

## Research Article

# Classification of Unit Groups of Five Radical Zero Completely Primary Finite Rings Whose First and Second Galois Ring Module Generators Are of the Order $p^k$ , $k = 2, 3, 4$

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Let  $R_0 = GR(p^{kr}, p^k)$  be a Galois maximal subring of  $R$  so that  $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ , where  $U, V, W$ , and  $Y$  are  $R_0/pR_0$  spaces considered as  $R_0$ -modules, generated by the sets  $\{u_1, \dots, u_e\}$ ,  $\{v_1, \dots, v_f\}$ ,  $\{w_1, \dots, w_g\}$ , and  $\{y_1, \dots, y_h\}$ , respectively. Then,  $R$  is a completely primary finite ring with a Jacobson radical  $Z(R)$  such that  $(Z(R))^5 = (0)$  and  $(Z(R))^4 \neq (0)$ . The residue field  $R/Z(R)$  is a finite field  $GF(p^r)$  for some prime  $p$  and positive integer  $r$ . The characteristic of  $R$  is  $p^k$ , where  $k$  is an integer such that  $1 \leq k \leq 5$ . In this paper, we study the structures of the unit groups of a commutative completely primary finite ring  $R$  with  $p^\psi u_i = 0$ ,  $\psi = 2, 3, 4$ ;  $p^\zeta v_j = 0$ ,  $\zeta = 2, 3$ ;  $pw_k = 0$ , and  $py_l = 0$ ;  $1 \leq i \leq e$ ,  $1 \leq j \leq f$ ,  $1 \leq k \leq g$ , and  $1 \leq l \leq h$ .

## 1. Introduction

Completely primary finite rings with identity  $1 \neq 0$  have been an active area of research in the recent years probably because they play a crucial role in the classification of finite rings. The unit groups of completely primary finite rings with maximal ideal  $Z(R)$  such that  $(Z(R))^3 = (0)$  with  $(Z(R))^2 \neq (0)$  have been classified by Chikunji [1–3]. Oduor and Onyango [4] constructed a class of completely primary finite rings in which  $(Z(R))^4 = (0)$  with  $(Z(R))^3 \neq (0)$  and determined the structures of their group of units for all the characteristics of the ring  $R$ . Recently, Were et al. [5] gave a construction of a completely primary finite ring satisfying the conditions  $(Z(R))^5 = (0)$ ;  $(Z(R))^4 \neq (0)$  and further determined the unit groups restricted to some conditions. This construction involved idealization of  $R_0$ -modules, whose choice was based on Wilson [6]. Were et al. [7] classified the group of units of five radical zero completely primary finite rings with variant orders of second Galois ring module generators. The structure of unit groups for an abelian group have been determined by various techniques.

Ayoub [8] studied groups which possess a particular type of series called  $j$ -diagram and determined the structure of the subgroups. In [9], Ayoub obtained various results based on the ideas regarding  $j$ -diagrams for the abelian  $p$ -groups. The  $j$ -diagram technique has been used by Alabiad and Alkhamees [10] to determine the structure of unit groups of a commutative chain ring where  $(p-1)|k$  by fixing Ayoub's approach and introduced a system of generators for the unit groups as well as enumerating the generators. In what follows,  $R$  shall denote a finite completely primary ring,  $Z(R)$  its maximal set of zero divisors (including zero). We shall also denote the coefficient Galois subring  $GR(p^{kr}, p^k)$  of characteristic  $p^k$  and order  $p^{kr}$  of the ring  $R$  by  $R_0$ . For the previous related work, we refer to [2, 3, 5, 10, 11–13].

The remaining part of this paper is organized as follows. In Section 2, we give the construction of five radical zero completely primary finite rings whose structure of unit groups are considered in this paper. In Section 3, we investigate and determine the structure of the group of units  $R^*$  of  $R$  for all characteristics  $p^k$ ,  $1 \leq k \leq 5$ , under the restrictions  $p^\psi u_i = 0$ ,  $p^\zeta v_j = 0$ ,  $pw_k = 0$ ,  $py_l = 0$ , where  $\psi = 2, 3, 4$ ;  $\zeta = 2, 3$ ;

$1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, \text{ and } 1 \leq l \leq h$ . Finally, in Section 4, we give the conclusion of this research and some areas which future researchers may dwell on. This is a continuation of the author's earlier research on the classification of unit groups of five radical zero commutative completely primary finite rings.

### 2. Preliminaries

The following result due to Wilson [6] formed the basis for the choice of the ring in this work as an  $R_0$ -module.

Let  $R$  be a completely primary finite ring of characteristic  $p^k$  with radical  $Z(R)$  such that  $R/Z(R) \cong GF(p^r)$ . Then, there exists an independent generating set  $b_1, b_2, \dots, b_m$  of  $R$  as a left  $GR(p^{kr}, p^r)$ -module such that

- (1)  $b_1 = 1; b_2, \dots, b_m \in Z(R)$
- (2)  $GR(p^{kr}, p^r)b_i$  is  $(GR(p^{kr}, p^r), GR(p^{kr}, p^r))$ -submodule of  $R$

This paper considers a specific case of the general construction of five radical zero commutative completely primary finite rings where the automorphism is the identity in  $R_0$  and the ring  $R$  is commutative.

**2.1. Construction.** Let  $R_0 = GR(p^{kr}, p^k)$  be a Galois ring of order  $p^{kr}$  and characteristic  $p^k$  where  $p$  is a prime integer,  $1 \leq k \leq 5$  and  $r \in \mathbb{Z}^+$ . Suppose  $U, V, W,$  and  $Y$  are  $R_0/pR_0$ -spaces considered  $R_0$  modules generated by  $e, f, g,$  and  $h$  elements, respectively, such that the corresponding generating sets are  $\{u_1, \dots, u_e\}, \{v_1, \dots, v_f\}, \{w_1, \dots, w_g\},$  and  $\{y_1, \dots, y_h\}$ , so that  $R = R_0 \oplus U \oplus V \oplus W \oplus Y$  is an additive abelian group. Then, on the additive group, we define multiplication by the following relations:

