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Research Article

On the Extensions of Zassenhaus Lemma and Goursat's Lemma to Algebraic Structures

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The Jordan–Hölder theorem is proved by using Zassenhaus lemma which is a generalization of the Second Isomorphism Theorem for groups. Goursat's lemma is a generalization of Zassenhaus lemma, it is an algebraic theorem for characterizing subgroups of the direct product of two groups $G_1 \times G_2$, and it involves isomorphisms between quotient groups of subgroups of G_1 and G_2. In this paper, we first extend Goursat's lemma to R-algebras, i.e., give the version of Goursat's lemma for algebras, and then generalize Zassenhaus lemma to rings, R-modules, and R-algebras by using the corresponding Goursat's lemma, i.e., give the versions of Zassenhaus lemma for rings, R-modules, and R-algebras, respectively.

1. Introduction

The Fundamental Homomorphism Theorem (or the First Isomorphism Theorem) provided by Noether [1] in 1927 shows that every homomorphism gives rise to an isomorphism and that quotient groups are merely constructions of homomorphic images. Even it has simple form, it expresses the important properties of quotient group. Noether emphasized the fundamental importance of this fact, and it has been widely used in the field of universal algebra and to prove the existence of some natural isomorphisms. The Diamond Isomorphism Theorem (or the Second Isomorphism Theorem) which is the consequence of the Fundamental Homomorphism Theorem is formulated in terms of subgroups of the normalizer and relates two quotient groups involving products and intersections of subgroups. After that, many researchers generalized the Second Isomorphism Theorem to other structures, such as Tamaschke [2, 3] generalized it to Schur semigroups [4] and Endam and Vilela [5] extended it to B-algebras introduced by Neggers and Kim [6, 7]. Sequentially, the First Isomorphism Theorem and Second Isomorphism Theorem are generalized to rings, vector spaces, R-modules, and R-algebras, respectively. In this paper, we suppose that the readers are familiar with the structures of groups, rings, R-modules, and R-algebras.

In 1934, Zassenhaus [8] found a new and beautiful proof of the Jordan–Hölder theorem via Zassenhaus lemma. Zassenhaus lemma (also called Butterfly lemma) is well known in group theory as a generalization of the Second Isomorphism Theorem for groups. Subsequently, many researchers extended Zassenhaus lemma to other structures, such as Teh [9, 10] extended it to universal algebras via employment of the graph theory, Wyler [11] extended it to categories under conditions, Ngaha Ngaha [12] considered the Second Isomorphism Theorem and Zassenhaus lemma in star-regular categories and described Zassenhaus lemma in the category of commutative Hopf algebras, and Prathomjit et al. [13] extended Zassenhaus lemma and the Schreier refinement theorem to the case of gyrogroups and then used these results to prove the Jordan–Hölder theorem for gyrogroups.

On the other hand, as we know, there is a natural question in group theory, that is, how many subgroups do exist for the general group G? If we want to count the number of subgroups of G, we should first express the form of subgroups of G. In 1889, Goursat [14] gave Goursat's lemma which is an algebraic theorem for characterizing subgroups of the direct product of two groups $G_1 \times G_2$, and involves isomorphisms between quotient groups of subgroups of G_1 and G_2. After that, many researchers expressed the subgroups in certain classes of groups; for instance, Usenko [15] introduced a reduced crossed homomorphism and used it to describe the subgroups of a semidirect product of groups and characterized the sub-semidirect products and semidirect products with a given structure of normal subgroups. Bauer et al. [16] obtained a description of the subgroups of a direct product $G_1 \times G_2 \times \cdots \times G_n$ of a finite number of groups by proving a generalization of Goursat's lemma. In 2009, Anderson and Camillo [17] proved Zassenhaus lemma by using Goursat's lemma, which means that Zassenhaus lemma is a corollary of Goursat's lemma. From Goursat's lemma, one can recover a more general version of Zassenhaus lemma. According to the above, in this paper, our main aim is to extend Goursat's lemma to R-algebras and give the expression of the form of subalgebras of general R-algebras. Further, we generalize Zassenhaus lemma to rings, R-modules, and R-algebras by using the corresponding Goursat's lemma, respectively, and show that the corresponding results are the generalization of the Second Isomorphism Theorem for rings, R-modules, and R-algebras, respectively.

In fact, Goursat's lemma (see Theorem 1) gave a way to describe the subgroups of a direct product $G_1 \times G_2$ which involves isomorphisms between quotient groups of subgroups of G_1 and G_2, i.e.,

$$H = \{(h_1, h_2) \in H_{12} \times H_{22} | f_H(h_1 H_{11}) = h_2 H_{21}\} \quad (1)$$

is a subgroup of $G_1 \times G_2$, where H_{i1} and H_{i2} are subgroups of G_i such that $H_{i1} \lhd H_{i2}$ for $i = 1, 2$, and $f_H\colon H_{12}/H_{11} \longrightarrow H_{22}/H_{21}$ is a group isomorphism. Subsequently, Anderson and Camillo (see Theorem 2) described the subrings of a direct product $R_1 \times R_2$ which involves isomorphisms between quotient rings of subrings of R_1 and R_2, i.e.,

$$T = \{(r_1, r_2) \in T_{12} \times T_{22} | f_T(r_1 + T_{11}) = r_2 + T_{21}\} \quad (2)$$

is a subring of $R_1 \times R_2$, where T_{i1} and T_{i2} are subrings of R_i such that T_{i1} is an ideal of T_{i2} for $i = 1, 2$, and $f_T\colon T_{12}/T_{11} \longrightarrow T_{22}/T_{21}$ is a ring isomorphism. For R-modules, Dickson (see Theorem 3) described the submodules of a direct product $M_1 \times M_2$ which involves isomorphisms between quotient R-modules of submodules of M_1 and M_2, i.e.,

$$M = \{(m_1, m_2) \in M_{12} \times M_{22} | f_M(m_1 + M_{11}) = m_2 + M_{21}\} \quad (3)$$

is a submodule of $M_1 \times M_2$, where M_{i1} and M_{i2} are R-submodules of M_i such that $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$, and $f_M\colon M_{12}/M_{11} \longrightarrow M_{22}/M_{21}$ is a R-module isomorphism.

