Research Article

Connected Degree of Fuzzifying Matroids

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Polya’s plausible reasoning methods are crucial not only in discovery of mathematics results, modeling methods, and data processing methods but also in many practical problems’ solving. This paper exemplifies how to use Polya’s plausible reasoning methods to generalize the popularized notion of 2-connectedness of graphs to a more universal notion of the connected degree of fuzzifying matroids. We introduce the connectedness of fuzzifying matroids, which is generalized from 2-connectedness of graphs, connectedness of matroids, and 2-connectedness of fuzzy graphs. Moreover, the connected degree of fuzzifying matroids is presented by considering the fuzziness degree of connectedness. It is proved that a fuzzifying matroid is connected, which is equivalent to its connectedness degree Con(M) is the biggest (i.e., Con(M) = 1). This, together with other properties of the connected degree of fuzzifying matroids, demonstrates the rationality of the proposed notion. Finally, we describe the concepts of this paper through some examples.

1. Introduction

Whitney in his fundamental paper [1] defined a matroid as an abstract generalization of a graph and a matrix. One of the great beauties of the subject of the matroid theory is that there are so many equivalent descriptions of a matroid. Bases, circuits, rank function, or closure operator is also sufficient to uniquely determine the matroid besides independent sets [2, 3]. In addition, it is well known that matroids are of great significance in combinatorial optimization and are just the structure that can use the greedy algorithm to find the optimal solution of some problems [4, 5]. Combined with fuzzy set theory [6], some fuzzy concepts of a matroid are given. One is using a family of fuzzy sets instead of that of sets, a Goetschel–Voxman fuzzy matroid was proposed in [7] and was later on widely studied by the various researchers [8–16]. The other is using a mapping from 2^E to [0, 1] rather than a family of sets, a fuzzifying matroid was introduced by Shi, and it can also be determined uniquely by its fuzzifying rank function [17, 18]. Subsequently, some other equivalent descriptions of a fuzzifying matroid were discussed including base-map and circuit-map [19], three kinds of fuzzifying operators [20, 21], fuzzifying nullity [22]. Also, as its applications, a fuzzifying matroid is just the structure of the fuzzifying greedy algorithm for fuzzy optimization problems [23, 24]. Thus, Shi’s fuzzification method maintains matroids’ features.

We know that there is an important connection between graphs and matroids. A graph (graphs in this paper are always loopless and without isolated vertices and have at least three vertices). G = (V, E) can induce a matroid (E, J_G) which is called its cycle matroid and denoted by M_G, where J_G = {A ⊆ E: A does not contain any cycle of G}. Motivated by analogies in graph theory, the connectedness of a matroid is defined. Here, M_G is connected, which is equivalent to G is 2-connected; hence, the connectedness for matroids corresponds directly to the idea of 2-connectedness for graphs. It is natural to ask that is there fuzzifying matroid concept that corresponds directly to the idea of connectedness, or 2-connectedness, for fuzzy graphs?
Our main aim is to give a concept of connectedness for fuzzifying matroids, which extends the corresponding notion for fuzzy graphs. We begin by recalling some basic notions and results to be used in this paper. In Section 3, by using Polya’s plausible reasoning method, we show the course of discovering the notion of connected degree of fuzzifying matroids. In Section 4, we generalize the concept of 2-connectedness from matroids to fuzzifying matroids, which corresponds directly to the idea 2-connectedness for fuzzy graphs. In Section 5, we consider the fuzziness degree of connectedness for fuzzifying matroids. Using the degree to which a set is a separator, the notion of connected degree for a fuzzifying matroid is introduced. We show that the relationship between the connected degree of a fuzzifying matroid and the connectedness of its corresponding level matroids and investigate the connected degree for the dual of a fuzzifying matroid. We close the section with various examples to illustrate the concepts of this paper.

2. Preliminaries

In this paper, we shall denote by $E$ a given finite set and by $2^E$ the set of all subsets of $E$. Let $\mathcal{A} \subseteq 2^E$ and define

\[ \text{Min}\mathcal{A} = \{ A \in \mathcal{A} | \text{A is minimal in } \mathcal{A} \}, \text{Opp}\mathcal{A} = \{ X \subseteq E | X \not\in \mathcal{A} \}, \text{Upp}\mathcal{A} = \{ X \subseteq E | \text{there exists } A \in \mathcal{A}, A \subseteq X \}. \]

We begin with the definition of a matroid.

Definition 1. (cf. [2, 3]) A matroid $M$ is a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ (called independent sets) such that (I1)–(I3) are satisfied, denoted by $M = (E, \mathcal{I})$.

(I1) $\mathcal{I} \neq \emptyset$;

(I2) $\mathcal{I}$ is a descending family;

(I3) For any two elements $A, B \in \mathcal{I}$ with $|A| < |B|$, one element $e \in B - A$ can be found so that $A \cup \{e\}$ belongs to $\mathcal{I}$.

It is important (for understanding the notion of fuzzifying matroid in the following Step 5) to notice that a matroid can also be defined equivalently as a finite set $E$ and a collection $\mathcal{C}$ of subsets of $E$ (called circuits) satisfying three circuit axioms [2]:

(C1) $\emptyset \not\in \mathcal{C}$;

(C2) $\mathcal{C}$ is an antichain;

(C3) for any two different elements $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$, one element $C_3 \in \mathcal{C}$ can be found to be included in $(C_1 \cup C_2) - \{e\}$. Specifically, take a matroid $(E, \mathcal{I})$ (i.e., which means that $\mathcal{I}$ satisfies (I1)–(I3)), then, $\mathcal{C}, \mathcal{I} = \text{Min}\text{Opp}(\mathcal{I})$ is the family of all circuits for it (i.e., $\mathcal{C}$ satisfies (C1)–(C3)); conversely, take a matroid $(E, \mathcal{C})$ (i.e., $\mathcal{C}$ satisfies (C1)–(C3)), then $\mathcal{I}, \mathcal{I} = \text{Opp}(\text{Upp}(\mathcal{C}))$ is the family of all independent sets for it (i.e., $\mathcal{I}$ satisfies (I1)–(I3)) and $\mathcal{C} = \mathcal{C}_{\mathcal{I}}$.