- (i) If  $k = 1$ , then  $u_i u_{i'} = u_i u_i = v_j, u_i v_j = v_j u_i = w_k, u_i w_k = w_k u_i = y_l, u_i y_l = y_l u_i = 0, v_j v_{j'} = v_j v_j = y_l, v_j w_k = w_k v_j = 0, v_j y_l = y_l v_j = 0, w_k w_{k'} = w_k w_k = 0, w_k y_l = y_l w_k = 0, y_l y_{l'} = y_l y_l = 0$
- (ii) If  $k = 2$ , then  $u_i u_{i'} = u_i u_i = pr_0 + pu_i + v_j, u_i v_j = v_j u_i = pu_i + w_k, u_i w_k = w_k u_i = pu_i + y_l, u_i y_l = y_l u_i = pu_i, v_j v_{j'} = v_j v_j = y_l, v_j w_k = w_k v_j = 0, v_j y_l = y_l v_j = 0, w_k w_{k'} = w_k w_k = 0, w_k y_l = y_l w_k = 0, y_l y_{l'} = y_l y_l = 0$
- (iii) If  $3 \leq k \leq 5$ , then  $u_i u_{i'} = u_i u_i = p^2 r_0 + pu_i + v_j, u_i v_j = v_j u_i = p^2 r_0 + pu_i + pv_j + w_k, u_i w_k = w_k u_i = p^2 r_0 + pu_i + pw_k + y_l, u_i y_l = y_l u_i = p^2 r_0 + pu_i, v_j v_{j'} = v_j v_j = p^2 r_0 + pv_j + y_l, v_j w_k = w_k v_j = p^2 r_0 + pv_j + pw_k, v_j y_l = y_l v_j = p^2 r_0 + pv_j, w_k w_{k'} = w_k w_k = p^2 r_0 + pw_k, w_k y_l = y_l w_k = p^2 r_0 + pw_k, y_l y_{l'} = y_l y_l = p^2 r_0$

Further  $u_i u_{i'} u_{i''} u_{i'''} u_{i''''} = 0, u_i r_0 = r_0 u_i, v_j r_0 = r_0 v_j, w_k r_0 = r_0 w_k, y_l r_0 = r_0 y_l$ , where  $r_0 \in R_0$  and  $1 \leq i, i' \leq e, 1 \leq j, j' \leq f, 1 \leq k, k' \leq g, 1 \leq l, l' \leq h$ . From the given multiplication in  $R$ , we see that if  $r_0 + \sum_{i=1}^e r_i u_i + \sum_{j=1}^f s_j v_j + \sum_{k=1}^g t_k w_k +$

$\sum_{l=1}^h z_l y_l$  and  $r'_0 + \sum_{i=1}^e r'_i u_i + \sum_{j=1}^f s'_j v_j + \sum_{k=1}^g t'_k w_k + \sum_{l=1}^h z'_l y_l$  are any two elements of  $R$ , then

$$\begin{aligned} & \left( r_0 + \sum_{i=1}^e r_i u_i + \sum_{j=1}^f s_j v_j + \sum_{k=1}^g t_k w_k + \sum_{l=1}^h z_l y_l \right) \\ & \cdot \left( r'_0 + \sum_{i=1}^e r'_i u_i + \sum_{j=1}^f s'_j v_j + \sum_{k=1}^g t'_k w_k + \sum_{l=1}^h z'_l y_l \right) \\ & = r_0 r'_0 + p^a \sum_{i,m=1}^e (r_i r'_m + pR_0) + \sum_{i=1}^e [r_0 r'_i + r_i r'_0 + pR_0] u_i \\ & + \sum_{j=1}^f \left[ (r_0 + pR_0) s'_j + s_j (r'_0 + pR_0) + \sum_{\nu, \mu=1}^e (r_\nu r'_\mu + pR_0) \right] v_j \\ & + \sum_{k=1}^g \left[ (r_0 + pR_0) t'_k + t_k (r'_0 + pR_0) + \sum_{i,j} (r_i + pR_0) s'_j \right. \\ & \left. + s_j (r'_i + pR_0) \right] w_k + \sum_{l=1}^h \left[ (r_0 + pR_0) z'_l + z_l (r'_0 + pR_0) \right. \\ & \left. + \sum_{i,k} (r_i + pR_0) t'_k + t_k (r'_i + pR_0) + \sum_{\kappa, \tau=1}^f (s_\kappa s'_\tau + pR_0) \right] y_l, \end{aligned} \tag{1}$$

where  $a = 1, 2, 3,$  or  $4$  depending on whether  $\text{Char } R_0 = p^2, p^3, p^4,$  or  $p^5$ . It can be verified that this multiplication turns  $R$  into a commutative ring with identity 1.

Notice that if  $R_0 = GR(p^r, p)$  where  $\text{Char } R = p$ , then the above multiplication reduces to

$$\begin{aligned} & \left( r_0 + \sum_{i=1}^e r_i u_i + \sum_{j=1}^f s_j v_j + \sum_{k=1}^g t_k w_k + \sum_{l=1}^h z_l y_l \right) \\ & \cdot \left( r'_0 + \sum_{i=1}^e r'_i u_i + \sum_{j=1}^f s'_j v_j + \sum_{k=1}^g t'_k w_k + \sum_{l=1}^h z'_l y_l \right) \\ & = r_0 r'_0 + \sum_{i=1}^e [r_0 r'_i + r_i r'_0] u_i + \sum_{j=1}^f \left[ (r_0) s'_j + s_j (r'_0) + \sum_{\nu, \mu=1}^e (r_\nu r'_\mu) \right] v_j \\ & + \sum_{k=1}^g \left[ (r_0) t'_k + t_k (r'_0) + \sum_{i,j} (r_i) s'_j + s_j (r'_i) \right] w_k \\ & + \sum_{l=1}^h \left[ (r_0) z'_l + z_l (r'_0) + \sum_{i,k} (r_i) t'_k + t_k (r'_i) + \sum_{\kappa, \tau=1}^f (s_\kappa s'_\tau) \right] y_l. \end{aligned} \tag{2}$$