Since an R-algebra has the ring structure and R-module structure, as a generalization, we consider Goursat's lemma for R-algebras (see Corollary 1) and describe the subalgebras of a direct product $A_1 \times A_2$ which involves isomorphisms between quotient R-algebras of subalgebras of A_1 and A_2, that is,

$$A = \{(a_1, a_2) \in A_{12} \times A_{22} | f_A(a_1 + A_{11}) = a_2 + A_{21}\} \quad (4)$$

is a subalgebra of $A_1 \times A_2$, where A_{i1} and A_{i2} are subalgebras of A_i such that A_{i1} is algebraic ideal of A_{i2} for $i = 1, 2$, and $f_A\colon A_{12}/A_{11} \longrightarrow A_{22}/A_{21}$ is an R-algebra isomorphism. Indeed, Lambek [18] gave Goursat's characterization of the subgroups of the direct product of two groups (also for a general class of algebras) under conditions by using graph theory. In this paper, we give a different form for the subalgebras of R-algebras. However, for other structures, such as Lie algebras, quantum cluster algebras [19], B-algebras, and free differential algebras [20], we still do not know their Goursat's characterization.

Furthermore, Anderson and Camillo [17] used Goursat's lemma to prove Zassenhaus lemma for groups (see Theorem 4) which is stated as follows:

$$\frac{N_1(H_1 \cap H_2)}{N_1(H_1 \cap N_2)} \cong \frac{N_2(H_1 \cap H_2)}{N_2(N_1 \cap H_2)}, \quad (5)$$

where $H_1, N_1, H_2,$ and N_2 are subgroups of G such that $N_1 \lhd H_1$ and $N_2 \lhd H_2$. As a generalization, we consider Zassenhaus lemma for rings (see Theorem 5), R-modules (see Theorem 6), and R-algebras (see Theorem 7), respectively, and obtain the following results:

(1) If $R_1, I_1, R_2,$ and I_2 are subrings of a ring R such that I_i is an ideal of R_i for $i = 1, 2$, then

$$\frac{I_1 + R_1 \cap R_2}{I_1 + R_1 \cap I_2} \cong \frac{I_2 + R_1 \cap R_2}{I_2 + I_1 \cap R_2}. \quad (6)$$

(2) If $M_1, N_1, M_2,$ *and* N_2 are submodules of R-module M satisfying $N_1 \subseteq M_1, N_2 \subseteq M_2$ for commutative ring R with identity, then

$$\frac{N_1 + M_1 \cap M_2}{N_1 + M_1 \cap N_2} \cong \frac{N_2 + M_1 \cap M_2}{N_2 + N_1 \cap M_2}. \quad (7)$$

(3) If A is an R-algebra and $B_1, C_1, B_2,$ *and* C_2 are subalgebras of A such that C_i is an algebraic ideal of B_i for $i = 1, 2$, then

$$\frac{C_1 + B_1 \cap B_2}{C_1 + B_1 \cap C_2} \cong \frac{C_2 + B_1 \cap B_2}{C_2 + C_1 \cap B_2}. \quad (8)$$

This paper is organized as follows. In Section 2, we first complete the proof of Goursat's lemma for R-modules and then extend Goursat's lemma to R-algebras, i.e., give the version of Goursat's lemma for algebras and give the form of subalgebras of R-algebras, which is different from Lambek's form [18]. Further, since Zassenhaus lemma is used to prove the Jordan–Hölder theorem and it is also the generalization the Second Isomorphism Theorem for groups, as a generalization, we extend Zassenhaus lemma to rings, R-modules, vector spaces, and R-algebras in terms of algebraic ideal in Section 3, i.e., give the versions of Zassenhaus lemma for rings, R-modules, and R-algebras, respectively.

2. Goursat's Lemma for Groups, Rings, R-Modules, and R-Algebras

Anderson and Camillo [17] gave an exposition of Goursat's lemma for groups and rings, and Dickson [21] gave Goursat's lemma for R-modules without proof, respectively. In this section, we complete the proof of Goursat's lemma for R-modules and give an exposition of Goursat's lemma for R-algebras as a corollary.

Theorem 1 (Goursat's lemma for groups, Theorem 4 in [17]). Let G_1 and G_2 be groups.

(1) *Let H be a subgroup of $G_1 \times G_2$, and*

$$\begin{aligned} H_{11} &= \{a \in G_1 \mid (a, 1) \in H\}, \\ H_{12} &= \{a \in G_1 \mid (a, b) \in H \text{ for some } b \in G_2\}, \\ H_{21} &= \{b \in G_2 \mid (1, b) \in H\}, \\ H_{22} &= \{b \in G_2 \mid (a, b) \in H \text{ for some } a \in G_1\}. \end{aligned} \tag{9}$$

Then, H_{i1} and H_{i2} are subgroups of G_i with $H_{i1} \triangleleft H_{i2}$ for $i = 1, 2$, and the map

$$f_H: \frac{H_{12}}{H_{11}} \longrightarrow \frac{H_{22}}{H_{21}}, aH_{11} \mapsto bH_{21} \tag{10}$$

is an isomorphism, where $(a, b) \in H$. Moreover, if $H \triangleleft G_1 \times G_2$, then $H_{i1}, H_{i2} \triangleleft G_i$ and $H_{i2}/H_{i1} \subseteq C(G_i/H_{i1})$, the center of G_i/H_{i1}.

(2) *Let H_{i1} an dH_{i2} be subgroups of G_i with $H_{i1} \triangleleft H_{i2}$ for $i = 1, 2$, and let $f: H_{12}/H_{11} \longrightarrow H_{22}/H_{21}$ be an isomorphism. Then,*

$$H = \{(a, b) \in H_{12} \times H_{22} \mid f(aH_{11}) = bH_{21}\} \tag{11}$$

is a subgroup of $G_1 \times G_2$. Furthermore, suppose $H_{i1}, H_{i2} \triangleleft G_i$ and $H_{i2}/H_{i1} \subseteq C(G_i/H_{i1})$ for $i = 1, 2$, then $H \triangleleft G_1 \times G_2$.

(3) *The constructions given in (1) and (2) are inverses to each other.*

Theorem 2 (Goursat's lemma for rings, Theorem 11 in [17]). Let R_1 and R_2 be rings.