Moreover, if $(E, \mathcal{I})$ is a matroid, its rank function $\rho_\mathcal{I}$ is defined as a mapping from $2^E$ to the set of natural numbers $\mathbb{N}$ by $\rho_\mathcal{I}(A) = \max \{|B| | B \subseteq A, B \in \mathcal{I}\}$ and can uniquely determine the matroid [3].

Next, we present some other notations, notions, and results needed in this paper. As usual, $[0, 1]^E$ denotes the collection of fuzzy subsets on $E$, where a fuzzy subset on $E$ is a mapping from $E$ to $[0, 1]$. If $A \in [0, 1]^E$ and $a \in [0, 1]$, we will often denote the set of $x \in E$ satisfying $A(x) \geq a$ and the set of $x \in E$ satisfying $A(x) > a$ by $A_{[a]}$ and $A_{(a)}$, respectively. Fuzzy natural numbers and their operations have been introduced [17, 22, 23, 25]. An antitone mapping $\lambda$ from the set of natural numbers $\mathbb{N}$ to $[0, 1]$ is called a fuzzy natural number if $\lambda(0) = 1$ and $\lambda(1) = \lambda(1 + k) = 0$. We will use $\mathbb{N}([0, 1])$ to denote the collection of fuzzy natural numbers. Let $\lambda, \mu$ be fuzzy natural numbers, their sum, written $\lambda + \mu$, is also a fuzzy natural numbers such that for every $n \in \mathbb{N}$, $(\lambda + \mu)(n) = \lambda(k) \mu(l) \cap (k + l)$, their subtraction is given by $\lambda - \mu = \{ v \in \mathbb{N} | (\lambda + \mu)(v) = 1 \}$. If we regard a natural number $m$ as a fuzzy natural number $\mu_m$ such that $\mu_m(n) = 1$ for $n \leq m$ and $\mu_m(n) = 0$ for $n > m + 1$, then, we can get following operational properties, for every $a \in [0, 1]$,

\[ (\lambda + \mu)(a) = \lambda(a) + \mu(a) \quad \text{ and } \quad (\lambda + \mu)(a) = \lambda(a) + \mu(a). \]

In [17, 18], the definition of a fuzzifying matroid on $E$ was given.

Definition 2. (cf. [17, 18]) A fuzzifying matroid $\mathcal{M}$ is a finite set $E$ and a mapping $\mu$ from $2^E$ to $[0, 1]$ (called fuzzy family of independent sets) such that (F1)–(F3) are satisfied,

written as $\mathcal{M} = (E, \mu)$. For any two subsets $A, B$ of $E$,

\[ (F1) \mu(A) = 1; \]

\[ (F2) \mu(A) \geq \mu(B) \text{ if } A \subseteq B; \]

\[ (F3) \mu(A \cup \{e\}) \geq \mu(A) \mu(B) \text{ if } |A| < |B|. \]

For a fuzzifying matroid, the following characterization theorem holds.

Theorem 1 (cf. [17]). If $\mu$ is a mapping from $2^E$ to $[0, 1]$, then, $(E, \mu)$ is a fuzzifying matroid, which is equivalent to each of the following statements.

(1) $\{ (E, \mu) | a \in [0, 1] \}$ is a family of matroids;

(2) $\{ (E, \mu) | a \in [0, 1] \}$ is a family of matroids.

It is also important to notice that a fuzzifying matroid can be defined equivalently as a pair $(E, C)$, where $C$ is a mapping from $2^E$ to $[0, 1]$ satisfying two axioms [19]:

(F1) for any $a \in [0, 1]$, $\mu_{C}(C) = 1$ if $C_{[a]}$ satisfies circuits axioms (C1)–(C3); (FC2) for any subset $C$ of $E$, $\mu_{C}(C) = a \neq 0$ implies $C \in \mu_{C}(C)$. In fact, let $(E, \mu)$ be a fuzzifying matroid (i.e., $\mu$ satisfies (F1)–(F3)), the mapping $C_{\mu} : 2^E \rightarrow [0, 1]$ defined by for any subset $C$ of $E$ such that $C_{\mu}(C) = 1 - \wedge \{ a \in [0, 1] | C \in \mu_{C} \}$, not only is the circuit-map of $(E, \mu)$ (i.e., $C$ satisfies (FC1)–(FC2)) but also $\mu(C_{\mu}) = \mu_{C_{\mu}}$ for any $a \in [0, 1]$. Conversely, let $C$ be a fuzzifying matroid (i.e., $C$ satisfies (FC1)–(FC2)), the mapping $\mu_{C}: 2^E \rightarrow [0, 1]$ defined by $\mu_{C}(A) = 1 - \wedge \{ C(B) | B \subseteq C \}$ (\forall A \in 2^E) satisfies (F1)–(F3) and $C = C_{\mu_{C}}$.

The fuzzifying rank function was also defined.

Definition 3 (cf. [17, 23]). A mapping $\mathbb{R}_{\mu}$ from $2^E$ to $\mathbb{N}([0, 1])$ is called fuzzifying rank function of a given fuzzifying matroid $(E, \mu)$, if for any subset $A$ of $E$ and any natural number $n$ such that $\mathbb{R}_{\mu}(A)(n) = \vee \{ \mu(B) | B \subseteq A, |B| \geq n \}$. 