Since the ring

$$\begin{aligned} R &= R_0 \oplus \sum_{i=1}^e R_0 u_i \oplus \sum_{j=1}^f R_0 v_j \oplus \sum_{k=1}^g R_0 w_k \\ &\oplus \sum_{l=1}^h R_0 y_l \text{ (as } R_0\text{-module),} \end{aligned} \tag{3}$$

we have its maximal ideal as

$$Z(R) = pR_0 \oplus \sum_{i=1}^e R_0 u_i \oplus \sum_{j=1}^f R_0 v_j \oplus \sum_{k=1}^g R_0 w_k \oplus \sum_{l=1}^h R_0 y_l, \quad (4)$$

and the finite abelian  $p$ -group given by

$$1 + Z(R) = 1 + pR_0 \oplus \sum_{i=1}^e R_0 u_i \oplus \sum_{j=1}^f R_0 v_j \oplus \sum_{k=1}^g R_0 w_k \oplus \sum_{l=1}^h R_0 y_l. \quad (5)$$

The unit groups of  $R$  is given by  $R^*$ , and it can be verified that

$$R^* = (R^*/1 + Z(R)) \times (1 + Z(R)) = \langle b \rangle \times (1 + Z(R)), \quad (6)$$

where  $\langle b \rangle$  is a cyclic group of order  $p^r - 1$ . The structure problem of  $R^*$  is reduced to that of  $1 + Z(R)$ . Using the ideas of Raghavendran [11] and Chikunji [1], we classify the unit groups of the rings constructed in this section.

**Proposition 1.** *Let  $R$  be a ring constructed above and of characteristic  $p$  with  $pu_i = 0, pv_j = 0, pw_k = 0,$  and  $py_l = 0$ . Then, its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_2^r)^g, & \text{if } p = 2, \\ \mathbb{Z}_{3^{r-1}} \times (\mathbb{Z}_9^r)^e \times (\mathbb{Z}_3^r)^f \times (\mathbb{Z}_3^r)^h, & \text{if } p = 3, \\ \mathbb{Z}_{p^{r-1}} \times (\mathbb{Z}_p^r)^e \times (\mathbb{Z}_p^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & \text{if } p > 3. \end{cases} \quad (7)$$

*Proof.* See Proposition 3.1 in [5]. □

### 3. Main Results

We now investigate the structures of the unit groups of finite completely primary rings with a maximal ideals  $Z(R)$  such that  $(Z(R))^5 = (0), (Z(R))^4 \neq (0)$  and with the characteristics  $p^k, 1 \leq k \leq 5$ , under the restrictions  $p^\psi u_i = 0, p^\zeta v_j = 0, pw_k = 0, py_l = 0$ , where  $\psi = 2, 3, 4; \zeta = 2, 3; 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g,$  and  $1 \leq l \leq h$ .

**Proposition 2.** *Let  $R$  be a ring constructed above and of characteristic  $p^2$  with  $p^2 u_i = 0, p^2 v_j = 0, pw_k = 0,$  and  $py_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (8)$$

*Proof.* Since  $R$  is commutative,  $R^* = \langle b \rangle \cdot (1 + Z(R)) \cong \langle b \rangle \times (1 + Z(R))$ , a direct product of the  $p$ -group  $1 + Z(R)$  by the cyclic group  $\langle b \rangle$ . Then it suffices to determine the structure of the subgroup  $1 + Z(R)$  of the unit group  $R^*$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\xi_1, \dots, \xi_r$

$\in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . We consider the two cases separately. □

*Case 1.  $p$  is even.* For each  $t = 1, \dots, r, (1 + 2\xi_t)^2 = 1, (1 + \xi_t u_i)^8 = 1, (1 + \xi_t v_j)^4 = 1, (1 + \xi_t w_k)^2 = 1$ . For nonnegative integers  $\alpha_t, \delta_t, \varphi_t,$  and  $\lambda_t$  with  $\alpha_t \leq 2, \delta_t \leq 8, \varphi_t \leq 4,$  and  $\lambda_t \leq 2$ , it is clear that

$$\prod_{t=1}^r \{(1 + 2\xi_t)^{\alpha_t}\} \cdot \prod_{i=1}^e \prod_{t=1}^r \{(1 + \xi_t u_i)^{\delta_t}\} \cdot \prod_{j=1}^f \prod_{t=1}^r \{(1 + \xi_t v_j)^{\varphi_t}\} \cdot \prod_{k=1}^g \prod_{t=1}^r \{(1 + \xi_t w_k)^{\lambda_t}\} = \{1\} \quad (9)$$

will imply that  $\alpha_t = 2, \delta_t = 8, \varphi_t = 4,$  and  $\lambda_t = 2$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + 2\xi_t)^\alpha : \alpha = 1, 2; \forall t = 1, \dots, r\}; B_{t,i} = \{(1 + \xi_t u_i)^\delta : \delta = 1, \dots, 8; \forall t = 1, \dots, r\}, (i = 1, \dots, e); C_{t,j} = \{(1 + \xi_t v_j)^\varphi : \varphi = 1, \dots, 4; \forall t = 1, \dots, r\}, (j = 1, \dots, f);$  and  $D_{t,k} = \{(1 + \xi_t w_k)^\lambda : \lambda = 1, 2; \forall t = 1, \dots, r\}, (k = 1, \dots, g);$  we see that  $A_t, B_{t,i}, C_{t,j},$  and  $D_{t,k}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + 2\xi_t \rangle, \langle 1 + \xi_t u_i \rangle, \langle 1 + \xi_t v_j \rangle,$  and  $\langle 1 + \xi_t w_k \rangle$  is trivial and that

$$\left| \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \right| \cdot \left| \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \right| \cdot \left| \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \right| \cdot \left| \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \quad (10)$$

coincides with  $|1 + Z(R)|$ , it follows that

$$1 + Z(R) = \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \times \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \cong \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g. \quad (11)$$