(1) *Let T be a subring of $R_1 \times R_2$, and*

$$\begin{aligned} T_{11} &= \{r \in R_1 \mid (r, 0) \in T\}, \\ T_{12} &= \{r \in R_1 \mid (r, s) \in T \text{ for some } s \in R_2\}, \\ T_{21} &= \{s \in R_2 \mid (0, s) \in T\}, \\ T_{22} &= \{s \in R_2 \mid (r, s) \in T \text{ for some } r \in R_1\}. \end{aligned} \tag{12}$$

Then, T_{i2} is a subring of R_i and T_{i1} is an ideal of T_{i2} for $i = 1, 2$. Moreover, the map $f_T: T_{12}/T_{11} \longrightarrow T_{22}/T_{21}$ defined by $f_T(r + T_{11}) = s + T_{21}$ for $(r, s) \in T$ is a ring isomorphism.

(2) *Suppose that T_{i2} is a subring of R_i and T_{i1} is an ideal of T_{i2} for $i = 1, 2$, and $f_T: T_{12}/T_{11} \longrightarrow T_{22}/T_{21}$ is a ring isomorphism. Then,*

$$T = \{(r, s) \in T_{12} \times T_{22} \mid f_T(r + T_{11}) = s + T_{21}\} \tag{13}$$

is a subring of $R_1 \times R_2$.

(3) *The construction given in (1) and (2) is inverse to each other.*

In Theorem 4 in [17], the authors stated Goursat's lemma for R-modules without proof (also see [21, 22]). In the following, we provide a proof of Goursat's lemma for R-modules by using the submodule criterion [23].

Theorem 3 (Goursat's lemma for R-modules). *Let R be a commutative ring with identity and M_1 and M_2 are R-modules.*

(1) *Let M be a submodule of $M_1 \times M_2$, and*

$$\begin{aligned} M_{11} &= \{m_1 \in M_1 \mid (m_1, 0) \in M\}, \\ M_{21} &= \{m_2 \in M_2 \mid (0, m_2) \in M\}, \\ M_{12} &= \{m_1 \in M_1 \mid (m_1, m_2) \in M \text{ for some } m_2 \in M_2\}, \\ M_{22} &= \{m_2 \in M_2 \mid (m_1, m_2) \in M \text{ for some } m_1 \in M_1\}, \end{aligned} \tag{14}$$

then M_{i1} and M_{i2} are submodules of M_i with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$, and the map $f_M: M_{12}/M_{11} \longrightarrow M_{22}/M_{21}$ given by $f_M(m_1 + M_{11}) = m_2 + M_{21}$ is a R-module isomorphism, where $(m_1, m_2) \in M$.

(2) *Suppose that M_{i1} and M_{i2} are submodules of M_i with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$ and the map $f_M: M_{12}/M_{11} \longrightarrow M_{22}/M_{21}$ is a R-module isomorphism, then*

$$M = \{(m_1, m_2) \in M_{12} \times M_{22} \mid f_M(m_1 + M_{11}) = m_2 + M_{21}\}, \tag{15}$$

is a submodule of $M_1 \times M_2$.

(3) *The construction given in (1) and (2) is inverse to each other.*

Proof. (1) From Theorem 1, it is obvious that $M_{i1} \subseteq M_{i2} \subseteq M_i$ for $i = 1, 2$. Since M is a submodule of $M_1 \times M_2$, we have

$$(m_{11}, 0) + r(m'_{11}, 0) = (m_{11} + rm'_{11}, 0) \in M, \tag{16}$$

for any $m_{11}, m'_{11} \in M_{11}$ and any $r \in R$. This means that $m_{11} + rm'_{11} \in M_{11}$. Thus, M_{11} is a submodule of M_1 following the submodule criterion [23]. On the other hand, for any $m_{12}, m'_{12} \in M_{12}$, there exist $m_2, m'_2 \in M_2$ such that $(m_{12}, m_2), (m'_{12}, m'_2) \in M$. Hence,

$$(m_{12}, m_2) + r(m'_{12}, m'_2) = (m_{12} + rm'_{12}, m_2 + rm'_2) \in M, \tag{17}$$

for any $r \in R$. It follows that $m_{12} + rm_{12}' \in M_{12}$, and then M_{12} is a submodule of M_1. Similarly, using the submodule criterion, we can obtain that M_{21} an dM_{22} are submodules of M_2 easily.

For the map $f_M\colon M_{12}/M_{11} \longrightarrow M_{22}/M_{21}$ given by $f_M(m_1 + M_{11}) = m_2 + M_{21}$, where $(m_1, m_2) \in M$, it is clear that M_{12}/M_{11} and M_{22}/M_{21} are R-modules and f_M is an additive group isomorphism by Theorem 1. So, it is enough to prove that $f_M(r(m_1 + M_{11})) = rf_M(m_1 + M_{11})$ for any $r \in R$ and $m_1 \in M_{12}$. Since $r(m_1, m_2) = (rm_1, rm_2) \in M$ for any $m_1 \in M_{12}, m_2 \in M_{22}$, and $r \in R$, we have $f_M(r(m_1 + M_{11})) = f_M\ (rm_1 + M_{11}) = rm_2 + M_{21} = rf_M(m_1 + M_{11})$. Hence, f_M is a R-module isomorphism.

(2) Suppose that M_{i1} and M_{i2} are submodules of M_i with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$, and the map $f_M\colon M_{12}/M_{11} \longrightarrow M_{22}/M_{21}$ is a R-module isomorphism. Since $M = \{(m_1, m_2) \in M_{12} \times M_{22} | f_M(m_1 + M_{11}) = m_2 + \ M_{21}\}$, we have $f_M(m_1 + M_{11}) = m_2 + M_{21}, f_M(m_1' + M_{11}) = m_2' + M_{21}$ for any $(m_1, m_2), (m_1', m_2') \in M$, and

$$\begin{aligned} &f_M((m_1 + M_{11}) + r(m_1' + M_{11})) = f_M((m_1 + rm_1') + M_{11})\\ &= f_M(m_1 + M_{11}) + rf_M(m_1' + M_{11})\\ &= m_2 + M_{21} + r(m_2' + M_{21})\\ &= (m_2 + rm_2') + M_{21}. \end{aligned} \tag{18}$$

This means that $(m_1 + rm_1', m_2 + rm_2') = (m_1, m_2) + r(m_1', m_2') \in M$. Therefore, M is a submodule of $M_1 \times M_2$ by the submodule criterion.

