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Research Article

δ_ω-Continuity and some Results on δ_ω-Closure Operator

Manjeet Singh, Asha Gupta, and Kushal Singh

Department of Applied Sciences, Punjab Engineering College, Chandigarh 160012, India

Correspondence should be addressed to Manjeet Singh; manjeetsingh.phdappsc@pec.edu.in

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Al-Jarrah et al. defined a new topological operator, namely, δ_ω-closure operator, and proved that it lies between the δ-closure operator and the usual closure operator. Al-Ghour et al. defined θ_ω-closure operator and discussed its properties. In this paper, it is proved that the δ_ω-closure operator lies between the θ_ω-closure operator and the usual closure operator. Also, sufficient conditions are given for the equivalence between the δ_ω-closure operator and the θ_ω-closure operator. Moreover, we define three new types of continuity, namely, δ_ω-continuity, ω-δ-continuity, and almost δ_ω-continuity, and discuss their properties. It is proved that the concepts of usual continuity and δ_ω-continuity are independent of each other. In addition, the relationships between different types of continuity have been investigated. Further, some examples and counter examples are given.

1. Introduction

Generating new topologies from old ones is a very interesting and useful concept in topology. Let (G, τ) be a topological space (abbreviated as ts) with no separation axioms assumed. The closure of a subset E of G is denoted by the symbol $\bar{E}$, and the interior of a subset E is denoted by the symbol int E. A subset E is regular open (regular closed) if int $(\bar{E}) = E(\overline{\text{int}\, E} = E)$. A point z in G is called a δ-cluster point [1] of a subset E if for every open subset O containing z, int $(\bar{O}) \cap E \neq \varnothing$. The collection of all δ-cluster points of a subset E is called δ-closure of E and is denoted by the symbol $Cl_\delta E$. A subset E is called δ-closed [1] if $Cl_\delta E = E$ and its complement is δ-open. The topology on G of all δ-open sets is denoted by the symbol τ_δ. The basis of τ_δ is the family of all regular open sets. A ts is called semiregular if $\tau_\delta = \tau$. The δ-closure operator and its related concepts have been studied and generalized by many authors [2–6]. All regular open sets, with the exception of the empty set, are included in the class of somewhere dense sets. Somewhere dense sets have been studied by many authors [7, 8]. In 1982, Hdeib [9] classified ω-closed sets as well as the complement of an ω-closed set, i.e., an ω-open set. τ_ω stands for the collection of all ω-open sets. The collection τ_ω forms a topology finer than the usual topology τ. Further, the ω-open and ω-closed sets and their properties have been studied by many authors [10–12]. In paper [13], ω-regularity has been defined as a generalization of regularity by Al-Ghour. In 2019, Al-Jarrah et al. [14] defined δ_ω-closure of a set by using ω-closure as follows: A point z in G is called δ_ω-contact point of a subset E if the interior of every ω-closed neighborhood (abbreviated as nbhd) of z intersects E. The set of all δ_ω-contact points of E is called the δ_ω-closure of E and is denoted by the symbol $Cl_{\delta_\omega} E$. The δ_ω-closure operator is a new topological operator which strictly lies between the usual closure and the δ-closure operators. Al Ghour et al. [13, 15] have defined new topological operators, namely, θ_ω-closure operator and $\theta_{\mathcal{N}}$-closure operator, and discussed their properties.

This paper is organized as follows: After the introduction, definitions and results that have been already defined and proved are included in preliminaries and will be utilized to support various findings in the following sections. In the next section, we have proved that δ_ω-closure operator lies between the usual closure operator and θ_ω-closure operator. Sufficient conditions are given for the equivalence of δ_ω-closure operator and θ_ω-closure operator, and also of δ_ω-closure operator and the usual closure operator. In last section, we define some new types of continuity, namely, δ_ω-continuity, ω-δ-continuity, and almost δ_ω-continuity. We observed that continuity and δ_ω-continuity are independent of each other. Moreover, we give some sufficient conditions for a δ_ω-continuous function to be continuous. In addition, we establish the relationships between different types of continuity and discuss some of their properties. Some examples and counter-examples are also provided.

2. Preliminaries

Definition 1 (see [9]). Let (G, τ) be a ts and $E \subseteq G$, then, the set of condensation points of E is denoted by the symbol $\mathrm{Cond}(E)$ and defined as $\mathrm{Cond}(E) = \{z \in G : O \cap E \text{ is uncountable}, O \in \tau \text{ and } z \in O\}$.

The set E is called ω-closed if $\mathrm{Cond}(E) \subseteq E$ and its complement is ω-open. τ_ω represents the collection of all ω-open sets.