*Case 2.  $p$  is odd.* For each  $t = 1, \dots, r, (1 + p\xi_t)^p = 1, (1 + \xi_t u_i)^{p^2} = 1, (1 + \xi_t v_j)^{p^2} = 1, (1 + \xi_t w_k)^p = 1,$  and  $(1 + \xi_t y_l)^p = 1$ . For nonnegative integers  $\beta_t, \alpha_t, \delta_t, \varphi_t,$  and  $\lambda_t$  with  $\beta_t \leq p, \alpha_t \leq p^2, \delta_t \leq p^2, \varphi_t \leq p,$  and  $\lambda_t \leq p$ , it is clear that

$$\prod_{t=1}^r \left\{ (1 + p\xi_t)^{\beta_t} \right\} \cdot \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + \xi_t u_i)^{\alpha_i} \right\} \cdot \prod_{j=1}^f \prod_{t=1}^r \left\{ (1 + \xi_t v_j)^{\delta_t} \right\} \cdot \prod_{k=1}^g \prod_{t=1}^r \left\{ (1 + \xi_t w_k)^{\varphi_t} \right\} \cdot \prod_{l=1}^h \prod_{t=1}^r \left\{ (1 + \xi_t y_l)^{\lambda_t} \right\} = \{1\} \quad (12)$$

will imply that  $\beta_t = p$ ,  $\alpha_t = p^2$ ,  $\delta_t = p^2$ ,  $\varphi_t = p$ , and  $\lambda_t = p$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + p\xi_t)^\beta : \beta = 1, \dots, p; \forall t = 1, \dots, r\}$ ;  $B_{t,i} = \{(1 + \xi_t u_i)^\alpha : \alpha = 1, \dots, p^2; \forall t = 1, \dots, r\}$  ( $i = 1, \dots, e$ );  $C_{t,j} = \{(1 + \xi_t v_j)^\delta : \delta = 1, \dots, p^2; \forall t = 1, \dots, r\}$ , ( $j = 1, \dots, f$ );  $D_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, \dots, p; \forall t = 1, \dots, r\}$ , ( $k = 1, \dots, g$ ); and  $E_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, \dots, p; \forall t = 1, \dots, r\}$ , ( $l = 1, \dots, h$ ), we see that  $A_t$ ,  $B_{t,i}$ ,  $C_{t,j}$ ,  $D_{t,k}$ , and  $E_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + p\xi_t \rangle$ ,  $\langle 1 + \xi_t u_i \rangle$ ,  $\langle 1 + \xi_t v_j \rangle$ ,  $\langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t y_l \rangle$  is trivial and that

$$\left| \prod_{t=1}^r \langle 1 + p\xi_t \rangle \right| \cdot \left| \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \right| \cdot \left| \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \right| \cdot \left| \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \cdot \left| \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \right| \quad (13)$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^r \langle 1 + p\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \\ &\times \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\ &\times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \\ &\cong \mathbb{Z}_p^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h. \end{aligned} \quad (14)$$

**Proposition 3.** Let  $R$  be a ring constructed above and of characteristic  $p^3$  with  $p^2 u_i = 0$ ,  $p^2 v_j = 0$ ,  $p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:

$$R^* \cong \begin{cases} \mathbb{Z}_{2r-1} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (15)$$

*Proof.* Since  $R$  is commutative,  $R^* = \langle b \rangle \cdot (1 + Z(R)) \cong \langle b \rangle \times (1 + Z(R))$ , a direct product of the  $p$ -group  $1 + Z(R)$  by the cyclic group  $\langle b \rangle$ . Then it suffices to determine the structure of the subgroup  $1 + Z(R)$  of the unit group  $R^*$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . We consider the two cases separately.  $\square$

*Case 1.  $p$  is even.* For each  $t = 1, \dots, r$ ,  $(1 + 2\xi_t)^2 = 1$ ,  $(1 + 4\xi_t)^2 = 1$ ,  $(1 + 2\xi_t u_i)^2 = 1$ ,  $(1 + \xi_t u_i + \xi_t v_j)^8 = 1$ ,  $(1 + \xi_t w_k)^2 = 1$ , and  $(1 + \xi_t y_l)^2 = 1$ . For nonnegative integers  $\eta_t, \beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\eta_t \leq 2$ ,  $\beta_t \leq 2$ ,  $\alpha_t \leq 2$ ,  $\delta_t \leq 8$ ,  $\varphi_t \leq 2$ , and  $\lambda_t \leq 2$ , it is clear that

$$\prod_{t=1}^r \left\{ (1 + 2\xi_t)^{\eta_t} \right\} \cdot \prod_{t=1}^r \left\{ (1 + 4\xi_t)^{\beta_t} \right\} \cdot \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + 2\xi_t u_i)^{\alpha_i} \right\} \cdot \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + \xi_t u_i + \xi_t v_j)^{\delta_t} \right\} \cdot \prod_{k=1}^g \prod_{t=1}^r \left\{ (1 + \xi_t w_k)^{\varphi_t} \right\} \cdot \prod_{l=1}^h \prod_{t=1}^r \left\{ (1 + \xi_t y_l)^{\lambda_t} \right\} = \{1\} \quad (16)$$

will imply that  $\eta_t = 2$ ,  $\beta_t = 2$ ,  $\alpha_t = 2$ ,  $\delta_t = 8$ ,  $\varphi_t = 2$ , and  $\lambda_t = 2$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + 2\xi_t)^\eta : \eta = 1, 2; \forall t = 1, \dots, r\}$ ;  $B_t = \{(1 + 4\xi_t)^\beta : \beta = 1, 2; \forall t = 1, \dots, r\}$ ;  $C_{t,i} = \{(1 + 2\xi_t u_i)^\alpha : \alpha = 1, 2; \forall t = 1, \dots, r\}$ , ( $i = 1, \dots, e$ );  $D_{t,i,j} = \{(1 + \xi_t u_i + \xi_t v_j)^\delta : \delta = 1, \dots, 8; \forall t = 1, \dots, r\}$ , ( $i = 1, \dots, e$ ), ( $j = 1, \dots, f$ );  $E_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, 2; \forall t = 1, \dots, r\}$ , ( $k = 1, \dots, g$ );  $F_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, 2; \forall t = 1, \dots, r\}$ , ( $l = 1, \dots, h$ ); we see that  $A_t$ ,  $B_t$ ,  $C_{t,i}$ ,  $D_{t,i,j}$ ,  $E_{t,k}$ , and  $F_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + 2\xi_t \rangle$ ,  $\langle 1 + 4\xi_t \rangle$ ,  $\langle 1 + 2\xi_t u_i \rangle$ ,  $\langle 1 + \xi_t u_i + \xi_t v_j \rangle$ ,  $\langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t y_l \rangle$  is trivial and that

$$\left| \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \right| \cdot \left| \prod_{t=1}^r \langle 1 + 4\xi_t \rangle \right| \cdot \left| \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \right| \cdot \left| \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \right| \cdot \left| \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \cdot \left| \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \right| \quad (17)$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned}
 1 + Z(R) &= \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \times \prod_{t=1}^r \langle 1 + 4\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \\
 &\quad \times \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \\
 &\quad \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \\
 &\cong \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h.
 \end{aligned} \tag{18}$$

