Research Article

Unified Approach to the Existence of Solutions for a Coupled System of Fractional Differential-Integral Equations with Infinite Points and Riemann–Stieltjes Integral Boundary Conditions

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In this article, by using the Schauder fixed point theorem, we first study the existence of solutions for a new coupled system of Caputo fractional differential equations with multipoint boundary value conditions under the assumption that the nonlinear term satisfies two types of the Carathéodory conditions. Using this result, we provide the existence of solutions of the system with infinite points and Riemann–Stieltjes integral boundary conditions, respectively, instead of doing it directly. Finally, we give three examples to illustrate the feasibility of main results.

1. Introduction

The fractional differential equation is an important branch of differential equations, which is widely used in mathematics, physics, engineering, and other fields, and it solves robotics, signal processing, and conversion problems [1–4]. In fact, the fractional differential equation is an important tool for mathematical modeling due to its memorability and genetic properties. Recently, nonlocal integer and fractional orders fractional differential equations have attracted attention of a large number of scholars, see [5–24]. Indeed, the nonlocal problems for (fractional) differential equations have received great attention, see [5–14] and the references therein. In this paper, we consider the coupled system of Caputo fractional differential equations

\[
\begin{align*}
\begin{cases}
^cD_0^\alpha x(t) = g_1(t, x(t), y(t), \int_0^t f_1(s, x(s), y(s)) \, ds, \int_0^t h_1(s, x(s), y(s)) \, ds), \quad a \cdot e \cdot t \in (0, 1),
\end{cases} \\
\begin{cases}
^cD_0^\beta y(t) = g_2(t, x(t), y(t), \int_0^t f_2(s, x(s), y(s)) \, ds, \int_0^t h_2(s, x(s), y(s)) \, ds), \quad a \cdot e \cdot t \in (0, 1),
\end{cases}
\end{align*}
\]

with the multipoint conditions

\[
\begin{align*}
\begin{cases}
x(0) + \sum_{i=1}^{m_1} a_i x(\tau_i) = x_0, \quad a_i > 0, \tau_i \in (0, 1),
\end{cases} \\
\begin{cases}
y(0) + \sum_{j=1}^{m_2} b_j y(\tau_j) = y_0, \quad b_j > 0, \tau_j \in (0, 1),
\end{cases}
\end{align*}
\]
where \( \mathcal{D}_0^\alpha \) and \( \mathcal{D}_0^\beta \) are the Caputo fractional derivatives, \( \alpha, \beta \in (0, 1] \), \( x_0, y_0 \in \mathbb{R} \). Furthermore, we consider system (1) with the following Riemann–Stieltjes integral conditions:

\[
\begin{align*}
&x(0) + \int_0^1 x(s)dl_1(s) = x_0, \\ &y(0) + \int_0^1 y(s)dl_2(s) = y_0,
\end{align*}
\]

where \( l_i : [0, 1] \to \mathbb{R} \) \( (i = 1, 2) \) is an increasing function, \( x_0, y_0 \in \mathbb{R} \). We also discuss system (1) with infinite-point boundary conditions

\[
\begin{align*}
&x(0) + \sum_{j=1}^{\infty} a_j x(\tau_j) = x_0, \quad a_j > 0, \quad \tau_j \in (0, 1), \quad 0 \leq \sum_{j=1}^{\infty} a_j < + \infty, \\
y(0) + \sum_{j=1}^{\infty} b_j y(\tau_j) = y_0, \quad b_j > 0, \quad \tau_j \in (0, 1), \quad 0 \leq \sum_{j=1}^{\infty} b_j < + \infty,
\end{align*}
\]

where \( x_0, y_0 \in \mathbb{R} \).

There are many methods for searching the existence of solutions of fractional boundary value problems, such as partial order methods \([15]\), topology degree theory \([10–14, 16–18]\), the critical point theorem \([19]\), and so on. Recently, the authors in \([10]\) studied the existence and continuous dependence of solutions by the Schauder fixed point theorem for the following coupled system:

\[
\begin{align*}
\frac{dx}{dt} &= g_1(t, y(t)), \quad t, y(s)ds \in (0, T], \\
\frac{dx}{dt} &= g_2(t, y(t)), \quad t, y(s)ds \in (0, T], \\
x_0 &= \sum_{i=1}^{\infty} a_i x(\tau_i), \quad a_i > 0, \quad \tau_i \in (0, T], \\
y_0 &= \sum_{j=1}^{\infty} b_j y(\tau_j), \quad b_j > 0, \quad \tau_j \in (0, T].
\end{align*}
\]

Moreover, the authors studied the following Riemann–Stieltjes integral conditions

\[
\begin{align*}
&x(0) + \int_0^T x(s)dh_1(s) = x_0, \\
y(0) + \int_0^T y(s)dh_2(s) = y_0,
\end{align*}
\]

and also discussed the following infinite-point conditions:

\[
\begin{align*}
&x(0) + \sum_{i=1}^{\infty} a_i x(\tau_i) = x_0, \quad a_i > 0, \quad \tau_i \in (0, T], \quad \sum_{i=1}^{\infty} a_i < + \infty, \\
y(0) + \sum_{j=1}^{\infty} b_j y(\tau_j) = y_0, \quad b_j > 0, \quad \tau_j \in (0, T], \quad \sum_{j=1}^{\infty} b_j < + \infty.
\end{align*}
\]

By using the Schauder fixed point theorem and Banach contraction mapping principle, the existence and uniqueness of solutions of the system are proved.

By applying the Guo–Krasnosel’skii fixed point theorem of the cone expansion-compression type, Zhang in \([11]\) considered the several local existence and multiplicity of positive solutions for the following problem:

\[
\begin{align*}
&\mathcal{D}_0^\alpha u(t) + q(t)f(t, u(t)) = 0, \quad t \in (0, 1), \\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\
&u^{(i)}(1) = \sum_{j=1}^{\infty} a_j u(\xi_j),
\end{align*}
\]

where \( \mathcal{D}_0^\alpha \) is Riemann–Liouville fractional derivative, \( \alpha > 2, n-1 < \alpha \leq n, t \in [1, n-2] \) is a fixed integer, \( a_j \geq 0, 0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < \) and \( a_j \xi_j^{\alpha-1} > 0, \triangle = (a-1)(a-2) \cdots (a-i), q_j : (0, 1) \to \mathbb{R}^+ \) and \( f \) may be singular with respect to both the time and space variables.

In \([12]\), the existence of solutions for the following problem was obtained by using the Schauder’s fixed point theorem

\[
\begin{align*}
&\frac{dx}{dt} = g(t, D^\alpha x(t)), \quad a \cdot e \cdot t \in (0, 1), \\
&\int_0^1 x(\phi(s))d\gamma(s) = x_0,
\end{align*}
\]

where \( D^\alpha \) is the Caputo fractional derivative and \( \alpha \in (0, 1], g : [0, 1] \to [0, 1] \) is an increasing function.

In this paper, we discuss the coupled system (1), in which the nonlinear terms \( g \) include two space variables \( x \) and \( y \) and the integral terms. By using the Schauder fixed point theorem, the existence of solutions for system (1) with the multipoint boundary conditions (2) is proved under the assumption that the nonlinear term satisfies two types of the Carathéodory conditions, which consists sublinear and linear conditions. Furthermore, based on the above results, we give the existence of solutions of system (1) with infinite point and Riemann–Stieltjes integral boundary conditions, respectively, which avoids proving again. Compared with the articles mentioned above, system (1) is more general. For example, the nonlinear terms in \([10]\) contain a single space variable, while we consider two space variables.