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Theorem 2 (cf. [17, 23]). The fuzzifying rank function $R_{II}$ for a fuzzifying matroid $(E, II)$ has the properties:

1. $R_{II}(A)[a] = S_{II}[a](A)(\forall A \in 2^E, a \in [0, 1]).$
2. $R_{II}(A)[a] = S_{II}[a](A)(\forall A \in 2^E, a \in [0, 1]).$

Furthermore, for any fuzzifying matroid $(E, II)$ (i.e., $II$ satisfies (F1)–(F3)), its fuzzifying rank function $R_{II}$ possesses some features, shown as, for any two subsets $A, B$ of $E$,

Step 1. Find out a characterization 2-connectedness of graphs (from all) start from which the notion described can be generalized: for a graph, written $G = (V, E)$, we call that it is 2-connected if its any two distinct edges are contained in a cycle [2, 3], i.e., for any two different elements $e_1, e_2$ in $E$, one element $C$ in $G$ (the set of all cycles of $G$) can be found to include $e_1$ and $e_2$, simultaneously.

Step 2. Since matroid concept can be regarded as a generalization of graph concept (for a graph $G$, the collection of its cycles is represented as $G$, which is exactly the family of circuits for the cycle matroid $G$ of $G$, thus, $G$ can also be denoted equivalently as $(E, G)$. So, we can think of $G$ as a matroid $G = (E, G)$, it is natural to generalize the notion of 2-connectedness from the class of graphs (written as $G$) to the class of matroids (written as $M$).

Step 3. It is also reasonable saying the special matroid $G = (E, G)$ is connected if for any two different elements $e_1, e_2$ in $E$, one element $C$ in $G$ can be found to include $e_1$ and $e_2$ simultaneously (see Step 1).

Generalization of this can be taken as a definition of connectedness of a matroid (see Step 4).

Step 4. We call that a matroid $M = (E, M)$ is connected if, for any two different elements $e_1, e_2$ in $E$, one element $C$ in $M$ can be found to include $e_1$ and $e_2$ simultaneously [2, 3].

Step 5. As a matroid is a special fuzzifying matroid (for any matroid $(E, \mathcal{S})$ or $(E, \mathcal{E})$, we do not distinguish between $(E, \mathcal{S})$ and $(E, 1_\mathcal{S})$ and between $(E, \mathcal{E})$ and $(E, 1_\mathcal{E})$, where $1_\mathcal{S}$ and $1_\mathcal{E}$ are characteristic functions of $\mathcal{S}$ and $\mathcal{E}$, respectively), we naturally extend the notion of connectedness from the class of matroids to that of fuzzifying matroids (written as $\mathcal{FM}$).

Step 6. It is reasonable saying the special fuzzifying matroid $(E, 1_\mathcal{E})$ is connected when the corresponding matroid $(E, \mathcal{E})$ is connected. In other words, for any $\{e_1, e_2\} \subseteq E$, $\forall [1_\mathcal{E}(C)e_1, e_2 \in C \mathcal{E}] = 1$. Generalization of this can be taken as a definition of connectedness of a fuzzifying matroid (see Step 7).

Step 7. A fuzzifying matroid $(E, C)$ is called connected (or 2-connected) if for any $\{e_1, e_2\} \subseteq E, \forall [C(C)e_1, e_2 \in C \mathcal{E}] = 1$ (see the following Theorem 4 for a characterization).

Step 8. Find out a characterization (i.e., the separator characterization) of the connectedness of matroids (from all) start from which the notion described can be generalized: A matroid $M$ is disconnected if and only if (briefly, iff) a proper subset $A$ of $E$ can be found to be a separator of $M$. Generalization of this can be taken as a definition of connected degree of a fuzzifying matroid (see Step 12).

Step 9. Recall and redesigne the definition of the separator (so that connectedness of a matroid can be generalized to connected degree of a fuzzifying matroid). Suppose that $M = (E, \mathcal{E})$ is a matroid, a subset $A$ of $E$ is called a separator of $M$ (in other words, the degree to which $A$ is a separator of $M$ is equal to 1, denoted by Sep$(A) = 1$) iff for any $C \in E$ there must be $C \subseteq A$ or $C \subseteq E - A$, that is, $C \notin \mathcal{E}$ whenever $C$ is a subset of $E$ not included in both $A$ and $E - A$ (i.e., $\land \{[1 - 1_\mathcal{E}(C)]\}C \subseteq E$ not included in both $A$ and $E - A$). Compare the values of Sep$(A)$ and $\land \{[1 - 1_\mathcal{E}(C)]\}$ and $\land \{[1 - 1_\mathcal{E}(C)]\}C$ is a subset of $E$ not included in both $A$ and $E - A$, we get Sep$(A) = \land \{[1 - 1_\mathcal{E}(C)]\}C$ is a subset of $E$ not
included in both $A$ and $E - A$. It is natural to generalize the equation from a matroid to a fuzzifying matroid (see Step 10).

Step 10. Given a fuzzifying matroid $M = (E, C)$ and a subset $A$ of $E$, we call that $\text{Sep}(A) = \{[1 - C(C)]C \mid C$ is a subset of $E$ not included in both $A$ and $E - A\}$ is the degree to which $A \in 2^E$ is a separator of $M$.

Step 11. A matroid $M$ is disconnected (that is to say the connected degree of $M$ is equal to 0, denoted by $\text{Con}(M) = 0$) if a proper subset $A$ of $E$ can be found to be a separator of $M$ (i.e., $\wedge \{1 - \text{Sep}(A) \mid A \in 2^E - \{\emptyset, E\}\} = 0$). Compare the values of $\text{Con}(M)$ and $\wedge \{1 - \text{Sep}(A) \mid A \in 2^E - \{\emptyset, E\}\}$, we get $\text{Con}(M) = \wedge \{1 - \text{Sep}(A) \mid A \in 2^E - \{\emptyset, E\}\}$. It is natural to generalize the equality from a matroid to a fuzzifying matroid (see Step 12).