Case 2.  $p$  is odd. For each  $t = 1, \dots, r$ ,  $(1 + p\xi_t)^{p^2} = 1$ ,  $(1 + \xi_t u_i)^{p^2} = 1$ ,  $(1 + \xi_t v_j)^{p^2} = 1$ ,  $(1 + \xi_t w_k)^p = 1$ , and  $(1 + \xi_t y_l)^p = 1$ . For nonnegative integers  $\beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\beta_t \leq p^2, \alpha_t \leq p^2, \delta_t \leq p^2, \varphi_t \leq p$ , and  $\lambda_t \leq p$ , it is clear that

$$\begin{aligned}
 &\prod_{t=1}^r \left\{ (1 + p\xi_t)^{\beta_t} \right\} \cdot \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + \xi_t u_i)^{\alpha_t} \right\} \\
 &\quad \cdot \prod_{j=1}^f \prod_{t=1}^r \left\{ (1 + \xi_t v_j)^{\delta_t} \right\} \cdot \prod_{k=1}^g \prod_{t=1}^r \left\{ (1 + \xi_t w_k)^{\varphi_t} \right\} \\
 &\quad \cdot \prod_{l=1}^h \prod_{t=1}^r \left\{ (1 + \xi_t y_l)^{\lambda_t} \right\} = \{1\}
 \end{aligned} \tag{19}$$

will imply that  $\beta_t = p^2, \alpha_t = p^2, \delta_t = p^2, \varphi_t = p$ , and  $\lambda_t = p$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + p\xi_t)^\beta : \beta = 1, \dots, p^2; \forall t = 1, \dots, r\}$ ;  $B_{t,i} = \{(1 + \xi_t u_i)^\alpha : \alpha = 1, \dots, p^2; \forall t = 1, \dots, r\}$ ,  $(i = 1, \dots, e)$ ;  $C_{t,j} = \{(1 + \xi_t v_j)^\delta : \delta = 1, \dots, p^2; \forall t = 1, \dots, r\}$ ,  $(j = 1, \dots, f)$ ;  $D_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, \dots, p; \forall t = 1, \dots, r\}$ ,  $(k = 1, \dots, g)$ ; and  $E_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, \dots, p; \forall t = 1, \dots, r\}$ ,  $(l = 1, \dots, h)$ , we see that  $A_t, B_{t,i}, C_{t,j}, D_{t,k}$ , and  $E_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + p\xi_t \rangle, \langle 1 + \xi_t u_i \rangle, \langle 1 + \xi_t v_j \rangle, \langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t y_l \rangle$  is trivial and that

$$\begin{aligned}
 &\left| \prod_{t=1}^r \langle 1 + p\xi_t \rangle \cdot \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \right| \\
 &\quad \cdot \left| \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \cdot \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \\
 &\quad \cdot \left| \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \right|
 \end{aligned} \tag{20}$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned}
 1 + Z(R) &= \prod_{t=1}^r \langle 1 + p\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \\
 &\quad \times \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\
 &\quad \times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \cong \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^e \\
 &\quad \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h.
 \end{aligned} \tag{21}$$

**Proposition 4.** Let  $R$  be a ring constructed above and of characteristic  $p^3$  with  $p^3 u_i = 0, p^2 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \tag{22}$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

Case 1.  $p$  is even;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 2;  $1 + 4\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 8;  $1 + \xi_t v_j$  of order 4;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Case 2.  $p$  is odd;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^2$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 3.

**Proposition 5.** Let  $R$  be a ring constructed above and of characteristic  $p^3$  with  $p^3 u_i = 0, p^3 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_2^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_4^r)^e \times (\mathbb{Z}_{16}^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^3}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \tag{23}$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

Case 1.  $p$  is even;  $1 \leq t \leq r$ ,  $1 \leq i \leq e$ ,  $1 \leq j \leq f$ ,  $1 \leq k \leq g$ ,  $1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 2;  $1 + 4\xi_t$  of order 2;  $1 + 2\xi_t u_i$  of order 4;  $1 + \xi_t u_i + \xi_t v_j$  of order 16;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t \gamma_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Case 2.  $p$  is odd;  $1 \leq t \leq r$ ,  $1 \leq i \leq e$ ,  $1 \leq j \leq f$ ,  $1 \leq k \leq g$ ,  $1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^2$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^3$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t \gamma_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 3.

**Proposition 6.** Let  $R$  be a ring constructed above and of characteristic  $p^4$  with  $p^2 u_i = 0$ ,  $p^2 v_j = 0$ ,  $p w_k = 0$ , and  $p \gamma_l = 0$ . Then its group of units is characterized as follows:

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \tag{24}$$

*Proof.* Since  $R$  is commutative,  $R^* = \langle b \rangle \cdot (1 + Z(R)) \cong \langle b \rangle \times (1 + Z(R))$ , a direct product of the  $p$ -group  $1 + Z(R)$  by the cyclic group  $\langle b \rangle$ . Then it suffices to determine the structure of the subgroup  $1 + Z(R)$  of the unit group  $R^*$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . We consider the two cases separately.  $\square$