Theorem 2 (see [9]). *Let (G, τ) be a ts and $E \subseteq G$, then*

(a) τ_ω forms a topology on G

(b) $\tau \subseteq \tau_\omega$ and $\tau_\omega \neq \tau$ in general

The closure of E in (G, τ_ω) is denoted by the symbol $\bar{E}^\omega$. Also, $\bar{E}^\omega \subseteq \bar{E}$ and $\bar{E}^\omega \neq \bar{E}$ in general.

Definition 3 (see [1]). Let (G, τ) be a ts and $E \subseteq G$, then, the set $\{z \in G : \bar{O} \cap E \neq \varnothing, O \in \tau \text{ and } z \in O\}$ is called the θ-closure of E and denoted by the symbol $Cl_\theta E$.

The set E is called θ-closed if $Cl_\theta E = E$ and its complement is θ-open. τ_θ represents the collection of all θ-open sets.

Theorem 4 (see [1]). *Let (G, τ) be a ts and $E \subseteq G$, then*

(a) τ_θ forms a topology on G

(b) $\tau_\theta \subseteq \tau$ and $\tau_\theta \neq \tau$ in general

Theorem 5 (see [1]). *A ts is regular if $\tau_\theta = \tau$.*

Definition 6 (see [13]). Let (G, τ) be a ts and $E \subseteq G$, then, the set $\{z \in G : \bar{O}^\omega \cap E \neq \varnothing, O \in \tau \text{ and } z \in O\}$ is called the θ_ω-closure of E and denoted by the symbol $Cl_{\theta_\omega} E$.

The set E is said to be θ_ω-closed if $Cl_{\theta_\omega} E = E$ and its complement is θ_ω-open. τ_{θ_ω} represents the collection of all θ_ω-open sets.

Theorem 7 (see [13]). *Let (G, τ) be a ts and $E \subseteq G$, then*

(a) τ_{θ_ω} forms a topology on G

(b) $\tau_\theta \subseteq \tau_{\theta_\omega} \subseteq \tau$ and $\tau_{\theta_\omega} \neq \tau$ in general

Theorem 8 (see [13]). *A ts is ω-regular if $\tau_{\theta_\omega} = \tau$.*

Definition 9 (see [14]). Let (G, τ) be a ts and $E \subseteq G$, then, the set $\{z \in G : \mathrm{int}\,(\bar{O}^\omega) \cap E \neq \varnothing, O \in \tau \text{ and } z \in O\}$ is called the δ_ω-closure of E and denoted by the symbol $Cl_{\delta_\omega} E$.

A set E is called δ_ω-closed if $Cl_{\delta_\omega} E = E$ and its complement is δ_ω-open. τ_{δ_ω} represents the collection of all δ_ω-open sets.

Theorem 10 (see [14]). *Let (G, τ) be a ts and let $E \subseteq G$. Then*

(a) $\bar{E} \subseteq Cl_{\delta_\omega} E \subseteq Cl_\delta E$

(b) If E is δ-closed, then E is δ_ω-closed

(c) If E is δ_ω-closed, then E is closed

Definition 11. A ts (G, τ) is called

(a) [16] Locally countable if for each $z \in G$, there exists an open subset O such that $z \in O$ and O is countable

(b) [13] Antilocally countable if every nonempty open subset is uncountable

(c) [17] Locally indiscrete if every open subset is closed

Lemma 12 (see [13]).

(a) If (G, τ) is an antilocally countable ts, then for all $E \in \tau_\omega$, $\bar{E}^\omega = \bar{E}$

(b) If (G, τ) is a locally countable ts, then τ_ω is a discrete topology

Definition 13 (see [13]). A ts is called ω-locally indiscrete if every open subset is ω-closed.

Theorem 14 (see [13]).

(a) Every ts that is locally indiscrete is ω-locally indiscrete

(b) Every ts that is locally countable is ω-locally indiscrete

Theorem 15 (see [14]). *Let (G, τ) be a ts and $E \subseteq G$. Then, $E \in \tau_{\delta_\omega}$ if for each $z \in E$, and there is $O \in \tau$ such that $z \in O \subseteq \mathrm{int}\,(\bar{O}^\omega) \subseteq E$.*

Theorem 16 (see[13]). *Let (G, τ) be an ω-locally indiscrete ts and let $E \subseteq G$. Then*

(a) $\bar{E} = Cl_{\theta_\omega}(E)$

(b) If E is closed in (G, τ), then, E is θ_ω-closed in (G, τ)

Theorem 17 (see [13]). *For any ts (G, τ), the following are equivalent:*

(a) (G, τ) is ω-regular

(b) For each subset E, $\bar{E} = Cl_{\theta_\omega}(E)$

Lemma 18 (see [1]). *Let (G,τ) be a ts. Then, for each $E \in \tau$, $\bar{E} = Cl_{\theta}(E)$.*

Theorem 19 (see [13]). *Let (G,τ) be a ts. Then, for each $E \in \tau_{\omega}$, $\bar{E} = Cl_{\theta_{\omega}}(E)$.*

Theorem 20 (see [13]).