The outline of this paper is organized as follows. In Section 2, we state some preliminary settings. In Section 3, we show the proof of existence of solutions. In Section 4, three examples are given in order to illustrate our results. In Section 5, we give the conclusions of this paper.

## 2. Preliminaries and Lemmas

In this section, we will provide some definitions and lemmas to be used in the following proofs.
Definition 1. (see [1]) Let \( f \in AC^n[0, 1] \). Then, the Caputo fractional derivative of order \( \alpha > 0 \) of \( f \) is given by the following:

\[
\begin{align*}
\mathcal{D}^\alpha \tau f(t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{n-\alpha+1}} \, ds, & n - 1 < \alpha < n, \\
\frac{f^{(n)}(s)}{\alpha}, & \alpha = n,
\end{cases}
\end{align*}
\]

which exists almost everywhere on \([0, 1]\).

Definition 2. (see [1]) For a function \( f : (0, +\infty) \rightarrow \mathbb{R} \), the Riemann–Liouville fractional integral of order \( \alpha > 0 \) of \( f \) is given by the following:

\[
\mathcal{I}^\alpha \tau f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds,
\]

provided that the right-hand side is pointwise defined on \((0, +\infty)\).

Lemma 1. (see [1]) For \( \alpha > 0 \), the fractional differential equation \( \mathcal{D}^\alpha \tau u(t) = 0 \) has a general solution

\[
u(t) = C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}, \quad C_i \in \mathbb{R}, 
\]

\[i = 1, 2, \ldots, n, \quad n - 1 < \alpha \leq n.
\]

Lemma 2. (see [1]) Assume that \( u \in C^n[0, 1] \), then

\[
\mathcal{I}^\alpha \mathcal{D}^\alpha \tau u(t) = \frac{1}{\alpha} \int_0^t (t-s)^{\alpha-1-\frac{1}{p}} f(s) \, ds,
\]

\( i = 1, 2, \ldots, n, \quad n - 1 < \alpha \leq n. \)

Lemma 3. (see [4]) Let \( f \in L^p[a, b], \ g \in L^q[a, b], \) where \( p > 1, \ (1/p) + (1/q) = 1. \) Then, \( f, g \in L^1[a, b], \) and

\[
\int_a^b |f(t)g(t)| \, dt \leq \|f\|_p \|g\|_q,
\]

where \( \|f\|_p = \left( \int_a^b |f(t)|^p \, dt \right)^{1/p} \).

Lemma 4. Let \( u \in C((0, 1) \cap L^{p_1}(0, 1)) \cap L^{p_2}(0, 1), \ (1/p_1) \in (0, \alpha), \ (1/p_2) \in (0, \beta). \) Then, the system

\[
\mathcal{D}^\alpha \mathcal{D}^\beta \tau x(t) = u(t), \quad a \cdot e \cdot t \in (0, 1),
\]

\[
\mathcal{D}^\alpha \mathcal{D}^\beta \tau y(t) = v(t), \quad a \cdot e \cdot t \in (0, 1),
\]

with the boundary conditions

\[
\begin{align*}
x(0) + \sum_{i=1}^m a_i x(\tau_i) &= x_0, \quad a_i > 0, \quad \tau_i \in (0, 1), \\
y(0) + \sum_{j=1}^m b_j y(\tau_j) &= y_0, \quad b_j > 0, \quad \tau_j \in (0, 1)
\end{align*}
\]

has a unique solution \((x, y)\), which can be expressed by

\[
\begin{align*}
x(t) &= \frac{1}{1 + \sum_{i=1}^m a_i} \left[ x_0 - \sum_{i=1}^m a_i \int_0^\tau (\tau_i - t)^{\alpha-1} u(s) \, ds \right] \\
&+ \int_0^\tau (t-s)^{\alpha-1} u(s) \, ds,
\end{align*}
\]

\[
\begin{align*}
y(t) &= \frac{1}{1 + \sum_{j=1}^m b_j} \left[ y_0 - \sum_{j=1}^m b_j \int_0^\tau (\tau_j - t)^{\beta-1} v(s) \, ds \right] \\
&+ \int_0^\tau (t-s)^{\beta-1} v(s) \, ds.
\end{align*}
\]

Proof. According to the known conditions and Lemma 3, we have

\[
\left| \int_0^\tau (t-s)^{\alpha-1} u(s) \, ds \right| \leq \frac{1}{\alpha} \left( \int_0^\tau (t-s)^{\alpha-1} \, ds \right)^{1-(1/p)} \left( \int_0^\tau |u(s)|^p \, ds \right)^{1/p} \leq \frac{1}{\alpha} \left( p_1 \alpha - 1 \right)^{1-(1/p)} \|u\|_{L^p},
\]

where \( u \in C((0, 1) \cap L^{p_1}(0, 1)) \) and \( (1/p_1) \in (0, \alpha) \), it can be seen \( (p_1 - 1)(\alpha - 1) > 0 \). Then,

\[
\int_0^\tau (t-s)^{\alpha-1} u(s) \, ds < \infty,
\]

in the same way

\[
\int_0^\tau (t-s)^{\beta-1} v(s) \, ds < \infty.
\]

Therefore, it is concluded that the integral exists.

Next, using Lemma 2 to integrate both sides at the same time, we have

\[
\begin{align*}
x(t) &= \int_0^\tau (t-s)^{\alpha-1} u(s) \, ds + C_1, \quad C_1 \in \mathbb{R}, \\
y(t) &= \int_0^\tau (t-s)^{\beta-1} v(s) \, ds + C_2, \quad C_2 \in \mathbb{R}.
\end{align*}
\]

By the boundary conditions (2), the expression of \( C_1, C_2 \) can be obtained as follows.
Consequently, substituting \( C_1, C_2 \) into formula (21), we obtain the expression of the solution as follows (17).

\[
\begin{aligned}
C_1 &= x_0 - \sum_{i=1}^{m} a_i \int_0^1 \left( (\tau_j - s)^{-1} / \Gamma (\alpha) \right) \mu (s) ds, \\
C_2 &= y_0 - \sum_{j=1}^{m} b_j \int_0^1 \left( (\tau_j - s)^{-1}/\Gamma (\beta) \right) \nu (s) ds.
\end{aligned}
\]

(22)

In order to prove later, we make some assumptions here as follows:

**(H)\(_i\)g_i: \([0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R} (i = 1, 2)\) satisfies the Carathéodory condition, i.e.,

(i) \( g_i (t, \cdot, \cdot, \cdot, \cdot) \) is measurable for any \( x_1, x_2, x_3, x_4 \in \mathbb{R} \).

(ii) \( g_i (t, \cdot, \cdot, \cdot, \cdot) \) is continuous for almost all \( t \in [0, 1] \).

(iii) There exist functions \( a_{i1}, b_{11}, b_{12}, b_{13}, b_{14} \in L^p \) \(([0, 1], \mathbb{R}^+), a_{i2}, b_{21}, b_{22}, b_{23}, b_{24} \in L^p \) \(([0, 1], \mathbb{R}^+))\) with \((1/p_i) \in (0, \alpha), (1/p_2) \in (0, \beta)\), such that

\[
\begin{aligned}
|g_1 (t, x_1, x_2, x_3, x_4)| &\leq a_{11} (t) + b_{11} (t)|x_1|^\alpha + b_{12} (t)|x_2|^\alpha, \\
&\quad + b_{13} (t)|x_3|^\alpha + b_{14} (t)|x_4|^\alpha, \\
&\quad + b_{23} (t)|x_3|^\beta + b_{24} (t)|x_4|^\beta. \\
\end{aligned}
\]

(23)

**(H)\(_i\)f_i: \([0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R} (i = 1, 2)\) satisfies the Carathéodory condition, i.e.,

(i) \( f_i (t, \cdot, \cdot, \cdot) \) is measurable for any \( x_1, x_2 \in \mathbb{R} \).