Case 1.  $p$  is even. For each  $t = 1, \dots, r$ ,  $(1 + 2\xi_t)^4 = 1$ ,  $(1 + 6\xi_t)^2 = 1$ ,  $(1 + 2\xi_t u_i)^2 = 1$ ,  $(1 + \xi_t u_i + \xi_t v_j)^8 = 1$ ,  $(1 + \xi_t w_k)^2 = 1$ , and  $(1 + \xi_t \gamma_l)^2 = 1$ . For nonnegative integers  $\eta_t, \beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\eta_t \leq 4$ ,  $\beta_t \leq 2$ ,  $\alpha_t \leq 2$ ,  $\delta_t \leq 8$ ,  $\varphi_t \leq 2$ , and  $\lambda_t \leq 2$ , it is clear that

$$\begin{aligned} & \prod_{t=1}^r \{(1 + 2\xi_t)^{\eta_t}\} \cdot \prod_{t=1}^r \{(1 + 6\xi_t)^{\beta_t}\} \cdot \prod_{i=1}^e \prod_{t=1}^r \{(1 + 2\xi_t u_i)^{\alpha_t}\} \\ & \cdot \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \{(1 + \xi_t u_i + \xi_t v_j)^{\delta_t}\} \\ & \cdot \prod_{k=1}^g \prod_{t=1}^r \{(1 + \xi_t w_k)^{\varphi_t}\} \cdot \prod_{l=1}^h \prod_{t=1}^r \{(1 + \xi_t \gamma_l)^{\lambda_t}\} = \{1\} \end{aligned} \tag{25}$$

will imply that  $\eta_t = 4$ ,  $\beta_t = 2$ ,  $\alpha_t = 2$ ,  $\delta_t = 8$ ,  $\varphi_t = 2$ , and  $\lambda_t = 2$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + 2\xi_t)^\eta : \eta = 1, \dots, 4; \forall t = 1, \dots, r\}$ ;  $B_t = \{(1 + 6\xi_t)^\beta : \beta = 1, 2; \forall t = 1, \dots, r\}$ ;  $C_{t,i} = \{(1 + 2\xi_t u_i)^\alpha : \alpha = 1, 2; \forall t = 1, \dots, r\}$ , ( $i = 1, \dots, e$ );  $D_{t,i,j} = \{(1 + \xi_t u_i + \xi_t v_j)^\delta : \delta = 1, \dots, 8; \forall t = 1, \dots, r\}$ , ( $i = 1, \dots, e$ ), ( $j = 1, \dots, f$ );  $E_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, 2; \forall t = 1, \dots, r\}$ , ( $k = 1, \dots, g$ ); and  $F_{t,l} =$

$\{(1 + \xi_t \gamma_l)^\lambda : \lambda = 1, 2; \forall t = 1, \dots, r\}$ , ( $l = 1, \dots, h$ ), we see that  $A_t, B_t, C_{t,i}, D_{t,i,j}, E_{t,k}$  and  $F_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + 2\xi_t \rangle$ ,  $\langle 1 + 6\xi_t \rangle$ ,  $\langle 1 + 2\xi_t u_i \rangle$ ,  $\langle 1 + \xi_t u_i + \xi_t v_j \rangle$ ,  $\langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t \gamma_l \rangle$  is trivial and that

$$\begin{aligned} & \left| \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \right| \cdot \left| \prod_{t=1}^r \langle 1 + 6\xi_t \rangle \right| \\ & \cdot \left| \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \right| \cdot \left| \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \right| \\ & \cdot \left| \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \cdot \left| \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t \gamma_l \rangle \right| \end{aligned} \tag{26}$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \times \prod_{t=1}^r \langle 1 + 6\xi_t \rangle \\ & \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \times \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \\ & \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\ & \times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t \gamma_l \rangle \cong \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \\ & \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h. \end{aligned} \tag{27}$$

Case 2.  $p$  is odd. For each  $t = 1, \dots, r$ ,  $(1 + p\xi_t)^p = 1$ ,  $(1 + \xi_t u_i)^{p^2} = 1$ ,  $(1 + \xi_t v_j)^{p^2} = 1$ ,  $(1 + \xi_t w_k)^p = 1$ , and  $(1 + \xi_t \gamma_l)^p = 1$ . For nonnegative integers  $\beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\beta_t \leq p^3$ ,  $\alpha_t \leq p^2$ ,  $\delta_t \leq p^2$ ,  $\varphi_t \leq p$ , and  $\lambda_t \leq p$ , it is clear that

$$\begin{aligned} & \prod_{t=1}^r \{(1 + p\xi_t)^{\beta_t}\} \cdot \prod_{i=1}^e \prod_{t=1}^r \{(1 + \xi_t u_i)^{\alpha_t}\} \\ & \cdot \prod_{j=1}^f \prod_{t=1}^r \{(1 + \xi_t v_j)^{\delta_t}\} \cdot \prod_{k=1}^g \prod_{t=1}^r \{(1 + \xi_t w_k)^{\varphi_t}\} \\ & \cdot \prod_{l=1}^h \prod_{t=1}^r \{(1 + \xi_t \gamma_l)^{\lambda_t}\} = \{1\} \end{aligned} \tag{28}$$

will imply that  $\beta_t = p^3$ ,  $\alpha_t = p^2$ ,  $\delta_t = p^2$ ,  $\varphi_t = p$ , and  $\lambda_t = p$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + p\xi_t)^\beta : \beta = 1, \dots, p^3; \forall t = 1, \dots, r\}$ ;  $B_{t,i} = \{(1 + \xi_t u_i)^\alpha : \alpha = 1, \dots, p^2; \forall t = 1, \dots, r\}$ , ( $i = 1, \dots, e$ );  $C_{t,j} = \{(1 + \xi_t v_j)^\delta : \delta = 1, \dots, p^2; \forall t = 1, \dots, r\}$ , ( $j = 1, \dots, f$ );  $D_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, \dots, p; \forall t = 1, \dots, r\}$ , ( $k = 1, \dots, g$ );

and  $E_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, \dots, p; \forall t = 1, \dots, r\}$ , ( $l = 1, \dots, h$ ), we see that  $A_t, B_{t,i}, C_{t,j}, D_{t,k}$  and  $E_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + p\xi_t \rangle, \langle 1 + \xi_t u_i \rangle, \langle 1 + \xi_t v_j \rangle, \langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t y_l \rangle$  is trivial and that

$$\left| \prod_{t=1}^r \langle 1 + p\xi_t \rangle \cdot \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \cdot \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \cdot \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \cdot \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \right| \quad (29)$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^r \langle 1 + p\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \\ &\times \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\ &\times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \cong \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \\ &\times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h. \end{aligned} \quad (30)$$

**Proposition 7.** *Let  $R$  be a ring constructed above and of characteristic  $p^4$  with  $p^3 u_i = 0, p^2 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (31)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 4;  $1 + 14\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 8;  $1 + \xi_t v_j$  of order 4;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^3$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of

order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 6.