(a) *Every open ω-closed subset in a ts is θ_{ω}-open*

(b) *Every countable open subset in a ts is θ_{ω}-open*

Definition 21 (see [18]). A function $f : G \longrightarrow H$ is called δ-continuous if for each $z \in G$ and each open nbhd P of $f(z)$, there is an open nbhd O of z such that $f(\text{int}(\bar{O})) \subseteq \text{int}(\bar{P})$.

Definition 22 (see [13]). A ts (G,τ) is ω-regular if for each open nbhd O of z there exists an open subset P such that $z \in P \subseteq \bar{P}^{\omega} \subseteq O$.

Definition 23 (see [19]). A function $f : G \longrightarrow H$ is called θ_{ω}-continuous if for each $z \in G$ and each open nbhd P of $f(z)$, there is an open nbhd O of z such that $f(\bar{O}) \subseteq \bar{P}^{\omega}$

Definition 24 (see [20]). A function $f : G \longrightarrow H$ is called almost open if for every regular open subset O of G, $f(O)$ is open in H.

Definition 25 (see [20]). Let (G,τ) and (H,σ) be two topological spaces (abbreviated as ts′s), and a function $f : G \longrightarrow H$ is called almost continuous if for each $z \in G$ and for each $P \subseteq H$ open nbhd of $f(z)$, there exists an open nbhd O of z such that $f(O) \subseteq \text{int}(\bar{P})$.

3. Some Results on δ_{ω}-Closure Operator

In this section, the relationships between usual closure operator, δ_{ω}-closure operator, and θ_{ω}-closure operator have been discussed.

Theorem 26. *Let (G,τ) be a ts and let $E \subseteq G$. Then*

(a) $Cl_{\delta_{\omega}} E \subseteq Cl_{\theta_{\omega}} E$

(b) *If E is θ_{ω}-closed, then E is δ_{ω}-closed*

Proof.

(a) To prove $Cl_{\delta_{\omega}} E \subseteq Cl_{\theta_{\omega}} E$, let $z \in Cl_{\delta_{\omega}} E$ and $O \in \tau$ with $z \in O$. Therefore, $\text{int}(\bar{O}^{\omega}) \cap E \neq \varnothing$. We know $\text{int}(\bar{O}^{\omega}) \subseteq \bar{O}^{\omega}$, and then, $\bar{O}^{\omega} \cap E \neq \varnothing$. We have $z \in Cl_{\theta_{\omega}} E$

(b) Let E be θ_{ω}-closed, then, $Cl_{\theta_{\omega}} E = E$. By using part (a) and Theorem 10(a), $Cl_{\delta_{\omega}} E = E$, and hence, E is δ_{ω}-closed.

The following results give the equivalence conditions for the δ_{ω}-closure operator with δ-closure and θ_{ω}-closure operators. □

Theorem 27. *Let (G,τ) be a ts and $E \subseteq G$. If (G,τ) is ω-locally indiscrete, then*

(a) $\bar{E} = Cl_{\delta_{\omega}}(E) = Cl_{\theta_{\omega}}(E)$

(b) *E is closed in (G,τ) if E is δ_{ω}-closed in (G,τ)*

(c) *E is δ_{ω}-closed in (G,τ) if E is θ_{ω}-closed in (G,τ)*

Proof.

(a) By Theorems 26, 10(a) and 16

(b) Suppose E is closed, then, $E = \bar{E}$. By part (a), $E = Cl_{\delta_{\omega}}(E)$, and therefore, E is δ_{ω}-closed. The converse follows from Theorem 10(c).

(c) Suppose E is δ_{ω}-closed, then $E = Cl_{\delta_{\omega}}(E)$. By part (a), $E = Cl_{\theta_{\omega}}(E)$, and therefore, E is θ_{ω}-closed. The converse follows from Theorem 26(b).

□

Corollary 28. *Let (G,τ) be a ts and $E \subseteq G$. If (G,τ) is ω-regular, then*

(a) $\bar{E} = Cl_{\delta_{\omega}}(E) = Cl_{\theta_{\omega}}(E)$

(b) *E is closed in (G,τ) if E is δ_{ω}-closed in (G,τ)*

(c) *E is δ_{ω}-closed in (G,τ) if E is θ_{ω}-closed in (G,τ)*

Remark 29. The statements of Theorem 27 hold even if (G,τ) is locally indiscrete or locally countable by using Theorem 14.

Theorem 30. *Let (G,τ) be a ts and $E \subseteq G$. If (G,τ) is antilocally countable, then*

(a) $Cl_{\delta}(E) = Cl_{\delta_{\omega}}(E)$

(b) *E is δ_{ω}-closed in (G,τ) if E is δ-closed in (G,τ)*

Proof.