(3) From the proofs of (1) and (2), we can easily obtain (3).

It is well known that an R-algebra A has the ring structure and R-module structure concurrently, and the operations of these two structures are compatible, i.e., $r(xy) = (rx)y = x(ry)$ for any $x, y \in A$ and $r \in R$. We firstly introduce some definitions about R-algebras for the commutative ring R with identity.

Definition 1 (see [23–25]). Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism $f\colon R \longrightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A, i.e., $f(r)$ commutes with every element of A for each $r \in R$. A subalgebra of an R-algebra A is a subring of A and a submodule of A. A left (respectively, right, two sided) algebraic ideal of an R-algebra A is a left (respectively, right, two sided) ideal of the ring A and a submodule of A.

If A and B are two R-algebras, an R-algebra homomorphism (respectively, isomorphism) is a ring homomorphism (respectively, isomorphism) $\varphi\colon A \longrightarrow B$ mapping 1_A to 1_B such that $\varphi(r \cdot a) = r \cdot \varphi(a)$ for all $r \in R$ and $a \in A$.

It is easy to check that if A is an R-algebra, then A is a ring with identity and has a natural left and right R-module structure defined by $r \cdot a = a \cdot r = f(r)a$, where $f(r)a$ is just the multiplication in the ring A. Every ring with identity is actually an $\mathbb{Z}$-algebra.

Since R-algebra has the ring structure and module structure by Definition 1, according to Theorem 2 and Theorem 3, we can obtain Goursat's lemma for R-algebras easily.

Corollary 1 (Goursat's lemma for R-algebras). *Let R be a commutative ring with identity and A_1 and A_2 be R-algebras.*

(1) Let A be a subalgebra of $A_1 \times A_2$, and

$$\begin{aligned} A_{11} &= \{a \in A_1 | (a, 0) \in A\},\\ A_{12} &= \{a \in A_1 | (a, b) \in A \text{ for some } b \in A_2\},\\ A_{21} &= \{b \in A_2 | (0, b) \in A\},\\ A_{22} &= \{b \in A_2 | (a, b) \in A \text{ for some } a \in A_1\}, \end{aligned} \tag{19}$$

then A_{i1} and A_{i2} are subalgebras of A_i such that A_{i1} is an algebraic ideal of A_{i2} for $i = 1, 2$, and the map

$$f_A\colon \frac{A_{12}}{A_{11}} \to \frac{A_{22}}{A_{21}}, a + A_{11} \mapsto b + A_{21}, \tag{20}$$

is an R-algebra isomorphism, where $(a, b) \in A$.

(2) Suppose that A_{i1} and A_{i2} are subalgebras of A_i such that A_{i1} is an algebraic ideal of A_{i2} for $i = 1, 2$, and $f_A\colon A_{12}/A_{11} \longrightarrow A_{22}/A_{21}$ is an R-algebra isomorphism, then

$$A = \{(a, b) \in A_{12} \times A_{22} | f_A(a + A_{11}) = b + A_{21}\} \tag{21}$$

is a subalgebra of $A_1 \times A_2$.

(3) The construction given in (1) and (2) is inverse to each other.

Example 1. Let $A = \mathbb{Z}_2[x]/(x^3 - 1)$, which is an $\mathbb{Z}_2$-algebra. We want to find out all the subalgebras of $A \times A$.

Step 1. Find out all the subalgebras A_i's $(1 \leq i \leq 6)$ and their corresponding algebraic ideals in A. Here, we omit the case when the corresponding algebraic ideal is A_i itself.

Case 1. The subalgebra $A_1 = \{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2\}$ with 8 elements of A, and the algebraic ideals of A_1 are as follows:

$$\begin{aligned} J_{11} &= \{0, 1 + x, 1 + x^2, x + x^2\},\\ J_{12} &= \{0, 1 + x + x^2\},\\ J_{13} &= \{0\}. \end{aligned} \tag{22}$$

Thus,

$$\frac{A_1}{J_{11}} = \left\{[0] = [1+x] = \left[1+x^2\right] = \left[x+x^2\right], [1] = [x] = \left[x^2\right] = \left[1+x+x^2\right]\right\},$$
$$\frac{A_1}{J_{12}} = \left\{[0] = \left[1+x+x^2\right], [1] = \left[x+x^2\right], [x] = \left[1+x^2\right], \left[x^2\right] = [1+x]\right\}, \quad (23)$$
$$\frac{A_1}{J_{13}} = \frac{R_1}{\{0\}} = A_1,$$

and A_1/J_{11} has 2 elements, A_1/J_{12} has 4 elements, and A_1 has 8 elements.

Case 2. The subalgebra $A_2 = \{0, 1, x+x^2, 1+x+x^2\}$ with 4 elements of A, and the algebraic ideals of A_2 are

$$J_{21} = \left\{0, x+x^2\right\},$$
$$J_{22} = \left\{0, 1+x+x^2\right\},$$
$$J_{23} = \{0\},$$
$$\frac{A_2}{J_{21}} = \left\{[0] = \left[x+x^2\right], [1] = \left[1+x+x^2\right]\right\}, \quad (24)$$
$$\frac{A_2}{J_{22}} = \left\{[0] = \left[1+x+x^2\right], [1] = \left[x+x^2\right]\right\},$$
$$\frac{A_2}{J_{23}} = \frac{A_2}{\{0\}} = A_2,$$

and A_2/J_{2i} has 2 elements for $i = 1, 2$, and A_2 has 4 elements.

Case 3. For the subalgebras $A_3 = \{0, 1+x, 1+x^2, x+x^2\}$, $A_4 = \{0, x+x^2\}$, $A_5 = \{0, 1+x+x^2\}$, and $A_6 = \{0, 1\}$, their algebraic ideals are only $\{0\}$ and A_i for $i = 3, 4, 5, 6$.

Step 2. According to Corollary 1, we first construct the isomorphism and then give the subalgebras of $A \times A$.

Case 4 (with 2 elements). Here, we only discuss one case. We have an isomorphism $A_2/J_{21} \cong A_4/\{0\}$, and the subalgebra of $A \times A$ is

$$\left\{(0,0), \left(x+x^2, 0\right), \left(1, x+x^2\right), \left(1+x+x^2, x+x^2\right)\right\}. \quad (25)$$

Case 5 (with 4 elements).