Step 12. Given a fuzzifying matroid $M = (E, C)$, we call $\text{Con}(M) = \wedge \{1 - \text{Sep}(A) \mid A \in 2^E - \{\emptyset, E\}\}$ is the connected degree of $M$.

4. Characterizations and Rationality of Connectedness of Fuzzifying Matroids

Although a fuzzifying matroid can be defined as both a pair $(E, II)$ and a pair $(E, C)$, we find that the circuit-map axioms (FC1)-(FC2) of a fuzzifying matroid is not a natural way to generalize the circuits axioms (C1)-(C3) of a matroid. Hence, in the subsequent discussion, we always suppose that a fuzzifying matroid and a matroid are pairs $(E, II)$ and $(E, \mathcal{I})$, respectively.

The following Definition 5, Theorem 4, and Corollary 1 (which characterize the connectedness of fuzzifying matroids, see Step 7 for the definition) show that the connectedness of fuzzifying matroids is in harmony with that of matroids.

Definition 5. A mapping $C_{\text{II}}$ from $2^E$ to $[0, 1]$ is the circuit-map of a given fuzzifying matroid $(E, II)$ which is said to be connected if for any two elements $e_1, e_2$ in $E$ such that $\forall \{C_{\text{II}}(C)|e_1, e_2 \in C \subseteq E\} = 1$.

Theorem 4. A fuzzifying matroid $(E, II)$ is connected, which is equivalent to $(E, II_{(0)})$ is connected.

Proof. If $(E, II)$ is connected, then, for any two elements $e_1, e_2$ in $E$, a subset $C$ of $E$ can be found to satisfy $C_{\text{II}}(C) = 1$ and include $e_1$ and $e_2$, and thus $C \in \text{Min}(C_{\text{II}}|1 = C_{\text{II}})$ by the property of $C_{\text{II}}$. This implies that $(E, II_{(0)})$ is connected. Conversely, if $(E, II_{(0)})$ is connected, then, for any two elements $e_1, e_2$ in $E$, one element $C \in C_{\text{II}}$ can be found to include $e_1$ and $e_2$, and thus $C_{\text{II}}(C) = 1 - \wedge \{a \in [0, 1]|e_1, e_2 \in C_{\text{II}}(C)| = 1 - 0 = 1$. Hence, $\forall \{C_{\text{II}}(C)|e_1, e_2 \in C \subseteq E\} = 1$, which implies that $(E, II)$ is connected.

Notice that, the fuzzifying matroid $(E, II)$ induced by a fuzzy family of circuits is introduced [27], it has the following properties, for any two elements $a, b$ in $[0, 1], a \leq b$ implies $\mathcal{C}_{II(a)} \subseteq \mathcal{C}_{II(b)}$. Thus, Theorem 4 deduces the following corollary.

Corollary 1. A fuzzifying matroid $(E, II)$ induced by a fuzzy family of circuits is connected if and only if $(E, II_{(a)})$ is connected $\forall a \in (0, 1)$.

A fuzzy graph [28] $\mathcal{G}$ is a set of nodes $V$ together with two fuzzy sets $\sigma$ and $\mu$ on $V$ and $V \times V$, respectively, satisfied for any pair of elements $x, y$ of $V$, $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$, written as $\mathcal{G} = (V, \sigma, \mu)$. Graph $\mathcal{G} = (V, 1_{\sigma}, 1_{\mu})$ is called the underline graph of $\mathcal{G}$, and we identify that $1_{\sigma(a)}$ and $1_{\mu(a)}$ are characteristic functions of $\sigma(a)$ and $\mu(a)$, respectively. When $\sigma$ and $\mu$ take no value in $(0, 1)$, $\mathcal{G}$ equals $\mathcal{G}$ and so it is a graph, which implies that the class of fuzzy graphs contains that of graphs. Furthermore, the concept of connectedness can be extended from graphs to more general a fuzzy graphs. In one way, a fuzzy graph is 2-connected, which is defined as its underline graph is 2-connected. Actually, each fuzzy graph $\mathcal{G} = (V, \sigma, \mu)$ can induce a fuzzifying matroid which is the set $E$ of edges in $\mathcal{G}$ and a mapping $II_{\mathcal{G}}$ from $2^E$ to $[0, 1]$ defined as for every subset $A$ of $E$ such that $II_{\mathcal{G}}(A) = \vee \{a \in [0, 1]|\mu(a) \in A$ and any cycle of $(V, \mu(a))$ is not included in $A\}$, called fuzzifying cycle matroid of $\mathcal{G}$. Therefore, the notion of 2-connectedness for fuzzy graphs can be generalized to that of connectedness for their fuzzifying cycle matroids, and then to that of connectedness for more general fuzzifying matroids (see the following Theorem 5) which means the connectedness of fuzzifying matroids is also in harmony with 2-connectedness of fuzzy graphs.

Lemma 1. $(E, II_{\mathcal{G}})$ and $(E, I_{\mathcal{G}})$ are the fuzzifying cycle matroid and the cycle matroid respectively, induced by a fuzzy graph $\mathcal{G} = (V, \sigma, \mu)$ and its underline graph $\mathcal{G}$, we have $I_{\mathcal{G}} = (II_{\mathcal{G}})(0)$.