**Proposition 8.** *Let  $R$  be a ring constructed above and of characteristic  $p^4$  with  $p^3 u_i = 0, p^3 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_4^r)^e \times (\mathbb{Z}_{16}^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^3}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (32)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 4;  $1 + 6\xi_t$  of order 2;  $1 + 2\xi_t u_i$  of order 4;  $1 + \xi_t u_i + \xi_t v_j$  of order 16;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^3$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^3$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 6.

**Proposition 9.** *Let  $R$  be a ring constructed above and of characteristic  $p^4$  with  $p^4 u_i = 0, p^2 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^4}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (33)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 4;  $1 + 6\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 16;  $1 + \xi_t v_j$  of order 4;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^3$ ;  $1 + \xi_t u_i$  of order

$p^4$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 6.

**Proposition 10.** *Let  $R$  be a ring constructed above and of characteristic  $p^4$  with  $p^4 u_i = 0$ ,  $p^3 v_j = 0$ ,  $p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_8^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^4}^r)^e \times (\mathbb{Z}_{p^3}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (34)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r$ ,  $1 \leq i \leq e$ ,  $1 \leq j \leq f$ ,  $1 \leq k \leq g$ ,  $1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 4;  $1 + 6\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 16;  $1 + \xi_t v_j$  of order 8;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r$ ,  $1 \leq i \leq e$ ,  $1 \leq j \leq f$ ,  $1 \leq k \leq g$ ,  $1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^3$ ;  $1 + \xi_t u_i$  of order  $p^4$ ;  $1 + \xi_t v_j$  of order  $p^3$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 6.

**Proposition 11.** *Let  $R$  be a ring constructed above and of characteristic  $p^5$  with  $p^2 u_i = 0$ ,  $p^2 v_j = 0$ ,  $p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (35)$$

*Proof.* Since  $R$  is commutative,  $R^* = \langle b \rangle \cdot (1 + Z(R)) \cong \langle b \rangle \times (1 + Z(R))$ , a direct product of the  $p$ -group  $1 + Z(R)$  by the cyclic group  $\langle b \rangle$ . Then it suffices to determine the structure of the subgroup  $1 + Z(R)$  of the unit group  $R^*$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . We consider the two cases separately.  $\square$

*Case 1.  $p$  is even.* For each  $t = 1, \dots, r$ ,  $(1 + 2\xi_t)^8 = 1$ ,  $(1 + 14\xi_t)^2 = 1$ ,  $(1 + 2\xi_t u_i)^2 = 1$ ,  $(1 + \xi_t u_i + \xi_t v_j)^8 = 1$ ,  $(1 + \xi_t w_k)^2 = 1$ , and  $(1 + \xi_t y_l)^2 = 1$ . For nonnegative integers  $\eta_t, \beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\eta_t \leq 8$ ,  $\beta_t \leq 2$ ,  $\alpha_t \leq 2$ ,  $\delta_t \leq 8$ ,  $\varphi_t \leq 2$ , and  $\lambda_t \leq 2$ , it is clear that

$$\begin{aligned} & \prod_{t=1}^r \left\{ (1 + 2\xi_t)^{\eta_t} \right\} \cdot \prod_{t=1}^r \left\{ (1 + 14\xi_t)^{\beta_t} \right\} \cdot \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + 2\xi_t u_i)^{\alpha_t} \right\} \\ & \cdot \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + \xi_t u_i + \xi_t v_j)^{\delta_t} \right\} \\ & \cdot \prod_{k=1}^g \prod_{t=1}^r \left\{ (1 + \xi_t w_k)^{\varphi_t} \right\} \cdot \prod_{l=1}^h \prod_{t=1}^r \left\{ (1 + \xi_t y_l)^{\lambda_t} \right\} = \{1\} \end{aligned} \quad (36)$$

will imply that  $\eta_t = 8$ ,  $\beta_t = 2$ ,  $\alpha_t = 2$ ,  $\delta_t = 8$ ,  $\varphi_t = 2$ , and  $\lambda_t = 2$  for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + 2\xi_t)^\eta : \eta = 1, \dots, 8; \forall t = 1, \dots, r\}$ ;  $B_t = \{(1 + 14\xi_t)^\beta : \beta = 1, 2; \forall t = 1, \dots, r\}$ ;  $C_{t,i} = \{(1 + 2\xi_t u_i)^\alpha : \alpha = 1, 2; \forall t = 1, \dots, r, (i = 1, \dots, e)\}$ ;  $D_{t,i,j} = \{(1 + \xi_t u_i + \xi_t v_j)^\delta : \delta = 1, \dots, 8; \forall t = 1, \dots, r, (i = 1, \dots, e), (j = 1, \dots, f)\}$ ;  $E_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, 2; \forall t = 1, \dots, r, (k = 1, \dots, g)\}$ ; and  $F_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, 2; \forall t = 1, \dots, r, (l = 1, \dots, h)\}$ , we see that  $A_t, B_t, C_{t,i}, D_{t,i,j}, E_{t,k}$ , and  $F_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + 2\xi_t \rangle$ ,  $\langle 1 + 14\xi_t \rangle$ ,  $\langle 1 + 2\xi_t u_i \rangle$ ,  $\langle 1 + \xi_t u_i + \xi_t v_j \rangle$ ,  $\langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t y_l \rangle$  is trivial and that

$$\begin{aligned} & \left| \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \cdot \prod_{t=1}^r \langle 1 + 14\xi_t \rangle \cdot \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \right. \\ & \cdot \left. \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \cdot \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right. \\ & \cdot \left. \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \right| \end{aligned} \quad (37)$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^r \langle 1 + 2\xi_t \rangle \times \prod_{t=1}^r \langle 1 + 14\xi_t \rangle \\ & \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + 2\xi_t u_i \rangle \times \prod_{j=1}^f \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i + \xi_t v_j \rangle \\ & \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\ & \times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t y_l \rangle \\ & \cong \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h. \end{aligned} \quad (38)$$