(ii) \( f_i (t, \cdot, \cdot, \cdot) \) is continuous for almost all \( t \in [0, 1] \).

(iii) There exist functions \( a_{i1}, c_{11}, c_{12}, c_{21}, c_{22} \in L^p \) \(([0, 1], \mathbb{R}^+), a_{i2}, c_{21}, c_{22} \in L^p \) \(([0, 1], \mathbb{R}^+))\) with \((1/p_i) \in (0, \alpha), (1/p_2) \in (0, \beta)\) such that

\[
\begin{aligned}
|f_1 (t, x_1, x_2)| &\leq a_{11} (t) + c_{11} (t)|x_1|^\alpha + c_{12} (t)|x_2|^\alpha, \\
f_2 (t, x_1, x_2) &\leq a_{12} (t) + c_{21} (t)|x_1|^\alpha + c_{22} (t)|x_2|^\alpha. \\
\end{aligned}
\]

(24)

**(H)\(_i\)h_i: \([0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R} (i = 1, 2)\) satisfies the Carathéodory condition, i.e.,

(i) \( h_i (t, \cdot, \cdot, \cdot, \cdot) \) is measurable for any \( x_1, x_2, x_3, x_4 \in \mathbb{R} \).

(iii) \( h_i (t, \cdot, \cdot, \cdot) \) is continuous for almost all \( t \in [0, 1] \).

(iii) There exist functions \( a_{i1}, d_{11}, d_{12} \in L^p \) \(([0, 1], \mathbb{R}^+), a_{i2}, d_{21}, d_{22} \in L^p \) \(([0, 1], \mathbb{R}^+))\) with \((1/p_i) \in (0, \alpha), (1/p_2) \in (0, \beta)\) such that

\[
\begin{aligned}
h_1 (t, x_1, x_2) &\leq a_{11} (t) + d_{11} (t)|x_1|^\alpha + d_{12} (t)|x_2|^\alpha, \\
h_2 (t, x_1, x_2) &\leq a_{12} (t) + d_{21} (t)|x_1|^\alpha + d_{22} (t)|x_2|^\alpha. \\
\end{aligned}
\]

(26)

\[
0 < \left( 1 + A^{-1} \sum_{j=1}^{m} a_j \right) \left( \frac{p - 1}{p \alpha - 1} \right)^{\alpha - 1/p} \left( \| a_{11} \|_{L^p} + \| a_{12} \|_{L^p} + \| c_{11} \|_{L^p} + \| c_{12} \|_{L^p} + \| d_{11} \|_{L^p} + \| d_{12} \|_{L^p} \right) \Gamma (\alpha),
\]

(27)

\[
0 < \left( 1 + B^{-1} \sum_{j=1}^{m} b_j \right) \left( \frac{p - 1}{p \beta - 1} \right)^{\beta - 1/p} \left( \| b_{11} \|_{L^p} + \| b_{12} \|_{L^p} + \| b_{21} \|_{L^p} + \| b_{22} \|_{L^p} + \| b_{23} \|_{L^p} + \| b_{24} \|_{L^p} \right) \Gamma (\beta),
\]
where \( A = 1 + \sum_{i=1}^{m} a_i, B = 1 + \sum_{j=1}^{m} b_j \).

\[
0 < \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{1-(1/P_1)} \left( 1 + \frac{(I_1 (1) - I_1 (0))}{1 + (I_1 (1) - I_1 (0))} \right) \left( \| a_{11} \|_{L^{P_1}} + \| b_{11} \|_{L^{P_1}} + \| a_{12} \|_{L^{P_1}} + \| b_{12} \|_{L^{P_1}} \right) \left( \| a_{11} \|_{L^{P_1}} + \| b_{11} \|_{L^{P_1}} + \| a_{12} \|_{L^{P_1}} + \| b_{12} \|_{L^{P_1}} \right) \left( \| a_{13} \|_{L^{P_1}} + \| d_{11} \|_{L^{P_1}} + \| d_{12} \|_{L^{P_1}} \right) \left( \| a_{23} \|_{L^{P_1}} + \| d_{21} \|_{L^{P_1}} + \| d_{22} \|_{L^{P_1}} \right) < \Gamma (\alpha),
\]

(28)

\[
0 < \left( \frac{P_2 - 1}{P_2 \beta - 1} \right)^{1-(1/P_2)} \left( 1 + \frac{(I_2 (1) - I_2 (0))}{1 + (I_2 (1) - I_2 (0))} \right) \left( \| a_{23} \|_{L^{P_2}} + \| d_{21} \|_{L^{P_2}} + \| d_{22} \|_{L^{P_2}} \right) \left( \| a_{22} \|_{L^{P_2}} + \| b_{23} \|_{L^{P_2}} + \| c_{21} \|_{L^{P_2}} + \| c_{22} \|_{L^{P_2}} + \| b_{24} \|_{L^{P_2}} \right) \left( \| a_{23} \|_{L^{P_2}} + \| d_{21} \|_{L^{P_2}} + \| d_{22} \|_{L^{P_2}} \right) \left( \| a_{22} \|_{L^{P_2}} + \| d_{21} \|_{L^{P_2}} + \| d_{22} \|_{L^{P_2}} \right) < \Gamma (\beta).
\]

(29)

3. Main Results

Let \( X = C[0, 1] \) be the Banach space equipped with the norm \( \| x \|_X = \max_{t \in [0,1]} |x(t)| \). Let \( Y = X \times X \) be the Banach space of ordered pairs \((x, y)\) equipped with the norm \( \|(x, y)\|_Y = \max\{\|x\|_X, \|y\|_X\} \). In the following part, from Lemma 3, we define the operator \( T: Y \rightarrow Y \) as follows:

\[
T(x, y) = (T_1(x, y), T_2(x, y)), \quad (x, y) \in Y,
\]

where

\[
T_1(x, y)(t) = \begin{bmatrix} \frac{(t - s)^{\alpha - 1}}{\Gamma (\alpha)} \int_0^t g_1 \left( s, x(s), y(s), \int_0^s f_1 (\theta, x(\theta), y(\theta)) d\theta, \int_0^s h_1 (\theta, x(\theta), y(\theta)) d\theta \right) ds \end{bmatrix}
\]

(31)

\[
+ \begin{bmatrix} \frac{(t - s)^{\beta - 1}}{\Gamma (\beta)} \int_0^t g_2 \left( s, x(s), y(s), \int_0^s f_2 (\theta, x(\theta), y(\theta)) d\theta, \int_0^s h_2 (\theta, x(\theta), y(\theta)) d\theta \right) ds \end{bmatrix}
\]

\[
T_2(x, y)(t) = \begin{bmatrix} \frac{(t - s)^{\alpha - 1}}{\Gamma (\alpha)} \int_0^t g_1 \left( s, x(s), y(s), \int_0^s f_1 (\theta, x(\theta), y(\theta)) d\theta, \int_0^s h_1 (\theta, x(\theta), y(\theta)) d\theta \right) ds \end{bmatrix}
\]