Proof. $\forall A \in (II_{\mathcal{G}})(0)$, i.e., $II_{\mathcal{G}}(A) > 0$, thus, a number $a$ in $(0, 1)$ can be found such that $\mu(a)$ contains $A$ and any cycle of $(V, \mu(a))$ is not included in $A$. Suppose that $A$ contains a cycle of $(V, \mu(a))$, then $A$ contains a cycle of $(V, \mu(a))$ since $A \subseteq \mu(a)$. There is a contradiction. Therefore, $\mu(a)$ contains $A$ and any cycle of $(V, \mu(a))$ is not included in $A$, i.e., $A \in I_{\mathcal{G}}$.

$\forall A \in I_{\mathcal{G}}$, i.e., $\mu(a)$ contains $A$ and any cycle of $(V, \mu(a))$ is not included in $A$. We can find a number $a$ in $(0, 1)$ satisfying $\mu(a) = \mu(a)$ because the range of $\mu$ is a finite set. Hence, $\mu(a)$ contains $A$ and any cycle of $(V, \mu(a))$ is not also included in $A$. This implies that $II_{\mathcal{G}}(A) \geq a > 0$, we have $A \in (II_{\mathcal{G}})(0)$.

By Theorem 4 and Lemma 1, we get the fuzzifying cycle matroid $(E, II_{\mathcal{G}})$ is connected iff $(E, II_{(a)})$ is connected iff $(E, I_{\mathcal{G}})$ is connected iff $\mathcal{G}$ is 2-connected. Hence, the following theorem holds.

Theorem 5. Fuzzifying cycle matroid $(E, II_{\mathcal{G}})$ induced by a fuzzy graph $\mathcal{G}$ is connected, which is equivalent to $\mathcal{G}$ is 2-connected.
5. Connected Degree of Fuzzifying Matroids

Recall that the connected degree of a fuzzifying matroid is defined by using the notion of the degree to which a set is a separator (see Steps 10 and 12). In this section, we study the connections between the notions of the degree to which a subset is a separator, connected degree, and connectedness of a fuzzifying matroid. We also obtain a generalization of a classical result on separators of matroids (see Theorem 7).

Definition 6. Define the degree to which \( A \in 2^E \) is a separator of a given fuzzifying matroid \( (E, \Pi) \) as \( \text{Sep}(A) = \wedge \{1 - C_{\Pi}(C)|C \text{ is a subset of } E \text{ not included in both } A \text{ and } E - A\} \).

Theorem 6. The following statements hold, see Definition 6 for this symbol \( \text{Sep}(A) \).

1. If \( \text{Sep}(A) > a \), then, \( A \) is a separator of \((E, \Pi_{(a)})\) \((a \in [0, 1])\).
2. If \( \text{Sep}(A) = a \), then, \( A \) is not a separator of \((E, \Pi_{(a)})\) \((a \in [0, 1])\).
3. If \( \text{Sep}(A) < a (a \in (0, 1)) \), uncertain.
4. \( \text{Sep}(A) = 0 \) if and only if \( A \) is not a separator of \((E, \Pi_{(0)})\).

Proof. (1) Suppose that \( A \) is not a separator of \((E, \Pi_{(a)})\), then, one element \( C \in \mathcal{C}_{\Pi_{(a)}} \) can be found to be not included in both \( A \) and \( E - A \); hence, \( C_{\Pi}(C) \geq 1 - a \) and then \( \text{Sep}(A) \leq a \) by the definitions of \( \Pi_{(a)} \) and \( \text{Sep}(A) \).

(2) Suppose that \( \text{Sep}(A) = a \in [0, 1] \), i.e., \( \wedge \{1 - C_{\Pi}(C)|C \text{ is a subset of } E \text{ not included in both } A \text{ and } E - A\} = a \), then, \( 1 - C_{\Pi}(C) = a \) for some subset \( C \) of \( E \) not included in both \( A \) and \( E - A \); which implies this subset \( C \in \text{Min}(C_{\Pi})\{1 - a\} = \mathcal{C}_{\Pi_{(a)}} \). In other words, \( C \in \mathcal{C}_{\Pi_{(a)}} \) for some subset \( C \) of \( E \) not included in both \( A \) and \( E - A \); i.e., \( A \) is not a separator of \((E, \Pi_{(a)})\).

(3) Take a counterexample. A finite set \( E = \{e_1, e_2, e_3\} \) together with a mapping \( \Pi \) from \( 2^E \) to \([0, 1]\) defined as \( \Pi(\emptyset) = \Pi(\{e_1\}) = 1, \Pi(\{e_2\}) = \Pi(\{e_3\}) = (1/2), \Pi(\{e_1, e_2\}) = \Pi(\{e_1, e_3\}) = \Pi(\{e_2, e_3\}) = \Pi(\{e_1, e_2, e_3\}) = (1/3) \), form a fuzzifying matroid. Then, \( \mathcal{C}_{\Pi_{(a)}} = \emptyset \) if \( 0 \leq a < (1/3) \), \( \{[e_1], [e_1, e_2], [e_1, e_3], [e_2, e_3]\} \) if \( (1/3) \leq a < (1/2) \), \( \{[e_1, e_2], [e_1, e_3], [e_2, e_3]\} \) if \( (1/2) \leq a < 1 \). By the definitions of \( \Pi_{(a)} \) and \( \mathcal{C}_{\Pi_{(a)}} \), and thus by the definition of \( C_{\Pi} \), \( C_{\Pi}(C) = 0 \) if \( C \subseteq \emptyset \), \( \{[e_1]\} \), \( \{[e_1, e_2]\} \), \( \{[e_1, e_3]\} \), \( \{[e_2, e_3]\} \), \( \{[e_1, e_2, e_3]\} \), \( 1/2 \) if \( C \subseteq \{[e_2]\} \), \( \{[e_1, e_3]\} \), \( 2/3 \) if \( C \subseteq \{[e_1, e_2]\} \), \( \{[e_1, e_3]\} \), \( \{[e_2, e_3]\} \). Let \( A = \{[e_1]\} \), then, by Definition 6, \( \text{Sep}(A) = \wedge \{1 - C_{\Pi}(C)|C = \{e_1, e_2\}, [e_1, e_3], [e_2, e_3]\} \{1 - (2/3)\wedge (1 - (2/3)) \wedge (1 - 0) = (1/3) < 0.4 < 0.5 \}. \) We can check that \( A \) is not a separator of \((E, \Pi_{(0.5)})\) although it is a separator of \((E, \Pi_{(0.4)})\).