(a) We have an isomorphism $f\colon A_1/J_{12} \longrightarrow A_3/\{0\}$ defined by $f([0]) = 0$, $f([1]) = x+x^2$, $f([x]) = 1+x$, $f([x^2]) = 1+x^2$, and the subalgebra of $A \times A$ is

$$\left\{(0,0), \left(1+x+x^2, 0\right), \left(1, x+x^2\right), \left(x+x^2, x+x^2\right), (x, 1+x), \left(1+x^2, 1+x\right), \left(x^2, 1+x^2\right), \left(1+x, 1+x^2\right)\right\}. \quad (26)$$

(b) We have an isomorphism $f\colon A_1/J_{12} \longrightarrow A_1/J_{12}$ which is an identity or defined by $f([0]) = [0]$, $f([1]) = [1]$, $f([x]) = [x^2]$, an d $f([x^2]) = [x]$, and the subalgebra of $A \times A$ for the identity case is

$$\left\{(0,0), \left(0, 1+x+x^2\right), \left(1+x+x^2, 0\right), \left(1+x+x^2, 1+x+x^2\right), (1,1), \left(1, x+x^2\right),\right.$$
$$\left(x+x^2, 1\right), \left(x+x^2, x+x^2\right), (x,x), \left(x, 1+x^2\right), \left(1+x^2, x\right), \left(1+x^2, 1+x^2\right), \quad (27)$$
$$\left.\left(x^2, x^2\right), \left(x^2, 1+x\right), \left(1+x, x^2\right), (1+x, 1+x)\right\},$$

and the subalgebra of $A \times A$ for the other case is

$$\left\{(0,0), \left(0, 1+x+x^2\right), \left(1+x+x^2, 0\right), \left(1+x+x^2, 1+x+x^2\right), (1,1),\right.$$
$$\left(1, x+x^2\right), \left(x+x^2, 1\right), \left(x+x^2, x+x^2\right), \left(x, x^2\right), (x, 1+x), \left(1+x^2, x^2\right), \quad (28)$$
$$\left.\left(1+x^2, 1+x\right), \left(x^2, x\right), \left(x^2, 1+x^2\right), (1+x, x), \left(1+x, 1+x^2\right)\right\}.$$

(c) For the isomorphism $f\colon A_3/\{0\} \longrightarrow A_3/\{0\}$ which is an identity or defined by $f(0)=0, f(x+x^2)=x+x^2, f(1+x)=1+x^2$, and $f(1+x^2)=1+x$, the subalgebra of $A\times A$ for the identity case is

$$\left\{(0,0),\left(x+x^2,x+x^2\right),(1+x,1+x),\left(1+x^2,1+x^2\right)\right\}, \tag{29}$$

and the subalgebra of $A\times A$ for the other case is

$$\left\{(0,0),\left(x+x^2,x+x^2\right),\left(1+x,1+x^2\right),\left(1+x^2,1+x\right)\right\}. \tag{30}$$

(d) For the isomorphism $f\colon A_2/\{0\} \longrightarrow A_2/\{0\}$ which is an identity or defined by $f(0)=0, f(1)=1, f(x+x^2)=1+x+x^2$, and $f(1+x+x^2)=x+x^2$, the subalgebra of $A\times A$ for the identity case is

$$\left\{(0,0),(1,1),\left(x+x^2,x+x^2\right),\left(1+x+x^2,1+x+x^2\right)\right\}, \tag{31}$$

and the subalgebra of $A\times A$ for the other case is

$$\left\{(0,0),(1,1),\left(x+x^2,1+x+x^2\right),\left(1+x+x^2,x+x^2\right)\right\}. \tag{32}$$

Case 6. (with 8 elements). The isomorphism $f\colon A_1/\{0\} \longrightarrow A_1/\{0\}$ is an identity, and the subalgebra of $A\times A$ is

$$\left\{(0,0),(1,1),(x,x),\left(x^2,x^2\right),\left(x+x^2,x+x^2\right),(1+x,1+x),\left(1+x^2,1+x^2\right),\left(1+x+x^2,1+x+x^2\right)\right\}. \tag{33}$$

3. The Zassenhaus Lemma for Groups, Rings, R-Modules, and R-Algebras

In group theory, it is well known that the four basic isomorphism theorems (Fundamental Homomorphism Theorem, Diamond Isomorphism Theorem, Freshman Theorem, and Correspondence Theorem) about homomorphism and their structure are very useful in the study of groups. All of these theorems have analogues in other algebraic structures, such as rings, vector spaces, R-modules, and R-algebras. Furthermore, for the Second Isomorphism Theorem, the set-theoretic version of the product formula (i.e., if G_1 and G_2 are subgroups of G, then there is a natural bijection between $|G_1/(G_1\cap G_2)|$ and $|G_1G_2/G_2|$), establishes a bijection which is the same as the bijection of the Second Isomorphism Theorem, but without the conditions of normality, and the bijection is purely at the set-theoretic level. In 1934, Zassenhaus gave Zassenhaus lemma which is a generalization of the Second Isomorphism Theorem for groups (see Lemma 5.10 in [26] and Lemma 4.52 in [27]). In this section, we give Zassenhaus lemma for rings, R-modules, and R-algebras, which is also a generalization of the Second Isomorphism Theorem for rings, R-modules, and R-algebras, respectively.