(31)
In the following part, for convenience, we set

\[ D_1 = A^{-1} |x_0| + \left( A^{-1} \sum_{i=1}^{m} a_i + 1 \right) \frac{1}{\Gamma(\alpha)} \left( \frac{p_1 - 1}{p_1 \alpha - 1} \right)^{1-(1/p_1)} \| a_{11} \|_{L^p}, \]

\[ D_2 = B^{-1} |y_0| + \left( B^{-1} \sum_{j=1}^{m} b_j + 1 \right) \frac{1}{\Gamma(\beta)} \left( \frac{p_1 - 1}{p_2 \beta - 1} \right)^{1-(1/p_2)} \| a_{21} \|_{L^p}, \]

\[ G_1 = \| b_{11} \|_{L^p} + \| b_{12} \|_{L^p} + \| b_{13} \|_{L^p} \left( \| a_{12} \|_{L^p} + \| b_{14} \|_{L^p} + \| c_{11} \|_{L^p} + \| c_{12} \|_{L^p} \right)^{\alpha}, \]

\[ G_2 = \| b_{22} \|_{L^p} + \| b_{23} \|_{L^p} + \| b_{24} \|_{L^p} \left( \| a_{22} \|_{L^p} + \| b_{24} \|_{L^p} + \| c_{21} \|_{L^p} + \| c_{22} \|_{L^p} \right)^{\beta}, \]

\[ W_1 = \| b_{11} \|_{L^p} + \| b_{12} \|_{L^p} + \| b_{13} \|_{L^p} \left( \| a_{12} \|_{L^p} + \| b_{14} \|_{L^p} + \| c_{11} \|_{L^p} + \| c_{12} \|_{L^p} \right), \]

\[ W_2 = \| b_{22} \|_{L^p} + \| b_{23} \|_{L^p} + \| b_{24} \|_{L^p} \left( \| a_{22} \|_{L^p} + \| b_{24} \|_{L^p} + \| c_{21} \|_{L^p} + \| c_{22} \|_{L^p} \right), \]

\[ E_1 = \left( A^{-1} \sum_{i=1}^{m} a_i + 1 \right) \frac{1}{\Gamma(\alpha)} \left( \frac{p_1 - 1}{p_1 \alpha - 1} \right)^{1-(1/p_1)} G_1, \]

\[ E_2 = \left( A^{-1} \sum_{j=1}^{m} b_j + 1 \right) \frac{1}{\Gamma(\beta)} \left( \frac{p_1 - 1}{p_2 \beta - 1} \right)^{1-(1/p_2)} G_2, \]

\[ Z_1 = \left( A^{-1} \sum_{i=1}^{m} a_i + 1 \right) \frac{1}{\Gamma(\alpha)} \left( \frac{p_1 - 1}{p_1 \alpha - 1} \right)^{1-(1/p_1)} W_1, \]

\[ Z_2 = \left( A^{-1} \sum_{j=1}^{m} b_j + 1 \right) \frac{1}{\Gamma(\beta)} \left( \frac{p_1 - 1}{p_2 \beta - 1} \right)^{1-(1/p_2)} W_2. \]

**Theorem 1.** Assume that the assumptions \((H_1)-(H_3)\) hold. Then, systems (1) and (2) have at least one solution.

**Proof.** Let us set up \( \bar{\alpha} = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \bar{\beta} = \max\{\beta_1, \beta_2, \beta_3, \beta_4\} \), then \( \bar{\alpha}, \bar{\beta} \in (0, 1) \). Accordingly, we infer that, there exists \( R_1 > 0 \) large enough such that

\[ D_1 + E_1 R_1^{\bar{\alpha}} \leq R_1, \]

\[ D_2 + E_2 R_1^{\bar{\beta}} \leq R_1, \]

without loss of generality, we may assume that \( R_1 > 1 \).

First, we prove that \( T: Q_{R_1} \rightarrow Q_{R_1} \), where \( Q_{R_1} = \{(x, y) \in Y: \|(x, y)\|_Y \leq R_1\} \). It should be noted that for any \((x, y) \in Q_{R_1}\),
\[
\frac{1}{\Gamma (\alpha )} \left[ \int_0^t (t-s)^{\alpha -1} g(s, x(s), y(s)) \left( \int_0^s f_1 (\theta, x(\theta), y(\theta)) d\theta \int_0^\theta h_1 (\theta, x(\theta), y(\theta)) d\theta \right) ds \right] \\
\leq \frac{1}{\Gamma (\alpha )} \left[ \int_0^t (t-s)^{\alpha -1} a_{11} (s) + b_{11} (s) y(s)^{\alpha_0} + b_{12} (s) \left( \int_0^s c_{11} (\theta) d\theta + \int_0^\theta c_{12} (\theta) d\theta \right) ds \right. \\
\left. + b_{13} (s) \left( \int_0^s b_{12} (\theta) d\theta + \int_0^\theta b_{13} (\theta) d\theta \right) \int_0^\theta c_{13} (\theta) d\theta ds \right] \\
\leq \frac{1}{\Gamma (\alpha )} \left[ \int_0^t (t-s)^{\alpha -1} a_{11} (s) ds + R_{\alpha}^{\alpha_0} \int_0^t (t-s)^{\alpha -1} b_{11} (s) ds \right. \\
\left. + R_{\alpha}^{\alpha_0} \int_0^t (t-s)^{\alpha -1} b_{12} (s) ds + R_{\alpha}^{\alpha_0} \int_0^t (t-s)^{\alpha -1} b_{13} (s) \left( \int_0^s c_{12} (\theta) d\theta + \sum_{i=1}^{n_0} \int_0^\theta c_{13} (\theta) d\theta \right) ds \right] \\
\leq \frac{1}{\Gamma (\alpha )} \left[ \int_0^t (t-s)^{\alpha -1} \left( \int_0^s a_{11} (s) ds \right)^{\alpha_0} \left( \int_0^\theta c_{12} (\theta) d\theta + \sum_{i=1}^{n_0} \int_0^\theta c_{13} (\theta) d\theta \right) \right] + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \int_0^t (t-s)^{\alpha -1} b_{13} (s) \\
\left[ \left( \int_0^s a_{11}^{\alpha_0} (\theta) d\theta \right) \int_0^s (t-s)^{-\alpha_{10}} ds \right] + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \int_0^t (t-s)^{\alpha -1} b_{14} (s) \\
\left[ \left( \int_0^s b_{12}^{\alpha_0} (\theta) d\theta \right) \int_0^s (t-s)^{-\alpha_{10}} ds \right] + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \int_0^t (t-s)^{\alpha -1} b_{13} (s) \\
\left[ \left( \int_0^s b_{13}^{\alpha_0} (\theta) d\theta \right) \int_0^s (t-s)^{-\alpha_{10}} ds \right] \\
\leq \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{11} \|_{L^\alpha} + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{12} \|_{L^\alpha} + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{13} \|_{L^\alpha} \\
+ \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{12} \|_{L^\alpha} + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{14} \|_{L^\alpha} \\
+ \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{13} \|_{L^\alpha} + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{13} \|_{L^\alpha} \\
+ \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{14} \|_{L^\alpha} + \frac{R_{\alpha}^{\alpha_0}}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{14} \|_{L^\alpha} \\
\leq \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{11} \|_{L^\alpha} + \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{12} \|_{L^\alpha} + \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{13} \|_{L^\alpha} \\
+ \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| a_{14} \|_{L^\alpha} + \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{11} \|_{L^\alpha} + \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{12} \|_{L^\alpha} \\
+ \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{13} \|_{L^\alpha} + \frac{1}{\Gamma (\alpha )} \left( \frac{P_1 - 1}{P_1 \alpha - 1} \right)^{\alpha_0} \| b_{14} \|_{L^\alpha}, t \in [0, 1].
\]
Thus, we have

\[
|T_1(x, y)(t)|
\]