(4) If \( \text{Sep}(A) = 0 \), then, \( A \) is not a separator of \((E, \Pi_{(0)})\) by (2). Conversely, if \( A \) is not a separator of \((E, \Pi_{(0)})\), then, \( \text{Sep}(A) \leq 0 \) by (1), i.e., \( \text{Sep}(A) = 0 \).

A classical result on separators of matroids: for any subset \( A \) of \( E \), it is a separator of a given matroid \((E, \mathcal{S})\), which is equivalent to the rank function \( \mathcal{R}_{\mathcal{S}} \) satisfies \( \mathcal{R}_{\mathcal{S}}(A) + \mathcal{R}_{\mathcal{S}}(E - A) = \mathcal{R}_{\mathcal{S}}(E) \). Its promotion is as follows:

\[ \text{Sep}(A) = 1; \]
\[ (2) \text{ A is a separator of } (E, \Pi_{(a)}) \text{ for each } a \in [0, 1); \]
\[ (3) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) ; \]
\[ (4) \text{ A is a separator of } (E, \Pi_{(a)}) \text{ for each } a \in (0, 1]. \]

Proof. (1)\( \Rightarrow \) (2). If \( \text{Sep}(A) = 1 \), then, for each \( a \in [0, 1) \), Sep \( A \) > \( a \), thus, \( A \) is a separator of \((E, \Pi_{(a)})\) by Theorem 6 (1). Conversely, if (2) holds, i.e., \( A \) is a separator of \((E, \Pi_{(a)})\) for each \( a \in (0, 1) \). Assume that \( \text{Sep}(A) = b \neq 1 \), then, \( A \) is not a separator of \((E, \Pi_{(b)})\) by Theorem 6 (2), which contradicts (2). Hence, \( \text{Sep}(A) = 1 \).

By Theorem 2 and the property of the addition of fuzzy natural numbers, we have

\[ \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \mathcal{R}_{\Pi}(A) + \mathcal{R}_{\Pi}(E - A) = \mathcal{R}_{\Pi}(E) \]

for each \( a \in (0, 1] \).

Definition 7. Con \( (\mathcal{M}) = \wedge \{1 - \text{Sep}(A)|A \in 2^E - \{\emptyset, E\}\} \) is called connected degree of a given fuzzifying matroid \( \mathcal{M} = (E, \Pi) \), where \( \text{Sep}(A) \) is defined in Definition 6.

The connection between the connected degree of a fuzzifying matroid and the degree to which a subset is a separator is described in the following theorem.

Theorem 8. The following statements are true, where symbols \( \text{Sep}(A) \) and \( \text{Con}(\mathcal{M}) \) are defined in Definition 6 and Definition 7, respectively.

1. If \( \text{Con}(\mathcal{M}) > a \), then, \( \text{Sep}(A) \neq 1 - a \) for each non-empty proper subset \( A \) of \( E \) \((a \in [0, 1))\).
2. If \( \text{Con}(\mathcal{M}) = a \), then, \( \text{Sep}(A) = 1 - a \) for some non-empty proper subset \( A \) of \( E \) \((a \in [0, 1))\).
3. If \( \text{Con}(\mathcal{M}) < a (a \in (0, 1)), \) uncertain.
4. \( \text{Con}(\mathcal{M}) = 0 \) if and only if \( \text{Sep}(A) = 1 \) for some non-empty proper subset \( A \) of \( E \).
(5) \( \text{Con}(M) = 1 \) if and only if \( \text{Sep}(A) = 0 \) for each nonempty proper subset \( A \) of \( E \).

**Proof**

(1) Suppose that there exists a nonempty proper subset \( A \) of \( E \) such that \( \text{Sep}(A) = 1 - a \), then \( \text{Con}(M) = \land \{ 1 - \text{Sep}(A) | A \in 2^E - (\emptyset, E) \} \leq a \).

(2) Suppose that \( \text{Con}(M) = a \in [0, 1] \), i.e., \( \land \{ 1 - \text{Sep}(A) | A \in 2^E - (\emptyset, E) \} = a \), then \( 1 - \text{Sep}(A) = a \) for some \( A \in 2^E - (\emptyset, E) \), that is, \( \text{Sep}(A) = 1 - a \) for some nonempty proper subset \( A \) of \( E \).