Theorem 4 (Zassenhaus lemma for groups, Lemma 4.52 in [27]). *Suppose that G is a group with subgroups H_1, N_1, H_2, and N_2 such that $N_1 \triangleleft H_1$ and $N_2 \triangleleft H_2$, then*

$$N_1(H_1\cap N_2) \triangleleft N_1(H_1\cap H_2), N_2(N_1\cap H_2) \triangleleft N_2(H_1\cap H_2), \tag{34}$$

and there is an isomorphism

$$\begin{aligned}\frac{N_1(H_1\cap H_2)}{N_1(H_1\cap N_2)} &\cong \frac{(H_1\cap H_2)}{(H_1\cap N_2)(N_1\cap H_2)}\\ &\cong \frac{N_2(H_1\cap H_2)}{N_2(N_1\cap H_2)}.\end{aligned} \tag{35}$$

Remark 1. Zassenhaus lemma for groups implies the Second Isomorphism Theorem for groups, i.e., if H and N are subgroups of a group G with $N \triangleleft G$, then

$$\frac{NH}{N} \cong \frac{H}{(H\cap N)}. \tag{36}$$

In fact, we can let $H = H_1\cap H_2$ and $N = N_1(H_1\cap N_2)$. Thus, according to Theorem 4, we have

$$\frac{N_1(H_1\cap H_2)}{N} \cong \frac{H}{(H_1\cap N_2)(N_1\cap H_2)}. \tag{37}$$

Indeed, for any $xy \in (H_1\cap N_2)(N_1\cap H_2)$ where $x \in H_1\cap N_2 \subseteq H_1\cap H_2$ and $y \in N_1\cap H_2 \subseteq H_1\cap H_2$, since $xy = y(y^{-1}xy) \in N_1(H_1\cap N_2)$, we have $xy \in (H_1\cap H_2) \cap N_1(H_1\cap N_2) = H\cap N$. Conversely, for any $n_1x \in (H_1\cap H_2)\cap N_1(H_1\cap N_2) = H\cap N$ where $n_1 \in N_1$ and $x \in H_1\cap N_2 \subseteq H_1\cap H_2$, there exists $h_2 \in H_2$ such that $n_1x = h_2 \in H_1\cap H_2$, i.e., $n_1 = h_2x^{-1} \in H_2$. It follows that $n_1x = x(x^{-1}n_1x) \in (H_1\cap N_2)(N_1\cap H_2)$. Thus,

$$\begin{aligned}(H_1\cap N_2)(N_1\cap H_2) &= (H_1\cap H_2)\cap N_1(H_1\cap N_2)\\ &= H\cap N.\end{aligned} \tag{38}$$

Further, it is easy to obtain that

$$N_1(H_1\cap H_2) = (N_1(H_1\cap N_2))(H_1\cap H_2) = NH. \tag{39}$$

Therefore, (37) can be written as $NH/N \cong H/(H \cap N)$.

Following Lemma 4.52 in [27], we can also set $H_1 = G, N_1 = N, H_2 = H$, and $N_2 = N \cap H$ and then obtain that $NH/N \cong H/(H \cap N)$.

Example 2. Consider the symmetric group S_4 with 24 elements which are as follows:

$$\{(1), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}. \quad (40)$$

Let

$$\begin{aligned} N_1 &= \{(1), (1324), (12)(34), (1423)\},\\ N_2 &= \{(1), (123), (132), (134), (143), (124), (142), (234), (243), (12)(34), (13)(24), (14)(23)\},\\ H_1 &= \{(1), (12), (34), (1324), (1423), (12)(34), (13)(24), (14)(23)\}, \end{aligned} \quad (41)$$

and $H_1, N_1,$ and N_2 are subgroups of S_4 with $N_1 \lhd H_1$. Let $H_2 = S_4$, then $N_2 \lhd H_2$. Thus,

$$\begin{aligned} N_1(H_1 \cap H_2) &= N_1 H_1 = H_1,\\ N_1(H_1 \cap N_2) &= \{(1), (12), (34), (1324), (1423), (12)(34), (13)(24), (14)(23)\} = H_1,\\ N_2(H_1 \cap H_2) &= N_2 H_1,\\ N_2(N_1 \cap H_2) &= N_2 N_1. \end{aligned} \quad (43)$$

then $N_2 H_1 = N_2 N_1$ by applying for Theorem 4. In fact, according to the definitions of $N_1, N_2,$ and H_1, we have

$$H_1 \cap N_2 = \{(1), (12)(34), (13)(24), (14)(23)\}, \quad (42)$$

By Theorem 4, we have $N_2 H_1 / N_2 N_1 = H_1 / H_1$, which means that $N_2 H_1 = N_2 N_1$.

As a generalization of the Second Isomorphism Theorem for rings, by using Theorem 2, we obtained Zassenhaus lemma for rings stated as follows.

Theorem 5 (Zassenhaus lemma for rings). *Let $R_1, I_1, R_2,$ and I_2 be subrings of a ring R such that I_i is an ideal of R_i for $i = 1, 2$, then $I_1 + R_1 \cap I_2$ (respectively, $I_2 + I_1 \cap R_2$) is an ideal of $I_1 + R_1 \cap R_2$ (respectively, $I_2 + R_1 \cap R_2$), and*

$$\frac{I_1 + R_1 \cap R_2}{I_1 + R_1 \cap I_2} \cong \frac{I_2 + R_1 \cap R_2}{I_2 + I_1 \cap R_2}. \quad (44)$$

Proof. Let $T = \{(a + c, b + c) \in R_1 \times R_2 | a \in I_1, b \in I_2, c \in R_1 \cap R_2\}$, then T is a subring of $R_1 \times R_2$. Indeed, for any $(a_1 + c_1, b_1 + c_1), (a_2 + c_2, b_2 + c_2) \in T$, where $a_i \in I_1, b_i \in I_2, c_i \in R_1 \cap R_2$ for $i = 1, 2$, since $R_1 \cap R_2$ is also a subring of R and I_i is an ideal of R_i for $i = 1, 2$, we have

$$\begin{aligned} (a_1 + c_1, b_1 + c_1) - (a_2 + c_2, b_2 + c_2) &\in T,\\ (a_1 + c_1, b_1 + c_1)(a_2 + c_2, b_2 + c_2) &\in T, \end{aligned} \quad (45)$$

which implies that T is a subring of $R_1 \times R_2$. According to Theorem 2, let

$$\begin{aligned} T_{12} &= \{r \in R_1 | (r, s) \in T \text{ for some } s \in R_2\},\\ T_{22} &= \{s \in R_2 | (r, s) \in T \text{ for some } r \in R_1\}, \end{aligned} \quad (46)$$

then $T_{12} = I_1 + R_1 \cap R_2, T_{22} = I_2 + R_1 \cap R_2$, and $T_{i2} = I_i + R_1 \cap R_2$ is a subring of R_i for $i = 1, 2$ clearly. Further, let

$$\begin{aligned} T_{11} &= \{r \in R_1 | (r, 0) \in T\},\\ T_{21} &= \{s \in R_2 | (0, s) \in T\}, \end{aligned} \quad (47)$$

then $T_{11} = I_1 + R_1 \cap I_2, T_{21} = I_2 + I_1 \cap R_2$, and T_{i1} is an ideal of T_{i2} for $i = 1, 2$. Following Theorem 2, we have $T_{12}/T_{11} \cong T_{22}/T21$, which shows that

$$\frac{I_1 + R_1 \cap R_2}{I_1 + R_1 \cap I_2} \cong \frac{I_2 + R_1 \cap R_2}{I_2 + I_1 \cap R_2}. \quad (48)$$

