689)
	II	1.08.73	1.823	1.14455	3.229
		(0.9358)	(0.9649)	(0.9473)	(0.946)
	III	5.38097	9.1965	5.68429	6.3552
		(0.9444)	(0.9699)	(0.9337)	(0.9521)

8. Conclusion

We discuss an iterative technique for deriving MLEs from an inverse power ishita lifetime distribution using progressively Type-II censored samples. Another estimation approach, based on Bayes estimates, is also considered. The Bayes estimates were produced using loss functions; however, they are not available in explicit form. Propose that the Bayes estimators and associated IPI distribution CIs be computed using the MCMC technique. Derive the posterior summaries of interest, such as credible intervals for the parameters, in a simple way using typical MCMC simulation methods for a Bayesian study of the model. The findings of this study used to the estimate problem of the IPID using complete data, which are usually progressive type-II censoring samples. The proposed methods are demonstrated with numerical examples [30, 31].

Data Availability

The Covid-19 data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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