(3) Consider the fuzzifying matroid \( M \) in [27], it is a set \( E = \{e_1, e_2, e_3, e_4, e_5\} \) and a mapping II from \( 2^E \) to \([0, 1]\) defined as \( II(A) = \lor \{ a \in [0, 1] | A \in \mathcal{F}_a \} \). Here, \( \mathcal{F}_a \) is constant in \([0, 1/3]\) (resp., \([1/3, 1/2]\), \([1/2, 1]\)), \( (E, J_0), (E, J_{1/3}), \) and \( (E, J_{1/2}) \) are matroids with the families of circuits \( C_0 = \{[e_1, e_2, e_3], [e_1, e_2, e_4], [e_1, e_3, e_4]\} \), and \( C_{1/3} = \{[e_2, e_3], [e_2, e_4, e_5], [e_3, e_4, e_5]\} \), respectively. Then, by the definition of \( C_0, C_{1/3} \) it is constant in \([0, 1/3]\) (resp., \([1/3, 1/2]\), \([1/2, 1]\)), \( (E, J_0), (E, J_{1/3}), \) and \( (E, J_{1/2}) \) are matroids with the families of circuits \( C_0 = \{[e_1, e_2, e_3], [e_1, e_2, e_4], [e_1, e_3, e_4]\} \), and \( C_{1/3} = \{[e_2, e_3], [e_2, e_4, e_5], [e_3, e_4, e_5]\} \), and \( 0 \) (otherwise). Thus, by Definition 9, \( \text{Sep}(A) = 1 \) if \( C \in \{[e_1, e_2, e_3], [e_1, e_2, e_4], [e_1, e_3, e_4]\} \), \( 1/2 \) (if \( C \in \{[e_2, e_3], [e_2, e_4, e_5], [e_3, e_4, e_5]\} \), and \( 0 \) (otherwise). Therefore, by Definition 7, \( \text{Con}(M) = (1/2) < (2/3) < (3/4) \). For \( a = (2/3), \) \( \text{Sep}(A) = (1/3) = 1 - a \) although \( \text{Sep}(A) < (1/4) = 1 - b \) for \( b = (3/4) \) and every nonempty proper subset \( A \) of \( E \).

(4) If \( \text{Con}(M) = 0 \), then, there exists a nonempty proper subset \( A \) of \( E \) such that \( \text{Sep}(A) = 1 \) by (2). Conversely, if \( \text{Sep}(A) = 1 \) for some nonempty proper subset \( A \) of \( E \), then, \( \text{Con}(M) \leq 0 \) by (1), i.e., \( \text{Con}(M) = 0 \).

(5) \( \text{Con}(M) = 1 \Leftrightarrow \land \{ 1 - \text{Sep}(A) | A \in 2^E - (\emptyset, E) \} = 1 \Leftrightarrow 1 - \text{Sep}(A) = 1 \) \( \land A \in 2^E - (\emptyset, E) \Leftrightarrow \text{Sep}(A) = 0 \), \( \land A \in 2^E - (\emptyset, E) \).

The following results show that the relationship between connected degree of a fuzzifying matroid and connectedness of its corresponding level matroids.

**Theorem 9.** The following statements are also true, among them, symbol \( \text{Con}(M) \) refers to Definition 7, which is the connected degree of a fuzzifying matroid \( M = (E, II) \).

1. If \( \text{Con}(M) = a \in [0, 1] \), then, \( (E, II_{(1-b)}) \) is not connected for any \( b > a \). However, the converse is not true.
2. If \( \text{Con}(M) = 1 \) if and only if \( (E, II_{(0)}) \) is connected.

**Proof**

(1) Suppose that \( b > a = \text{Con}(M) \), by Definition 7, there exists \( A \in 2^E - (\emptyset, E) \) such that \( 1 - \text{Sep}(A) < b \), and thus \( \text{Sep}(A) > 1 - b \); hence, \( A \) is a separator in \( (E, II_{(1-b)}) \) by Theorem 6. This implies that \( (E, II_{(1-b)}) \) is not a connected matroid.

(2) It follows from Theorems 6 (4) and 8 (5).

An immediate consequence of Theorems 5 and 9 is the following Theorem 10 which shows the connections between the connected degree of a fuzzifying matroid and connectedness of a fuzzifying matroid.

**Theorem 10.** A fuzzifying matroid \( M \) is connected, which is equivalent to its connected degree \( \text{Con}(M) = 1 \).

The rest of this section discusses the fuzzy analogs of the results: A separator in a matroid is also a separator in its dual, and the connectedness of a matroid is equivalent to that of its dual.

**Theorem 11.** The symbols \( M \) and \( M^* \) represent a fuzzifying matroid \( (E, II) \) and its dual \( (E, II^*) \), if \( (II^*)^* = II \), then, the following statements hold.

1. \( \text{Sep}^*(A) = \text{Sep}(A) \) for each \( A \in 2^E \), where \( \text{Sep}(A) \) and \( \text{Sep}^*(A) \) are the degrees to which \( A \) is a separator of \( M \) and \( M^* \).
2. \( \text{Con}(M^*) = \text{Con}(M) \).
3. \( M \) is connected, which is equivalent to \( M^* \) is connected.

**Proof**

(1) Suppose that \( \text{Sep}(A) = a \), then, \( A \) is not a separator in \( (E, II_{(a)}) \) by Theorem 6 (2), so it is not in the dual matroid \( (E, (II^*)_{(a)}) \), nor in \( (E, (II^*)_{(a)}) \) by \( (II^*)^* = II \) and Theorem 3. By Theorem 6 (1), \( \text{Sep}^*(A) \leq \text{Sep}(A) \), i.e., \( \text{Sep}^*(A) \leq \text{Sep}(A) \) and \( \text{Con}(M^*) = \text{Con}(M) \).

(2) It is an immediate consequence of (1) by the definition of connected degree for fuzzifying matroids.

(3) It is obvious by Theorem 10 and (2).

**Remark 1.** The characterization of the disconnectedness for a given fuzzifying matroid \( M = (E, II) \) induced by a fuzzy family of circuits in [27] is as follows: \( M \) is disconnected if and only if there is a nonempty proper subset of \( A \) such that \( R_{II}(A) + R_{II}(E - A) = R_{II}(E) \). From Theorems 7 and 8, it can also be characterized by the connected degree in this paper, \( M \) is disconnected if and only if \( \text{Con}(M) = 0 \), that is, \( M \) is connected if and only if \( \text{Con}(M) \neq 0 \). From this, we can see the connected degree introduced in this paper unifies the connectedness in [27].

We now describe the concepts of this paper by various examples.
Example 1. A simple class of fuzzifying matroids are fuzzifying uniform matroids. It is a fuzzifying matroid \((E, II)\) satisfying for any \(A, B \in 2^E\), \(|A| = |B|\) implies \(II(A) = II(B)\) [29].