□

Remark 2. The Zassenhaus lemma for rings implies the second isomorphism theorem for rings, i.e., if S is a subring of a ring R and I an ideal of R, then

$$(S + I)/I \cong S/(S \cap I). \quad (49)$$

In fact, let $I_1 = I$, $R_1 = R$, $I_2 = I \cap S$, and $R_2 = S$, since $I + I \cap S = I$, $I \cap S + S = S$, and $I \cap S + I \cap S = I \cap S$, we have

$$\begin{aligned} I_1 + R_1 \cap R_2 &= I_1 + S = I + S,\\ I_1 + R_1 \cap I_2 &= I_1 + I_2 = I + I \cap S = I,\\ I_2 + R_1 \cap R_2 &= I_2 + S = I \cap S + S = S,\\ I_2 + I_1 \cap R_2 &= I_2 + I \cap S = I \cap S + I \cap S = I \cap S. \end{aligned} \tag{50}$$

According to Theorem 5, we have $(S + I)/I \cong S/(S \cap I)$.

For R-modules, we know that a module is a mathematical object in which things can be added together commutatively by multiplying coefficients and in which most of the rules of manipulating vectors hold. A module is abstractly very similar to a vector space although in modules coefficients are taken in rings that are much more general algebraic objects than the fields used in vector spaces. R-modules can be thought of as generalizations of vector spaces and abelian groups. We will also see that they can be regarded as "representations" of a ring.

For the Second Isomorphism Theorem for R-modules, by using Goursat's lemma for R-modules, we have the following generalized Zassenhaus lemma.

Theorem 6 (Zassenhaus lemma for R-modules). *Let R be a commutative ring with identity and M an R-module. Suppose that $M_1, N_1, M_2,$ and N_2 are submodules of M satisfying $N_1 \subseteq M_1, N_2 \subseteq M_2$, then $N_1 + M_1 \cap N_2$ (respectively, $N_2 + N_1 \cap M_2$) is a submodule of $N_1 + M_1 \cap M_2$ (respectively, $N_2 + M_1 \cap M_2$), and*

$$\frac{N_1 + M_1 \cap M_2}{N_1 + M_1 \cap N_2} \cong \frac{N_2 + M_1 \cap M_2}{N_2 + N_1 \cap M_2}. \tag{51}$$

Proof. Let $L = \{(a + c, b + c) \in M_1 \times M_2 | a \in N_1, b \in N_2, c \in M_1 \cap M_2\}$; since $M_1 \cap M_2$ is also a submodule of M and N_i is a submodule of M_i for $i = 1, 2$, we have

$$\begin{aligned} &(a_1 + c_1, b_1 + c_1) + r(a_2 + c_2, b_2 + c_2)\\ &= (a_1 + ra_2 + c_1 + rc_2, b_1 + rb_2 + c_1 + rc_2) \in L, \end{aligned} \tag{52}$$

for any $r \in R, a_i \in N_1, b_i \in N_2, c_i \in M_1 \cap M_2, i = 1, 2$, which implies that L is a submodule of $M_1 \times M_2$ by the submodule criterion [23]. Following Theorem 3, let

$$\begin{aligned} M_{11} &= \{m_1 \in M_1 | (m_1, 0) \in L\},\\ M_{21} &= \{m_2 \in M_2 | (0, m_2) \in L\},\\ M_{12} &= \{m_1 \in M_1 | (m_1, m_2) \in L \text{ for some } m_2 \in M_2\},\\ M_{22} &= \{m_2 \in M_2 | (m_1, m_2) \in L \text{ for some } m_1 \in M_1\}, \end{aligned} \tag{53}$$

as in the proof of Theorem 5, we have

$$\begin{aligned} M_{11} &= N_1 + M_1 \cap N_2,\\ M_{12} &= N_1 + M_1 \cap M_2,\\ M_{21} &= N_2 + N_1 \cap M_2,\\ M_{22} &= N_2 + M_1 \cap M_2, \end{aligned} \tag{54}$$

and M_{i1} and M_{i2} are submodules of M_i with $M_{i1} \subseteq M_{i2}$ for $i = 1, 2$. Thus, by Theorem 3, we have

$$\frac{N_1 + M_1 \cap M_2}{N_1 + M_1 \cap N_2} \cong \frac{N_2 + M_1 \cap M_2}{N_2 + N_1 \cap M_2}. \tag{55}$$

□

Remark 3. Zassenhaus lemma for modules implies the Second Isomorphism Theorem for modules, i.e., if K_1 and K_2 are submodules of a R-module M, then $(K_1 + K_2)/K_1 \cong K_2/(K_1 \cap K_2)$. In fact, let $N_1 = K_1$, $M_1 = M, N_2 = K_1 \cap K_2$, and $M_2 = K_2$; since $K_1 + K_1 \cap K_2 = K_1, K_2 + K_1 \cap K_2 = K_2$, $K_1 \cap K_2 + K_1 \cap K_2 = K_1 \cap K_2$, we have

$$\begin{aligned} N_1 + M_1 \cap M_2 &= K_1 + K_2,\\ N_1 + M_1 \cap N_2 &= K_1 + K_1 \cap K_2 = K_1,\\ N_2 + M_1 \cap M_2 &= K_1 \cap K_2 + K_2 = K_2,\\ N_2 + N_1 \cap M_2 &= K_1 \cap K_2 + K_1 \cap K_2 = K_1 \cap K_2. \end{aligned} \tag{56}$$

Following Theorem 6, we have $(K_1 + K_2)/K_1 \cong K_2/(K_1 \cap K_2)$.