(a) By Theorem 10(a), we have $Cl_{\delta_{\omega}}(E) \subseteq Cl_{\delta}(E)$. Now to prove $Cl_{\delta}(E) \subseteq Cl_{\delta_{\omega}}(E)$, let $z \in Cl_{\delta}(E)$ and open subset O such that $z \in O$. By definition, $\text{int}(\bar{O}) \cap E \neq \varnothing$. Given (G,τ) is antilocally countable, then by

Lemma 12(a), $\bar{O}^{\omega} = \bar{O}$, and hence, int $(\bar{O}^{\omega}) \cap E \neq \varnothing$. Therefore, we get $z \in Cl_{\delta_\omega}(E)$

(b) Suppose E is δ_ω-closed, then $E = Cl_{\delta_\omega}(E)$. By part (a), $E = Cl_\delta(E)$. Therefore, E is δ-closed. The converse follows from Theorem 10(b).

The following result gives some properties of δ_ω-closure operator. □

Theorem 31. *Let (G, τ) be a ts and let $E, F \subseteq G$. Then*

(a) If $E \subseteq F \subseteq G$, then $Cl_{\delta_\omega} E \subseteq Cl_{\delta_\omega} F$

(b) For every $E, F \subseteq G$, $Cl_{\delta_\omega}(E \cup F) = Cl_{\delta_\omega}(E) \cup Cl_{\delta_\omega}(F)$

(c) For every $E \subseteq G$, $Cl_{\delta_\omega} E$ is a closed subset in (G, τ)

(d) For every $E \in \tau_\omega$, $Cl_{\delta_\omega} E = \bar{E}$

(e) For every $E \in \tau$, $Cl_\delta E = Cl_{\delta_\omega} E = Cl_{\theta_\omega} E = \bar{E}$

Proof.

(a) Let $z \in Cl_{\delta_\omega} E$ and $O \in \tau$ with $z \in O$. Since $z \in Cl_{\delta_\omega} E$, then int $(\bar{O}^{\omega}) \cap E \neq \varnothing$. Since $E \subseteq F$, therefore, int $(\bar{O}^{\omega}) \cap F \neq \varnothing$. Hence, $z \in Cl_{\delta_\omega} F$

(b) By part (a), we get $Cl_{\delta_\omega}(E \cup F) \supseteq Cl_{\delta_\omega}(E) \cup Cl_{\delta_\omega}(F)$. Now to prove $Cl_{\delta_\omega}(E \cup F) \subseteq Cl_{\delta_\omega}(E) \cup Cl_{\delta_\omega}(F)$. Let $z \notin Cl_{\delta_\omega}(E) \cup Cl_{\delta_\omega}(F)$, and there are two open sets O, P containing z such that int $(\bar{O}^{\omega}) \cap E = \varnothing$ and int $(\bar{P}^{\omega}) \cap F = \varnothing$. Now, we have $z \in O \cap P \in \tau$ and

$$\begin{aligned} &\text{int}\left(\overline{O \cap P}^{\omega}\right) \cap (E \cup F) \\ &= \left(\text{int}\left(\overline{O \cap P}^{\omega}\right) \cap E\right) \cup \left(\text{int}\left(\overline{O \cap P}^{\omega}\right) \cap F\right) \subseteq \left(\text{int}\left(\bar{O}^{\omega}\right) \cap E\right) \cup \left(\text{int}\left(\bar{P}^{\omega}\right) \cap F\right) \\ &= \varnothing \cup \varnothing = \varnothing. \end{aligned} \tag{1}$$

Hence, we get $z \notin Cl_{\delta_\omega}(E \cup F)$.

(c) To prove $G - Cl_{\delta_\omega}(E) \in \tau$. Let $z \in G - Cl_{\delta_\omega}(E)$, there is an open set O containing z such that int $(\bar{O}^{\omega}) \cap E = \varnothing$. Therefore, $O \cap Cl_{\delta_\omega}(E) = \varnothing$. Hence, we get $G - Cl_{\delta_\omega}(E) \in \tau$

(d) By Theorems 19, 10, and 26

(e) Follows from part (d) and Lemma 18

□

Theorem 32 (see [14]). *Let (G, τ) be a ts, then, $\tau_\delta \subseteq \tau_{\delta_\omega} \subseteq \tau$.*

The equality in Theorem 32 does not hold, as the following examples show.

Example 1. Let (R, τ) be a ts where R is real line and $\tau = \{\varnothing\} \cup \{O \subseteq R : \text{complement of } O \text{ is countable}\}$. The regular open subsets of ts τ are $\{\varnothing, R\}$, and then, $\tau_\delta = \{\varnothing, R\}$. Also (R, τ) is antilocally countable ts. Therefore, by Theorem 30, we have $\tau_\delta = \tau_{\delta_\omega}$. Then, $\tau_{\delta_\omega} = \{\varnothing, R\}$. Hence, we get $\tau_{\delta_\omega} \neq \tau$.