Consider the example that a fuzzifying uniform matroid \(\mathcal{M}\) is a finite set \(E = \{e_1, e_2, e_3\}\) and a mapping \(\mathcal{M}\) from \(2^E\) to \([0, 1]\) defined as \(II(\emptyset) = 1\), \(II(\{e_1\}) = II(\{e_2\}) = II(\{e_3\}) = 0.5\), \(II(\{e_1, e_2\}) = II(\{e_1, e_3\}) = 0.3\), \(II(\{e_1, e_2, e_3\}) = 0.2\). So, by the definition of circuit-map, \(C_{II}(C) = 0\) (if \(C \in \{\emptyset\}\)), \(0.5\) (if \(C \in \{\{e_1\}, \{e_2\}, \{e_3\}\}\)), \(0.7\) (if \(C \in \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}\)), and \(0.8\) (if \(C \in \{\{e_1, e_2, e_3\}\}\)). By Definition 6, we have \(Sep(C) = \\{1 - C_{II}(C)\}\) \(C = \{e_1, e_2, e_3\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}\), and \(0.8\). Similarly, we also have \(Sep(A) = 0.2\) if \(A \in \{\{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}\).

Example 2. Another simple and more general class of fuzzifying matroids is fuzzifying paving matroids. It is a fuzzifying matroid \((E, II)\) satisfying for any \(A, B \in 2^E\), \(|A| < |B|\) implies \(II(A) \geq II(B)\) [29].

Consider the fuzzifying matroid \(\mathcal{M} = (E, II)\) given in the proof of Theorem 6 (3), which is a fuzzifying paving matroid rather than a fuzzifying uniform matroid. Similarly, we also have \(Sep(A) = (1/3)\) if \(A = \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_2, e_3\}\), or \(\{e_1, e_3\}\). Hence, \(Con(\mathcal{M}) = \\{1 - Sep(A)\}\) \(A \in 2^E - \{\emptyset, E\}\) = 0.8.

Example 3. Similarly, we can use the examples of fuzzifying acyclic matroids and fuzzifying simple matroids introduced in [29] to illustrate the concepts of this paper, which are omitted here.

Example 4. The very important class of fuzzifying matroids are derived from fuzzy graphs, which have been discussed in Section 4 in detail. In addition, we can also proved that \(\Pi_{\mathcal{G}}(A) = \lambda_{x \in A} \alpha(x)\) (if \(A\) does not contain any cycle), 0 (otherwise). Note that when \(\delta(e) = 1\) for any \(e \in V\), a fuzzy graph is abbreviated as \(\mathcal{G} = (V, \mu)\).

Consider the example that a fuzzifying cycle matroid \((E, II)\) induced by \(\mathcal{G} = (V, \mu)\) in Figure 1, and record it as \(\mathcal{M}_G\), where \(E = \{e_1, e_2, e_3, e_4\}\) is the edges set in \((V, \mu)\) and \(\mu(e_1) = 0.1, \mu(e_2) = 0.2, \mu(e_3) = 0.3, \mu(e_4) = 0.4\). Then, \(\Pi_{\mathcal{G}}(A) = 0\) (if \(A \in \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}\}\), \(0.1\) (if \(A \in \{\{e_1\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_4\}\}\)), \(0.2\) (if \(A \in \{\{e_2\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\}\)), \(0.3\) (if \(A \in \{\{e_2\}, \{e_3, e_4\}\}\)), \(0.4\) (if \(A \in \{\{e_4\}\}\)), and \(1\) (if \(A \in \{\emptyset\}\)). By the definition of circuit-map, \(C_{II}(C) = 1\) (if \(C \in \{\{e_1, e_2, e_3\}\}\)), \(0.9\) (if \(C \in \{\{e_1\}\}\)), \(0.8\) (if \(C \in \{\{e_2\}\}\)), \(0.7\) (if \(C \in \{\{e_3\}\}\)), \(0.6\) (if \(C \in \{\{e_4\}\}\)), and 0 (otherwise). By Definition 6, \(Sep(\{e_1, e_2, e_3\}) = \{1 - C_{II}(C)\} = \{e_1, e_2, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\).

6. Conclusions

This paper exemplifies an application of Polya’s plausible reasoning method (mainly analogous and generalization) in matroid theory, where we obtain some new results on 2-connectedness of fuzzifying matroids but just start from the 2-connectedness of graphs (a notion popularized for undergraduate students). We show some properties of connectedness and degree of connectedness of fuzzifying matroids and the reasonability of the definition of connected degree and enrich the theory of fuzzifying matroids. The course of discovering the notion of connected degree of fuzzifying matroids and related results is given clearly and relatively understandably which may be beneficial to mathematics enthusiasts and researchers in other areas who are interested to discover. There are also some work to be performed further. For example, (1) For the notion of connectedness, a fuzzy graph is either connected or not, we can consider the fuzziness degree of connectedness for a fuzzy graph analogously. (2) We can define and study connected degree of Goetschel–Voxman fuzzy matroids analogously; we can even define and study connected degree of \([0, 1]\)-matroids introduced in [18, 30] (a generalization of Goetschel–Voxman fuzzy matroids) analogously. (3) As we know, as an application of fuzzifying matroids, the fuzzifying greedy algorithm was presented in [23, 24], fuzzifying matroids are precisely the structures for which the very simple and efficient fuzzifying greedy algorithm works. Whether connectedness of fuzzifying matroids could be applied to some fuzzy combinatorial problems, we shall consider this problem in the future. Of course, applications in decision-making problems as in [31–34] should also be considered; (4) We are also extending our work to \(m\)-polar fuzzifying matroids by imitating reference [15].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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