\[
\leq A^{-1}|x_0| + \sum_{i=1}^{m} a_i \left| \int_0^t \frac{(t-s)^{a_i-1}}{\Gamma(a_i)} g_i(s, x(s), y(s)) \, ds \right| \int_0^t |f_i(\theta, x(\theta), y(\theta))| \, d\theta \int_0^t |h_i(\theta, x(\theta), y(\theta))| \, d\theta \, ds
\]

\[
\leq A^{-1}|x_0| + \left( A^{-1} \sum_{i=1}^{m} a_i + 1 \right) \frac{1}{\Gamma(a)} \left( \frac{p_1 - 1}{p_1 \alpha - 1} \right)^{1-\left(\frac{1}{p_1}\right)} \left| a_{i1} \right|_{L^p}
\]

\[
+ \left( A^{-1} \sum_{i=1}^{m} a_i + 1 \right) \frac{1}{\Gamma(a)} \left( \frac{p_1 - 1}{p_1 \alpha - 1} \right)^{1-\left(\frac{1}{p_1}\right)} G_i R_i^\alpha = D_1 + E_i R_i^\alpha \leq R_1, \quad t \in [0, 1],
\]

which means \(\|T_1(x, y)\|_X \leq R_1\). In the same way, we get that

\[
\|T_2(x, y)\|_X \leq D_2 + E_i R_i^\alpha \leq R_1.
\]

Based on (35) and (36), we conclude that

\[
\|T(x, y)\|_Y = \max\{\|T_1(x, y)\|_X, \|T_2(x, y)\|_X\} \leq R_1.
\]

This proves that \(T: Q_{R_1} \longrightarrow Q_{R_1}\).

Second, we prove that operator \(T: Q_{R_1} \longrightarrow Q_{R_1}\) is continuous. Let \((x_n, y_n, (x, y)) \in Q_{R_1}\), and \((x_n, y_n) \longrightarrow (x, y) (n \longrightarrow \infty)\) in \(Y\). Utilizing the assumptions \((H_1)-(H_3)\), we have

\[
g_i(s, x_n(s), y_n(s), \int_0^s f_i(\theta, x_n(\theta), y_n(\theta)) \, d\theta, \int_0^s h_i(\theta, x_n(\theta), y_n(\theta)) \, d\theta)
\]

\[
\longrightarrow g_i(s, x(s), y(s), \int_0^s f_i(\theta, x(\theta), y(\theta)) \, d\theta, \int_0^s h_i(\theta, x(\theta), y(\theta)) \, d\theta), \quad a \cdot e \cdot s \in [0, 1], \quad i = 1, 2.
\]

On the other hand, we have

\[
\left| g_i(s, x_n(s), y_n(s), \int_0^s f_i(\theta, x_n(\theta), y_n(\theta)) \, d\theta, \int_0^s h_i(\theta, x_n(\theta), y_n(\theta)) \, d\theta) - g_i(s, x(s), y(s), \int_0^s f_i(\theta, x(\theta), y(\theta)) \, d\theta, \int_0^s h_i(\theta, x(\theta), y(\theta)) \, d\theta) \right|
\]

\[
\leq 2a_{i1}(s) + 2R_i^\alpha (b_{i1}(s) + b_{i2}(s) + b_{i3}(s) + b_{i4}(s)), \quad s \in [0, 1], \quad i = 1, 2.
\]

Based on \((H_1)\), we know \(2a_{i1} + 2R_i^\alpha (b_{i1} + b_{i2} + b_{i3} + b_{i4}) \in L^1[0, 1]\). Thus, by the Lebesgue dominated convergence theorem, we have

\[
\lim_{n \longrightarrow \infty} T_i(x_n, y_n) = T_i(x, y) \quad \text{in} \quad X, \quad i = 1, 2.
\]

Accordingly, we conclude that \(T(x_n, y_n) \longrightarrow T(x, y)\) in \(Y\) as \(n \longrightarrow \infty\), that is, the operator \(T: Q_{R_1} \longrightarrow Q_{R_1}\) is continuous.

In the following part, we prove that \(\{T(Q_{R_1})\}\) is equi-continuous. Based on \((H_1)\), we know \(a_{i1}, b_{i1}, b_{i2}, b_{i3}, b_{i4}\)
\( b_{i4} \in L^1[0,1]. \) Thus, for any \((x, y) \in Q_{R_i}, \) \( t_1, t_2 \in [0,1] \) \((t_1 < t_2),\) we have

\[
|T_i(x, y)(t_2) - T_i(x, y)(t_1)| \leq \left| \int_0^{t_1} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} \left| \int_0^s \left[ g_i(s, x(s), y(s)) \int_0^s f_i(\theta, x(\theta), y(\theta)) d\theta \int_0^s h_i(\theta, x(\theta), y(\theta)) d\theta \right] ds \right| \right| ds \\
- \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \left| \int_0^s \left[ g_i(s, x(s), y(s)) \int_0^s f_i(\theta, x(\theta), y(\theta)) d\theta \int_0^s h_i(\theta, x(\theta), y(\theta)) d\theta \right] ds \right| ds,
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \left( \left\| a_{i1} \right\|_{L^1} + R_i \left( \left\| b_{11} \right\|_{L^1} + \left\| b_{12} \right\|_{L^1} + \left\| b_{13} \right\|_{L^1} \right) \left\| c_{i1} \right\|_{L^1} + \left\| c_{i2} \right\|_{L^1} \right) \left( t_2^{\alpha} - t_1^{\alpha} \right), \quad i = 1, 2.
\]

Since the function \( t^\alpha \) is continuous on \([0,1],\) we get that, for any \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that, for any \( t_1, t_2 \in [0,1], \) \(|t_1 - t_2| < \delta,\)

\[
|t_2^{\alpha} - t_1^{\alpha}| < \varepsilon.
\]

Consequently, for any \( t_1, t_2 \in [0,1], \) \(|t_1 - t_2| < \delta,\) we have

\[
|T_i(x, y)(t_2) - T_i(x, y)(t_1)| < \varepsilon,
\]

which means that \( \{T_i(Q_{R_i})\} \) \((i = 1, 2)\) is equicontinuous on \([0,1],\) and also, \( \{T(Q_{R_i})\} \) is equicontinuous on \([0,1].\) Based on (35) and (36), we know that the set \( \{T(Q_{R_i})\} \) is uniformly bounded. According to the Arzel \( \alpha^*-\)Ascoli theorem, we prove that operator \( T \) is a completely continuous operator. Thus, by the Schauder fixed point theorem, we deduce that systems (1) and (2) have at least one solution \((x, y) \in Q_{R_1},\)

which can be expressed as follows:
\[
\begin{aligned}
x(t) &= \frac{1}{1 + \sum_{i=1}^{m} a_i} \left[ x_0 - \sum_{i=1}^{m} a_i \int_0^{(t_i - s)^{\alpha_i}} f_1 \left( \theta, x(\theta), y(\theta) \right) d\theta \right]
+ \int_0^{(t - s)^{\alpha_i}} g_1 \left( s, x(s), y(s) \right) \, ds,
\\
y(t) &= \frac{1}{1 + \sum_{j=1}^{m} b_j} \left[ y_0 - \sum_{j=1}^{m} b_j \int_0^{(t_j - s)^{\beta_j}} g_2 \left( s, x(s), y(s) \right) \, ds \right]
+ \int_0^{(t - s)^{\beta_j}} g_2 \left( s, x(s), y(s) \right) \, ds.
\end{aligned}
\]

Theorem 2. Assume that the assumptions (H_1)-(H_3) hold. Then, systems (1) and (3) have at least one solution.