Example 2. Let (Z, τ) be a ts where Z is set of integer and $\tau = \{\varnothing\} \cup \{O \subseteq Z : \text{complement of } O \text{ is finite}\}$. The regular open subsets of ts τ are $\{\varnothing, Z\}$, and then, $\tau_\delta = \{\varnothing, Z\}$. Also (Z, τ) is locally countable ts. Therefore, by Remark 29, we have $\tau = \tau_{\delta_\omega}$. Hence, $\tau_\delta \neq \tau_{\delta_\omega}$.

If ts (G, τ) is semiregular, then $\tau_\delta = \tau_{\delta_\omega} = \tau$.

Theorem 33. *Let (G, τ) be a ts, then, $\tau_{\theta_\omega} \subseteq \tau_{\delta_\omega} \subseteq \tau$.*

Proof. To prove $\tau_{\theta_\omega} \subseteq \tau_{\delta_\omega}$. Let $E \in \tau_{\theta_\omega}$, then, $G - E$ is θ_ω-closed and by Theorem 26 (b), $G - E$ is δ_ω-closed. Hence, $E \in \tau_{\delta_\omega}$.

The equality in Theorem 33 does not hold, as the following example shows. □

Example 3. Let (R, τ) be a ts where R is the real line and $\tau = \{\varnothing, R, N, Q^c, N \cup Q^c\}$. By Example 2.26 of [13], $\tau_{\theta_\omega} = \{\varnothing, R, N\}$. Now, to prove $Q^c \in \tau_{\delta_\omega}$ by using Theorem 15, we have to find $O \in \tau$ for each $z \in Q^c$ such that $z \in O \subseteq \text{int}\,(\bar{O}^{\omega}) \subseteq Q^c$. Let $O = Q^c$, then, $\bar{Q}^{c\omega} = R - N$. Now, int $(\bar{Q}^{c\omega}) = \text{int}\,(R - N) = Q^c$. So, $z \in O \subseteq \text{int}\,(\bar{O}^{\omega}) \subseteq Q^c$ is true for $O = Q^c$, and this implies $Q^c \in \tau_{\delta_\omega}$. But $Q^c \notin \tau_{\theta_\omega}$. Hence, $\tau_{\delta_\omega} \neq \tau_{\theta_\omega}$.

If ts (G, τ) is ω-regular (ω-locally indiscrete or locally indiscrete or locally countable), then $\tau_{\theta_\omega} = \tau_{\delta_\omega} = \tau$. If ts (G, τ) is antilocally countable, then $\tau_\delta = \tau_{\delta_\omega}$. If ts (G, τ) is regular, then $\tau_\theta = \tau_\delta = \tau_{\theta_\omega} = \tau_{\delta_\omega} = \tau$. If a ts is antilocally countable and ω-regular, then $\tau_\theta = \tau_\delta = \tau_{\theta_\omega} = \tau_{\delta_\omega} = \tau$. It can be easily seen by Example 2 and Example 3 that τ_δ and τ_{θ_ω} are incomparable.

Theorem 34.

(a) Every open ω-closed subset in a ts is δ_ω-open

(b) Every countable open subset in a ts is δ_ω-open

Proof. Directly follow from Theorems 33 and 20. □

4. δ_ω-Continuity

In this section, we define some new types of continuity and discuss their relationships.

Definition 35. A function $f : G \longrightarrow H$ is called δ_ω-continuous if for each $z \in G$ and each open subset P of H containing $f(z)$, there is an open subset O of G containing z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^{\omega})$.

Theorem 36. *Every δ_ω-continuous function is δ-continuous.*

Proof. Let (G, τ) and (H, σ) be two ts$'$s and $f : G \longrightarrow H$ be a δ_ω-continuous function. Let z be in G and let $P \subseteq H$ open nbhd of $f(z)$. By assumption, there exists $O \subseteq G$ an open nbhd of z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega) \subseteq \text{int}\,(\bar{P})$. This implies that f is δ-continuous.

The converse of the Theorem 36 does not hold, as the following example shows. □

Example 4. Let a function $f : (Z, \tau_{\text{ind}}) \longrightarrow (Z, \tau_{\text{cof}})$ (Z be the set of integers) be defined as $f(z) = z$, where τ_{ind} is indiscrete topology on Z and τ_{cof} is finite complement topology on Z. Then, f is δ-continuous but not δ_ω-continuous. Let $z \in Z$ and $P \in \tau_{cof}$ such that $f(z) = z \in P$. Take $O = Z$ such that $z \in O \in \tau_{cof}$ and $f(\text{int}\,(\bar{O})) = Z \subseteq \text{int}\,(\bar{P}) = Z$. Hence, f is δ-continuous. Now to prove f is not δ_ω-continuous, let $z = 0$ and $P = Z - \{1, 2\}$. Then, $P \in \tau_{cof}$ with $f(0) = 0 \in P$. Suppose there exists $O \in \tau_{\text{ind}}$ such that $0 \in O \in \tau_{\text{ind}}$ and $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$, but open subset in τ_{ind} containing 0 is Z. Take $O = Z$, then $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$ is not true because $f(\text{int}\,(\bar{O})) = Z$ and $\text{int}\,(\bar{P}^\omega) = Z - \{1, 2\}$. This implies that f is not δ_ω-continuous.