Similarly, we can easily obtain Zassenhaus lemma for vector spaces, which is also a generalization of the Second Isomorphism Theorem for vector spaces; that is, if V is a vector space and V_1 and V_2 are linear subspaces of V, then $(V_1 + V_2)/V_2 \cong V_1/(V_1 \cap V_2)$. By Theorem 6, we have the following corollary.

Corollary 2 (Zassenhaus lemma for vector spaces). *Let V be a vector space and $U_1, V_1, U_2,$ and V_2 are linear subspaces of V satisfying $V_1 \subseteq U_1, V_2 \subseteq U_2$, then $V_1 + U_1 \cap V_2$ (respectively, $V_2 + V_1 \cap U_2$) is a subspace of $V_1 + U_1 \cap U_2$ (respectively, $V_2 + U_1 \cap U_2$), and*

$$\frac{V_1 + U_1 \cap U_2}{V_1 + U_1 \cap V_2} \cong \frac{V_2 + U_1 \cap U_2}{V_2 + V_1 \cap U_2}. \tag{57}$$

For the Second Isomorphism Theorem for R-algebras, by using Corollary 1, we have the following generalized Zassenhaus lemma.

Theorem 7 (Zassenhaus lemma for R-algebras). *Suppose that A is an R-algebra and $B_1, C_1, B_2,$ and C_2 are subalgebras of A such that C_i is an algebraic ideal of B_i for $i = 1, 2$, then $C_1 + B_1 \cap C_2$ (respectively, $C_2 + C_1 \cap B_2$) is an algebraic ideal of $C_1 + B_1 \cap B_2$ (respectively, $C_2 + B_1 \cap B_2$), and*

$$\frac{C_1 + B_1 \cap B_2}{C_1 + B_1 \cap C_2} \cong \frac{C_2 + B_1 \cap B_2}{C_2 + C_1 \cap B_2}. \tag{58}$$

Proof. Since $B_1 \cap B_2$ is a subring and a submodule of A, it tells that $B_1 \cap B_2$ is a subalgebra of A. Let $\Gamma = \{(a + c, b + c) \in B_1 \times B_2 | a \in C_1, b \in C_2, c \in B_1 \cap B_2\}$; since C_i is an algebraic ideal of B_i for $i = 1, 2$, then Γ is a subring and a submodule of $B_1 \times B_2$ from Theorem 5 and Theorem 6, which implies that Γ is a subalgebra of $B_1 \times B_2$. Following Theorem 3, let

$$\begin{aligned}\Gamma_{11} &= \{\gamma_1 \in B_1 | (\gamma_1, 0) \in \Gamma\},\\ \Gamma_{21} &= \{\gamma_2 \in B_2 | (0, \gamma_2) \in \Gamma\},\\ \Gamma_{12} &= \{\gamma_1 \in B_1 | (\gamma_1, \gamma_2) \in \Gamma \text{ for some } \gamma_2 \in B_2\},\\ \Gamma_{22} &= \{\gamma_2 \in B_2 | (\gamma_1, \gamma_2) \in \Gamma \text{ for some } \gamma_1 \in B_1\},\end{aligned} \tag{59}$$

then we have

$$\begin{aligned}\Gamma_{11} &= C_1 + B_1 \cap C_2, \Gamma_{12} = C_1 + B_1 \cap B_2,\\ \Gamma_{21} &= C_2 + C_1 \cap B_2, \Gamma_{22} = C_2 + B_1 \cap B_2,\end{aligned} \tag{60}$$

and Γ_{i1} an dΓ_{i2} are subalgebras of B_i such that Γ_{i1} is an algebraic ideal of Γ_{i2} for $i = 1, 2$ according to Theorem 5 and Theorem 6. Thus, by Theorem 3, we have

$$\frac{C_1 + B_1 \cap B_2}{C_1 + B_1 \cap C_2} \cong \frac{C_2 + B_1 \cap B_2}{C_2 + C_1 \cap B_2}. \tag{61}$$

□

Remark 4. Zassenhaus lemma for R-algebra implies the Second Isomorphism Theorem for R-algebra, i.e., if A is an R-algebra and B and J are subalgebras of A such that J is an algebraic ideal of A, then

$$\frac{J + B}{J} \cong \frac{B}{B \cap J}. \tag{62}$$

In fact, let $C_1 = J$, $B_1 = A$, $C_2 = J \cap B$, and $B_2 = B$; since $J + J \cap B = J, B + J \cap B = B, J \cap B + J \cap B = J \cap B$, we have

$$\begin{aligned}C_1 + B_1 \cap B_2 &= J + B,\\ C_1 + B_1 \cap C_2 &= J + J \cap B = J,\\ C_2 + B_1 \cap B_2 &= J \cap B + B = B,\\ C_2 + C_1 \cap B_2 &= J \cap B + J \cap B = J \cap B.\end{aligned} \tag{63}$$

Following Theorem 7, we have $(J + B)/J \cong B/(B \cap J)$.

Example 3. Let $C_1 = (1 + x) = \{(1 + x) f(x) | f(x) \in \mathbb{Z}[x]\}$, $B_1 = \mathbb{Z}[x]$ which is an $\mathbb{Z}$-algebra, $C_2 = (x^2)$, and $B_2 = (x^2, 5) = \{x^2 f(x) + 5g(x) | f(x), g(x) \in \mathbb{Z}[x]\}$, then C_i is an algebraic ideal of B_i for $i = 1, 2$, and

$$\begin{aligned}C_1 + B_1 \cap B_2 &= (1 + x) + \left(x^2, 5\right) = \left(1 + x, x^2, 5\right),\\ C_1 + B_1 \cap C_2 &= (1 + x) + \left(x^2\right) = \left(1 + x, x^2\right),\\ C_2 + B_1 \cap B_2 &= \left(x^2\right) + \left(x^2, 5\right) = \left(x^2, 5\right),\\ C_2 + C_1 \cap B_2 &= \left(x^2\right) + (1 + x) \cap \left(x^2, 5\right).\end{aligned} \tag{64}$$

According to Theorem 7, we have

$$\frac{\left(1 + x, x^2, 5\right)}{\left(1 + x, x^2\right)} \cong 5\mathbb{Z} \cong \frac{\left(x^2, 5\right)}{\left[\left(x^2\right) + \left(x^2, 5\right) \cap (1 + x)\right]}. \tag{65}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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