Proof. First, we make an interval partition: \(0 = t_0 < t_1 < \cdots < t_m = 1\) such that \(t_i \in (t_{i-1}, t_i)\) \(t_j \in (t_{j-1}, t_j)\) (\(i, j = 1, 2, \ldots, m\)). Let \(a_i = l_1(t_i) - l_1(t_{i-1})\), \(b_j = l_2(t_j) - l_2(t_{j-1})\). Then, we have

\[
\begin{aligned}
x(0) + \sum_{i=1}^{m} a_i x(t_i) &= x(0) + \sum_{i=1}^{m} (l_1(t_i) - l_1(t_{i-1})) x(t_i) = x_0,
\\
y(0) + \sum_{j=1}^{m} b_j y(t_j) &= y(0) + \sum_{j=1}^{m} (l_2(t_j) - l_2(t_{j-1})) y(t_j) = y_0.
\end{aligned}
\]

By Theorem 1, we know, systems (1) and (2) have a solution \((x, y) \in Q_{R_1}\), which can be expressed in (44). Let \(m \to \infty\) in (45), we then have

\[
\begin{aligned}
x(0) + \lim_{m \to \infty} \sum_{i=1}^{m} (l_1(t_i) - l_1(t_{i-1})) x(t_i) &= x(0) + \int_0^{1} x(s) \, dl_1(s) = x_0,
\\
y(0) + \lim_{m \to \infty} \sum_{j=1}^{m} (l_2(t_j) - l_2(t_{j-1})) y(t_j) &= y(0) + \int_0^{1} y(s) \, dl_2(s) = y_0,
\end{aligned}
\]

that is, the boundary conditions (3). Therefore, we take the limit to the solution (44), and we then get the solution of systems (1) and (3), which can be expressed as follows:
\[ x(t) = \lim_{m \to \infty} \frac{1}{1 + \sum_{j=1}^{m} a_j} \left[ x_0 - \sum_{i=1}^{m} a_i \int_0^{\tau_i} \left( \frac{(t \tau_i - s)^{n-1}}{\Gamma(n)} \right) g_1 \left( s, x(s), y(s), \int_0^s f_1(\theta, x(\theta), y(\theta))d\theta, \int_0^s h_1(\theta, x(\theta), y(\theta))d\theta \right) ds \right] \]

\[ + \int_0^{t-1} \left( t - s \right)^{\beta-1} \frac{1}{\Gamma(\beta)} g_1 \left( s, x(s), y(s), \int_0^s f_1(\theta, x(\theta), y(\theta))d\theta, \int_0^s h_1(\theta, x(\theta), y(\theta))d\theta \right) ds \]

\[ x_m(t) = \frac{1}{1 + \sum_{j=1}^{m} a_j} \left[ x_0 - \sum_{i=1}^{m} a_i \int_0^{\tau_i} \left( \frac{(t \tau_i - s)^{n-1}}{\Gamma(n)} \right) g_1 \left( s, x_m(s), y_m(s), \int_0^s f_1(\theta, x_m(\theta), y_m(\theta))d\theta, \int_0^s h_1(\theta, x_m(\theta), y_m(\theta))d\theta \right) ds \right] \]

\[ + \int_0^{t-1} \left( t - s \right)^{\beta-1} \frac{1}{\Gamma(\beta)} g_1 \left( s, x_m(s), y_m(s), \int_0^s f_1(\theta, x_m(\theta), y_m(\theta))d\theta, \int_0^s h_1(\theta, x_m(\theta), y_m(\theta))d\theta \right) ds \]

\[ y_m(t) = \frac{1}{1 + \sum_{j=1}^{m} b_j} \left[ y_0 - \sum_{i=1}^{m} b_j \int_0^{\tau_i} \left( \frac{(t \tau_i - s)^{\beta-1}}{\Gamma(\beta)} \right) g_2 \left( s, x_m(s), y_m(s), \int_0^s f_2(\theta, x_m(\theta), y_m(\theta))d\theta, \int_0^s h_2(\theta, x_m(\theta), y_m(\theta))d\theta \right) ds \right] \]

\[ + \int_0^{t-1} \left( t - s \right)^{\beta-1} \frac{1}{\Gamma(\beta)} g_2 \left( s, x_m(s), y_m(s), \int_0^s f_2(\theta, x_m(\theta), y_m(\theta))d\theta, \int_0^s h_2(\theta, x_m(\theta), y_m(\theta))d\theta \right) ds \]

\[ y(t) = \lim_{m \to \infty} \frac{1}{1 + \sum_{j=1}^{m} b_j} \left[ y_0 - \sum_{i=1}^{m} b_j \int_0^{\tau_i} \left( \frac{(t \tau_i - s)^{\beta-1}}{\Gamma(\beta)} \right) g_2 \left( s, x(s), y(s), \int_0^s f_2(\theta, x(\theta), y(\theta))d\theta, \int_0^s h_2(\theta, x(\theta), y(\theta))d\theta \right) ds \right] \]

\[ + \int_0^{t-1} \left( t - s \right)^{\beta-1} \frac{1}{\Gamma(\beta)} g_2 \left( s, x(s), y(s), \int_0^s f_2(\theta, x(\theta), y(\theta))d\theta, \int_0^s h_2(\theta, x(\theta), y(\theta))d\theta \right) ds \]

Theorem 3. Assume that the assumptions \((H_1)-(H_3)\) hold. Then, systems (1) and (4) have at least one solution.

Proof. Based on Theorem 1, we know, for any \(m \in N\), the solution of systems (1) and (2) can be expressed as follows:
Let \( m \to \infty \) in (48), we then have

\[
\lim_{m \to \infty} x_m(t) = \lim_{m \to \infty} \frac{1}{1 + \sum_{i=1}^{m} a_i} \left[ x_0 - \sum_{i=1}^{m} a_i \int_{0}^{\tau_i} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds \right]
\]

\[
+ \lim_{m \to \infty} \int_{0}^{\tau} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds,
\]

\[
\lim_{m \to \infty} y_m(t) = \lim_{m \to \infty} \frac{1}{1 + \sum_{j=1}^{m} b_j} \left[ y_0 - \sum_{j=1}^{m} b_j \int_{0}^{\tau_j} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds \right]
\]

\[
+ \lim_{m \to \infty} \int_{0}^{\tau} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds.
\]

(49)

Based on the convergence of \( \sum_{i=1}^{\infty} a_i \) and \( \sum_{j=1}^{\infty} b_j \), it is easy to check that

\[
\sum_{i=1}^{\infty} a_i \int_{0}^{\tau_i} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds
\]

\[
\sum_{j=1}^{\infty} b_j \int_{0}^{\tau_j} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds,
\]

are convergent. Thus, by using the Lebesgue-dominated convergence theorem, we know that the existence of the limits for the right-hand side in (44). On the other hand, we notice that the boundary conditions (2) can be transformed to the boundary conditions (4) as \( m \to \infty \), we obtain that the limits of (44) is the solution of systems (1) and (4), which can be expressed as follows:

\[
\begin{align*}
\begin{cases}
x(t) = \frac{1}{1 + \sum_{i=1}^{\infty} a_i} \left[ x_0 - \sum_{i=1}^{\infty} a_i \int_{0}^{\tau_i} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds \right] \\
+ \left( \int_{0}^{\tau} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds, \right),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
y(t) = \frac{1}{1 + \sum_{j=1}^{\infty} b_j} \left[ y_0 - \sum_{j=1}^{\infty} b_j \int_{0}^{\tau_j} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds \right] \\
+ \left( \int_{0}^{\tau} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds, \right).
\end{cases}
\end{align*}
\]

(51)

**Theorem 4.** Assume that the assumptions \((H'_1)-(H'_4)\) and \((H_4)\) hold. Then, systems (1) and (2) have at least one solution.