The following result gives sufficient criteria for a δ-continuous to be δ_ω-continuous.

Theorem 37. *Let (G, τ) and (H, σ) be two ts$'$s, and let $f : G \longrightarrow H$ be a δ-continuous function with (H, σ) antilocally countable, then, f is δ_ω-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be any open nbhd of $f(z)$. By assumption, f is δ-continuous, so there exists $O \subseteq G$ an open nbhd of z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P})$. Given (H, σ) is antilocally countable, then by Lemma 12, we have $\bar{P} = \bar{P}^\omega$, and thus, $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$. This implies that f is δ_ω-continuous.

The independence of continuity and δ-continuity have been observed by Noiri [18]. In the following Examples 5 and 6, it is shown that the concepts of continuity and δ_ω-continuity are independent of each other. □

Example 5. Let G be the real line with the usual topology and H the real line with the co-countable topology. Let $f : G \longrightarrow H$ be the function defined by $f(z) = z$. The ts H is antilocally countable and by Example 4.4 of [18], f is δ-continuous but not continuous, and then, by Theorem 37, f is δ_ω-continuous.

Example 6. Let N be the set of natural number with topology $\{\varnothing, N, \{1\}\}$, and let a function $f : N \longrightarrow N$ be defined as $f(z) = z$. It can be easily seen that f is a continuous function. To check that f is δ_ω-continuous, take $z = 1$ and $P = \{1\}$, and then, P is open and $f(z) = z \in P$. Also N is locally countable, and then, $\bar{P}^\omega = P$. Now for an open subset O with $1 \in O$, we have $O = \{1\}$ or $O = N$. In both possibilities, $\bar{O} = N$ and $f(\text{int}\,(\bar{O})) = N \subseteq \text{int}\,(\bar{P}^\omega) = \{1\}$ which is not possible. This implies that f is not δ_ω-continuous.

The following results from 38 to 41 give sufficient criteria for a δ_ω-continuous function to be continuous.

Theorem 38. *Let (G, τ) and (H, σ) be two ts$'$s and let $f : G \longrightarrow H$ be a δ_ω-continuous function with (H, σ) locally countable, then, f is continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is δ_ω-continuous, so there exists $O \subseteq G$ open nbhd of z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$. Since (H, σ) is locally countable, then by Lemma 12(b), we have τ_ω is the discrete topology. Therefore, $P = \bar{P}^\omega$, and thus, $f(O) \subseteq f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega) \subseteq P$. This implies that f is continuous. □

Theorem 39. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be a δ_ω-continuous function with (H, σ) ω-locally indiscrete, then f is continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is δ_ω-continuous, so there exists $O \subseteq G$ an open nbhd of z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$. Since (H, σ) is ω-locally indiscrete, then P is ω-closed and $\bar{P}^\omega = P$. Thus $f(O) \subseteq f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega) \subseteq P$. Hence f is continuous. □

Corollary 40. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be a δ_ω-continuous function with (H, σ) locally indiscrete, then, f is continuous.*

Theorem 41. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be a δ_ω-continuous function with (H, σ) ω-regular, then f is continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, (H, σ) is an ω-regular ts, then by Definition 22, there exists N of H open subset such that $f(z) \in N \subseteq \bar{N}^\omega \subseteq P$. Given f is δ_ω-continuous, there exists $O \subseteq G$ an open subset containing z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{N}^\omega)$. Thus, we have $f(O) \subseteq f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{N}^\omega) \subseteq P$. This implies that f is continuous.

The next result gives sufficient criteria for a θ_ω-continuous function to be δ_ω-continuous. □

Theorem 42. *If $f : (G, \tau) \longrightarrow (H, \sigma)$ is a θ_ω-continuous function and almost-open, then f is δ_ω-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is θ_ω-continuous, so there exists $O \subseteq G$ an open nbhd of z such that $f(\bar{O}) \subseteq \bar{P}^\omega$. Therefore, $f(\text{int}\,(\bar{O})) \subseteq f(\bar{O}) \subseteq \bar{P}^\omega$. Since f is almost-open, then $\text{int}\,(f(\text{int}\,(\bar{O}))) = f(\text{int}\,(\bar{O}))$ and thus $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P}^\omega)$. Hence, f is δ_ω-continuous. □

Corollary 43. *If $f : (G, \tau) \longrightarrow (H, \sigma)$ is a θ_ω-continuous function and open, then f is δ_ω-continuous.*

Definition 44. Let (G, τ) and (H, σ) be two ts's and a function $f : G \longrightarrow H$ is called ω - δ-continuous function if for each $z \in G$ and for each $P \subseteq H$ open nbhd of $f(z)$, there is an open subset O of G containing z such that $f(\text{int}\,(\bar{O}^{\omega})) \subseteq \text{int}\,(\bar{P})$.

Theorem 45. *Every ω - δ-continuous function is almost continuous.*

Proof. Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be an ω-δ-continuous function. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is ω-δ-continuous, so there is an open nbhd O in G containing z such that $f(O) \subseteq f(\text{int}\,(\bar{O}^{\omega})) \subseteq \text{int}\,(\bar{P})$. Hence, f is almost continuous.

The following results give sufficient criteria for the almost continuous function to be ω-δ-continuous. □

Theorem 46. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost continuous with $(G, \tau)\omega$-locally indiscrete, then f is ω - δ-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is almost continuous, so there is an open nbhd $O \subseteq G$ of z such that $f(O) \subseteq \text{int}\,(\bar{P})$. Since (G, τ) is ω-locally indiscrete, then O is ω-closed and $\bar{O}^{\omega} = O$. Thus, $f(\text{int}\,(\bar{O}^{\omega})) = f(O) \subseteq \text{int}\,(\bar{P})$. Hence, f is ω-δ-continuous.

The following corollaries can be easily proved by Theorems 14 and 46. □

Corollary 47. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost continuous with (G, τ) locally indiscrete, then f is ω - δ-continuous.*

Corollary 48. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost continuous with (G, τ) locally countable, then f is ω - δ-continuous.*

Theorem 49. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost continuous with $(G, \tau)\omega$-regular, then f is ω - δ-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is almost continuous, so there is an open nbhd $N \subseteq G$ of z such that $f(N) \subseteq \text{int}\,(\bar{P})$. Since (G, τ) is ω-regular, then there is an open subset O in G containing z such that $\text{int}\,(\bar{O}^{\omega}) \subseteq \bar{O}^{\omega} \subseteq N$. Therefore, $f(\text{int}\,(\bar{O}^{\omega})) \subseteq f(N) \subseteq \text{int}\,(\bar{P})$. Hence, f is ω-δ-continuous. □

Theorem 50. *Every δ-continuous function is ω - δ-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is δ-continuous, so there is an open nbhd $O \subseteq G$ of z such that $f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P})$. Thus, $f(\text{int}\,(\bar{O}^{\omega})) \subseteq f(\text{int}\,(\bar{O})) \subseteq \text{int}\,(\bar{P})$. Hence, f is ω-δ-continuous.

The converse of the Theorem 50 does not hold, as the following example shows. □

Example 7. The function f defined in Example 4.5 of [18], shows that f is continuous but not δ-continuous, and then, f is almost continuous by definition. Also, G is locally countable, so by Corollary 48, f is ω - δ-continuous.

The following result gives sufficient criteria for a ω-δ-continuous function to be δ-continuous.

Theorem 51. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be ω - δ-continuous with (G, τ) antilocally countable, then f is δ-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is ω-δ-continuous, so there is an open nbhd O of z such that $f(\text{int}\,(\bar{O}^{\omega})) \subseteq \text{int}\,(\bar{P})$. Since (G, τ) is antilocally countable, then by Lemma 12, $\bar{O}^{\omega} = \bar{O}$. Therefore, $f(\text{int}\,(\bar{O})) = f(\text{int}\,(\bar{O}^{\omega})) \subseteq \text{int}\,(\bar{P})$. Hence, f is δ-continuous. □

Definition 52. A function $f : G \longrightarrow H$ is called almost δ_{ω}-continuous if for each $z \in G$ and each open subset P of H containing $f(z)$, there is an open subset O of G containing z such that $f(O) \subseteq \text{int}\,(\bar{P}^{\omega})$.

Theorem 53. *Every almost δ_{ω}-continuous function is almost continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is almost δ_{ω}-continuous, so there is an open nbhd O of z such that $f(O) \subseteq \text{int}\,(\bar{P}^{\omega})$. By Theorem 2, $\bar{P}^{\omega} \subseteq \bar{P}$, and therefore, $f(O) \subseteq \text{int}\,(\bar{P}^{\omega}) \subseteq \text{int}\,(\bar{P})$. Hence, f is almost continuous.