**Proof.** It follows from \((H_4)\) that \( Z_1, Z_2 < 1 \). Thus, there exists \( R'_1 > 0 \) large enough such that

\[
D_1 + Z_1R'_1 \leq R'_1, \quad D_2 + Z_2R'_1 \leq R'_1.
\]

Similarly as (35), we obtain that, for any \( (x, y) \in Q_{R'_1} \),

\[
\sum_{i=1}^{\infty} a_i \int_{0}^{\tau_i} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} g_1\left(s, x(s), y(s), \int_{0}^{s} f_1(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_1(\theta, x(\theta), y(\theta))d\theta\right)ds
\]

\[
\sum_{j=1}^{\infty} b_j \int_{0}^{\tau_j} (t-s)^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta+1)} g_2\left(s, x(s), y(s), \int_{0}^{s} f_2(\theta, x(\theta), y(\theta))d\theta, \int_{0}^{s} h_2(\theta, x(\theta), y(\theta))d\theta\right)ds
\]

(52)
Thus, we obtain that \( \|T(x, y)\|_Y = \max\{\|T_1(x, y)\|_X, \|T_2(x, y)\|_Y\} \leq R_1' \). By the same way as Theorem 1, we get that \( T \) is a completely continuous operator. Then, by the Schauder fixed point theorem, we deduce that there exists at least one solution of systems (1) and (2). \( \square \)

**Theorem 5.** Assume that the assumptions \((H_1), (H_2), (H_3)\) hold. Then, systems (1) and (3) have at least one solution.

**Proof.** First, we make an interval partition: \( 0 = t_0 < t_1 < \ldots < t_m = 1 \) such that \( \tau_j \in (t_{j-1}, t_j), \tau_j \in (t_{j-1}, t_j) (i, j = 1, 2, \ldots, m) \). Let \( a_j = l_1(t_j) - l_1(t_{j-1}), b_j = l_2(t_j) - l_2(t_{j-1}) \). In this case, we obtain as follows:

\[
\sum_{i=1}^{m} a_i = l_1(1) - l_1(0), \sum_{j=1}^{m} b_j = l_2(1) - l_2(0). \tag{54}
\]

So, the assumption \((H_4)\) implies that \((H_4)\) is true. Thus, based on Theorem 4, we know that systems (1) and (2) have a solution in \( Q_R' \). Accordingly, by Theorem 2, we take the limit to the solution of systems (1) and (2), the solution of systems (1) and (3) is then obtained. \( \square \)

**Theorem 6.** Assume that the assumptions \((H_1'), (H_2), (H_3)\), \((H_6)\) hold. Then, systems (1) and (4) have at least one solution.

**Proof.** The proof is similar to that of Theorem 3. So, we omit details. \( \square \)

**4. Examples**

**Example 1.** In this section, we consider the following system:

\[
\begin{align*}
\begin{cases}
\frac{\varepsilon^7 D^{3/4} x(t)}{t^{(1/5)}} &= t^{-(1/5)} + t x(t) + \frac{y(t)}{t^{(1/2)}(x(t) + 1)} + 2t \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \tag{25/5}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\frac{\varepsilon^7 D^{3/4} y(t)}{t^{(1/5)}} &= t^{-(1/2)} x^{(1/5)} \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \tag{1/7}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\frac{\varepsilon^7 D^{3/4} y(t)}{t^{(1/5)}} &= t^{-(34)} + 4t^{-1/2} x^{(1/5)}(t) y^{(1/7)}(t) \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \left( \int_0^t \left( 2s^{-1/4} x^{(1/3)}(s) + s^{-1/2} y(s) \right) ds \right) \tag{1/3}
\end{cases}
\end{align*}
\]

(55)
with the following boundary conditions:
\[
\begin{align*}
x(0) + \frac{1}{2} x(1) + \frac{1}{4} x(2) + \frac{1}{6} x(3) &= 1, \\
y(0) + \frac{1}{4} y(1) + \frac{1}{2} y(2) + \frac{1}{4} y(3) &= 4.
\end{align*}
\] (56)

Let \( \alpha = (2/3) \), \( \beta = (3/4) \),
\[
g_1(t, x_1, x_2, x_3, x_4) = t^{-\alpha/3} + t^{\alpha/3} x_1^{(1/3)} + 2t^2 (x_3 x_4)^{(2/5)},
\]
\[
g_2(t, x_1, x_2, x_3, x_4) = t^{-\beta/3} + 4t^{-\beta/3} x_1^{(1/3)} x_2^{(1/3)} x_3^{(1/3)} + t x_4 + t^{\alpha/3} (x_2 + 1).
\]

Then, systems (55) and (56) can be transformed to systems (1) and (2). In addition, we obtain that
\[
\begin{align*}
[g_1(t, x_1, x_2, x_3, x_4)] &\leq t^{-\alpha/3} + t^{\alpha/3} x_1^{(1/3)} + t^{\alpha/3} (x_3 x_4)^{(2/5)}, \\
[g_2(t, x_1, x_2, x_3, x_4)] &\leq t^{-\beta/3} + 4t^{-\beta/3} x_1^{(1/3)} x_2^{(1/3)} x_3^{(1/3)} + t x_4 + t^{\alpha/3} (x_2 + 1).
\end{align*}
\] (57)

Thus, we get that the conditions (H_i)- (H_j) are satisfied. Therefore, by Theorem 1, we infer that systems (55)-(56) have at least one solution.

Example 2. We discuss the following system:
\[
\begin{align*}
D^{\gamma}_{x} x(t) &= t^2 + 2t x(t)^{(1/3)} + t^{\gamma/2} y(t)^{(1/3)} \\
&+ t^{-\alpha/3} \left( \int_0^t s^{-\alpha/2} (x(s) y(s)^{(1/3)}) ds \right)^{(1/3)}, \\
D^{\gamma}_{y} y(t) &= t^{-\beta/3} + \frac{x^4(t)}{1 + [x(t)]^{(1/3)} [y(t)]^{(1/3)}}, \\
&+ t^{\alpha/3} \left( \int_0^t s + x(s)^{(1/3)} + y(s)^{(1/3)} ds \right)^{(1/3)}, \\
&+ t^{-\beta/3} \left( \int_0^t \ln(1 + [x(s)]^{(1/3)} [y(s)]^{(1/3)}) ds \right)^{(1/2)},
\end{align*}
\] (58)

with the following boundary condition:
\[
\begin{align*}
x(0) + \int_0^1 x(s) ds &\leq 1, \\
y(0) + \int_0^1 y(s) ds &\leq 4,
\end{align*}
\] (59)
where

\[
\begin{aligned}
l_1(t) &= \begin{cases} 
1, & t \in [0, \frac{1}{2}), \\
2, & t \in [\frac{1}{2}, 1], \\
1, & t \in [0, \frac{1}{2}), \\
\frac{1}{2}, & t \in [\frac{1}{2}, 1], \\
3, & t \in [\frac{1}{2}, 1]. 
\end{cases}
\end{aligned}
\]

\[
l_2(t) = \begin{cases} 
1, & t \in [0, \frac{1}{2}), \\
2, & t \in [\frac{1}{2}, 1]. 
\end{cases}
\]