The converse of the Theorem 53 does not hold, as the following example shows. □

Example 8. Consider a function $f : (N, \tau) \longrightarrow (N, \tau_{\text{cof}})$ defined by $f(z) = z$, where τ is indiscrete topology and τ_{cof} is finite complement topology on N. To check f is almost continuous, let $z \in N$ and $P \in \tau_{\text{cof}}$ such that $f(z) = z \in P$. Then, for every $P \in \tau_{\text{cof}}$, $\bar{P} = N$. Choose $O = N$, then $z \in O \in \tau$ and $f(O) = f(N) = N \subseteq \text{int}\,(\bar{P}) = \text{int}\,(N) = N$. Now, to check f is not almost δ_{ω}-continuous, let $z = 1$ and $P = N - \{2\}$. Since τ_{cof} is locally countable, then, $\bar{P}^{\omega} = P$. So $O \in \tau$ containing z is N, then, $f(O) = f(N) = N \subseteq \text{int}\,(\bar{P}^{\omega}) = \text{int}\,(P) = \text{int}\,(N - \{2\}) = N - \{2\}$ which is not true. Hence, f is not almost δ_{ω}-continuous.

The following result gives sufficient criteria for the almost continuous function to be almost δ_{ω}-continuous:

Theorem 54. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost continuous with (H, σ) antilocally countable, then f is almost δ_{ω}-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is almost continuous, so there is an open nbhd O of z such that $f(O) \subseteq \text{int}\,(\bar{P})$. Since (H, σ) is antilocally countable, then $\bar{P}^{\omega} = \bar{P}$. Therefore, $f(O) \subseteq \text{int}(\bar{P}) = \text{int}(\bar{P}^{\omega})$. Hence, f is almost δ_{ω}-continuous. □

Theorem 55. *Every continuous function is almost δ_{ω}-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is continuous, so there is an open nbhd O of z such that $f(O) \subseteq P$. Thus, $f(O) \subseteq P \subseteq \text{int}(\bar{P}^{\omega})$. Hence, f is almost δ_{ω}-continuous.

The converse of the Theorem 55 does not hold, as the following example shows. □

Example 9. Consider a function $f : (R, \tau) \longrightarrow (R, \tau_{\text{coc}})$ defined by $f(z) = z$, where τ is the usual topology and τ_{coc} is the countable complement topology on R. It is easy to check that f is discontinuous. To check, f is almost δ_{ω}-continuous. Let $z \in R$ and $P \in \tau_{coc}$ such that $f(z) = z \in P$. Since (R, τ_{coc}) is antilocally countable, then, $\bar{P}^{\omega} = \bar{P}$. Therefore, for any open subset, $P \in \tau_{\text{coc}}$ such that $\bar{P}^{\omega} = \bar{P} = R$. Take $O = R$ such that $z \in O \in \tau$ and $f(O) = R \subseteq \text{int}(\bar{P}^{\omega}) = \text{int}(R) = R$. Hence, f is almost δ_{ω}-continuous.

The following result gives sufficient criteria for the almost δ_{ω}-continuous function to be continuous.

Theorem 56. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost δ_{ω}-continuous with $(H, \sigma)\omega$-locally indiscrete, then f is continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is almost δ_{ω}-continuous, so there is an open nbhd O of z such that $f(O) \subseteq \text{int}(\bar{P}^{\omega})$. Since (H, σ) is ω-locally indiscrete, then $\bar{P}^{\omega} = P$. Therefore, $f(O) \subseteq \text{int}(\bar{P}^{\omega}) = P$. Hence, f is continuous.

The following corollaries can be easily proved by Theorems 14 and 56. □

Corollary 57. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost δ_{ω}-continuous with (H, σ) locally indiscrete, then f is continuous.*

Corollary 58. *Let (G, τ) and (H, σ) be two ts's and let $f : G \longrightarrow H$ be almost δ_{ω}-continuous with (H, σ) locally countable, then f is continuous.*

Theorem 59. *Every δ_{ω}-continuous function is almost δ_{ω}-continuous.*

Proof. Let z be in G and let $P \subseteq H$ be an open nbhd of $f(z)$. By assumption, f is δ_{ω}-continuous, so there is an open nbhd O in G containing z such that $f(\text{int}(\bar{O})) \subseteq \text{int}(\bar{P}^{\omega})$. Thus, $f(O) \subseteq f(\text{int}(\bar{O})) \subseteq \text{int}(\bar{P}^{\omega})$. Hence, f is almost δ_{ω}-continuous.

The converse of the Theorem 59 does not hold, as the following example shows the same. □

Example 10. In Example 6, f is continuous but not δ_{ω}-continuous. Also, by Theorem 55, every continuous function is almost δ_{ω}-continuous.

5. Conclusion

In this work, we continue the research on δ_{ω}-closure operator, and some properties of δ_{ω}-closure operator are discussed. Also, we give some sufficient conditions for the equivalence of δ_{ω}-closure and θ_{ω}-closure operators. Further, we define some new types of continuity, namely, δ_{ω}-continuity, ω-δ-continuity, and almost δ_{ω}-continuity. Also, some examples and counter examples are given.

Data Availability

No data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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