Let \( \alpha = (2/3), \beta = (2/3), \)

\[
g_1(t, x, x_2, x_3, x_4) = t^2 + 2tx_1^{(1/3)} + t^{(1/2)}x_2^{(1/5)} + t^{(1/3)}x_3^{(1/3)} + t^{(1/2)}x_4^{(1/5)},
\]

\[
g_2(t, x, x_2, x_3, x_4) = t^{-(3/4)} + \frac{x_4^2}{1 + |x_1|^{(1/4)}} |x_2|^{(1/4)} + t^{(1/3)}x_3^{(1/4)} + t^{(1/3)}x_4^{(1/2)},
\]

\[
f_1(t, x_1, x_2) = t^{(1/2)} (x_1x_2)^{(1/3)},
\]

\[
f_2(t, x_1, x_2) = t + x_1^{(1/3)} + \frac{x_2}{2 + t} x_2,
\]

\[
h_1(t, x_1, x_2) = \frac{(x_1x_2)^{(4/5)}}{4 + t + \sqrt{|x_1| + |x_2|}},
\]

\[
h_2(t, x_1, x_2) = \ln \left( 1 + |x_1|^{(1/5)} |x_2|^{(1/5)} \right).
\]

Then, systems (60) and (61) can be transformed to systems (1) and (3). In addition, we obtain that

\[
\begin{aligned}
\|g_1(t, x_1, x_2, x_3, x_4)\| &\leq t^2 + 2|t|x_1^{(1/3)} + t^{(1/2)}|x_2|^{(1/5)} + t^{(1/3)}|x_3|^{(1/3)} + t^{(1/2)}|x_4|^{(1/5)}, \\
\|g_2(t, x_1, x_2, x_3, x_4)\| &\leq t^{-(3/4)} + \frac{|x_4^2|}{1 + |x_1|^{(1/4)}} |x_2|^{(1/4)} + t^{(1/3)}|x_3|^{(1/4)} + t^{(1/3)}|x_4|^{(1/2)}, \\
\|f_1(t, x_1, x_2)\| &\leq t^{(1/2)} |x_1x_2|^{(1/3)} + t^{(1/2)} |x_2|^{(1/2)} + t^{(1/3)} |x_3|^{(1/4)} + t^{(1/3)} |x_4|^{(1/2)}, \\
\|f_2(t, x_1, x_2)\| &\leq t + (1 + t)|x_1|^{(1/3)} + (1 + t)|x_2|^{(1/2)}, \\
\|h_1(t, x_1, x_2)\| &\leq t^{-(1/3)} |x_1|^{(1/3)} + \frac{1}{t} |x_2|^{(1/5)}, \\
\|h_2(t, x_1, x_2)\| &\leq t + t^{-(1/3)} |x_1|^{(1/3)} + \frac{1}{t} |x_2|^{(1/3)}. 
\end{aligned}
\]

Let \((1/p_1) = (1/4), (1/p_2) = (1/2), a_{11}(t) = t^2, b_{11}(t) = 2t, b_{12}(t) = t^{(1/2)}, b_{13}(t) = t^{(1/3)}, b_{14}(t) = t^{(1/13)}, a_{21}(t) = t^{(2/3)}, b_{21}(t) = t, b_{22}(t) = t, b_{23}(t) = t^{(1/3)}, b_{24}(t) = t^{(1/3)}, a_{12}(t) = t, c_{14}(t) = t^{(1/2)}, c_{12}(t) = t^{(1/2)}, a_{22}(t) = t, c_{21}(t) = t^{(1/3)}, c_{22}(t) = t, a_{13}(t) = 2t, d_{11}(t) = 1 + td_{12}(t) = 1 + t, a_{23}(t) = t, d_{21}(t) = t^{(1/2)}, d_{22}(t) = t^{(1/2)}).

It then holds that \((1/p_1) = (0, \alpha) (\alpha = (2/3)), (1/p_2) = (0, \beta) (\beta = (2/3)), a_{11}, b_{11}, b_{12}, b_{13}, b_{14}, a_{12}, a_{13}, d_{11}, d_{12} \in L^{p_1}([0, 1], \mathbb{R}^{d_1}), a_{21}, b_{21}, b_{22}, b_{23}, b_{24}, a_{22}, c_{21}, c_{22}, a_{23}, d_{21}, d_{22} \in L^{p_2}([0, 1], \mathbb{R}^{d_2}), \)

\[
\begin{aligned}
|g_1(t, x_1, x_2, x_3, x_4)| &\leq a_{11}(t) + b_{11}(t)|x_1|^{\alpha_1} + b_{12}(t)|x_2|^{\alpha_2}, \\
|g_2(t, x_1, x_2, x_3, x_4)| &\leq a_{21}(t) + b_{21}(t)|x_1|^{\beta_1} + b_{22}(t)|x_2|^{\beta_2} + b_{23}(t)|x_3|^{\beta_3} + b_{24}(t)|x_4|^{\beta_4}, \\
|f_1(t, x_1, x_2)| &\leq a_{12}(t) + c_{11}(t)|x_1| + c_{12}(t)|x_2|, \\
|f_2(t, x_1, x_2)| &\leq a_{22}(t) + c_{21}(t)|x_1| + c_{22}(t)|x_2|, \\
h_{11}(t, x_1, x_2) &\leq a_{13}(t) + d_{11}(t)|x_1| + d_{12}(t)|x_2|, \\
h_{21}(t, x_1, x_2) &\leq a_{23}(t) + d_{21}(t)|x_1| + d_{22}(t)|x_2|.
\end{aligned}
\]

Thus, we get that the conditions \((H_1)-(H_d)\) are satisfied. Therefore, by Theorem 2, we infer that systems (60)-(61) have at least one solution.

**Example 3.** We discuss the following system:

\[
\begin{aligned}
&cD^{(3/4)} x(t) = t^{-(2/3)} + 2t^{-(1/2)} x(t)^{(1/4)} y(t)^{(1/4)} \\
&+ t \left( \int_0^t s^{-(1/4)} x(s)^{(1/3)} + \frac{y(s)^2}{2 + y(s)^{(1/3)}} ds \right)^{(1/4)} \\
&+ t^{(1/2)} \left( \int_0^t s^{-(1/2)} x(s)^{(1/3)} + y(s)^{(1/3)} ds \right)^{(1/3)} + ty(t)^{(1/4)} \\
&+ t^{(1/2)} \left( \int_0^t s^{-(1/3)} x(s)^{(1/3)} + y(s)^{(1/3)} ds \right)^{(1/5)} \\
&+ t^{(1/2)} \left( \int_0^t s^{-(1/4)} y(s)^{(1/4)} ds \right)^{(1/7)},
\end{aligned}
\]

with the following boundary condition:

\[
\begin{aligned}
\left( x(0) + \sum_{i=1}^{\infty} \frac{1}{i+1} x \left( \frac{1}{3} \right) \right) = 1, \\
y(0) + \sum_{i=1}^{\infty} \frac{1}{i+1} x \left( \frac{1}{3} \right) = 5.
\end{aligned}
\]

Let \( \alpha = (3/4), \beta = (2/5), \).
Thus, we infer that the conditions \((H_1)-(H_3)\) are satisfied. Therefore, by Theorem 3, we infer that systems (66)-(67) have at least one solution.

5. Conclusions

In this paper, by using the Schauder fixed point theorem, the existence of solutions for the coupled system (1) with multipoint boundary conditions (2) is proved, in which the nonlinear term satisfies two types of the Carathéodory condition. Under these two conditions, the power exponent of the control function of the nonlinear term includes sublinear and linear cases, which makes the problem we study more general. Furthermore, we consider the Riemann–Stieltjes integral and infinite-point boundary conditions around it. Based on the results of systems (1)-(2), the existence of solutions for the last two systems is obtained directly. Finally, three examples are given to illustrate our theoretical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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