*Case 2.  $p$  is odd.* For each  $t = 1, \dots, r$ ,  $(1 + p\xi_t)^{p^4} = 1$ ,  $(1 + \xi_t u_i)^{p^2} = 1$ ,  $(1 + \xi_t v_j)^{p^2} = 1$ ,  $(1 + \xi_t w_k)^p = 1$ , and



$(1 + \xi_t \gamma_l)^p = 1$ . For nonnegative integers  $\beta_t, \alpha_t, \delta_t, \varphi_t$ , and  $\lambda_t$  with  $\beta_t \leq p^4, \alpha_t \leq p^2, \delta_t \leq p^2, \varphi_t \leq p$ , and  $\lambda_t \leq p$ , it is clear that

$$\begin{aligned} & \prod_{t=1}^r \left\{ (1 + p\xi_t)^{\beta_t} \right\} \cdot \prod_{i=1}^e \prod_{t=1}^r \left\{ (1 + \xi_t u_i)^{\alpha_t} \right\} \\ & \cdot \prod_{j=1}^f \prod_{t=1}^r \left\{ (1 + \xi_t v_j)^{\delta_t} \right\} \cdot \prod_{k=1}^g \prod_{t=1}^r \left\{ (1 + \xi_t w_k)^{\varphi_t} \right\} \quad (39) \\ & \cdot \prod_{l=1}^h \prod_{t=1}^r \left\{ (1 + \xi_t \gamma_l)^{\lambda_t} \right\} = \{1\} \end{aligned}$$

will imply that  $\beta_t = p^4, \alpha_t = p^2, \delta_t = p^2, \varphi_t = p$ , and  $\lambda_t = p$ , for all  $t = 1, \dots, r$ .

If we set  $A_t = \{(1 + p\xi_t)^\beta : \beta = 1, \dots, p^4; \forall t = 1, \dots, r\}$ ;  $B_{t,i} = \{(1 + \xi_t u_i)^\alpha : \alpha = 1, \dots, p^2; \forall t = 1, \dots, r, (i = 1, \dots, e)\}$ ;  $C_{t,j} = \{(1 + \xi_t v_j)^\delta : \delta = 1, \dots, p^2; \forall t = 1, \dots, r, (j = 1, \dots, f)\}$ ;  $D_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, \dots, p; \forall t = 1, \dots, r, (k = 1, \dots, g)\}$ ; and  $E_{t,l} = \{(1 + \xi_t \gamma_l)^\lambda : \lambda = 1, \dots, p; \forall t = 1, \dots, r, (l = 1, \dots, h)\}$ , we see that  $A_t, B_{t,i}, C_{t,j}, D_{t,k}$ , and  $E_{t,l}$  are all cyclic subgroups of  $1 + Z(R)$ , and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups  $\langle 1 + p\xi_t \rangle, \langle 1 + \xi_t u_i \rangle, \langle 1 + \xi_t v_j \rangle, \langle 1 + \xi_t w_k \rangle$ , and  $\langle 1 + \xi_t \gamma_l \rangle$  is trivial and that

$$\begin{aligned} & \left| \prod_{t=1}^r \langle 1 + p\xi_t \rangle \right| \cdot \left| \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \right| \\ & \cdot \left| \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \right| \cdot \left| \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \right| \quad (40) \\ & \cdot \left| \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t \gamma_l \rangle \right| \end{aligned}$$

coincides with  $|1 + Z(R)|$ , it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^r \langle 1 + p\xi_t \rangle \times \prod_{i=1}^e \prod_{t=1}^r \langle 1 + \xi_t u_i \rangle \\ & \times \prod_{j=1}^f \prod_{t=1}^r \langle 1 + \xi_t v_j \rangle \times \prod_{k=1}^g \prod_{t=1}^r \langle 1 + \xi_t w_k \rangle \\ & \times \prod_{l=1}^h \prod_{t=1}^r \langle 1 + \xi_t \gamma_l \rangle \\ & \cong \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^2}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h. \quad (41) \end{aligned}$$

**Proposition 12.** *Let  $R$  be a ring constructed above and of characteristic  $p^5$  with  $p^3 u_i = 0, p^2 v_j = 0, p w_k = 0$ , and  $p \gamma_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times \left(\mathbb{Z}_8^r\right)^e \times \left(\mathbb{Z}_4^r\right)^f \times \left(\mathbb{Z}_2^r\right)^g \times \left(\mathbb{Z}_2^r\right)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^3}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \quad (42)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 8;  $1 + 14\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 8;  $1 + \xi_t v_j$  of order 4;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t \gamma_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^4$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t \gamma_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 11.

**Proposition 13.** *Let  $R$  be a ring constructed above and of characteristic  $p^5$  with  $p^3 u_i = 0, p^3 v_j = 0, p w_k = 0$ , and  $p \gamma_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times \left(\mathbb{Z}_4^r\right)^e \times \left(\mathbb{Z}_{16}^r\right)^{e+f} \times \left(\mathbb{Z}_2^r\right)^g \times \left(\mathbb{Z}_2^r\right)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^3}^r\right)^e \times \left(\mathbb{Z}_{p^3}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \quad (43)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 8;  $1 + 14\xi_t$  of order 2;  $1 + 2\xi_t u_i$  of order 4;  $1 + \xi_t u_i + \xi_t v_j$  of order 16;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t \gamma_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

*Case 2.  $p$  is odd;*  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^4$ ;  $1 + \xi_t u_i$  of order  $p^3$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t \gamma_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 3.

**Proposition 14.** *Let  $R$  be a ring constructed above and of characteristic  $p^5$  with  $p^4 u_i = 0, p^2 v_j = 0, p w_k = 0$ , and  $p \gamma_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^4}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (44)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 8;  $1 + 14\xi_t$  of order 2;  $1 + \xi_t u_i$  of order 16;  $1 + \xi_t v_j$  of order 4;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.*

*Case 2.  $p$  is odd;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^4$ ;  $1 + \xi_t u_i$  of order  $p^4$ ;  $1 + \xi_t v_j$  of order  $p^2$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 11.*

**Proposition 15.** *Let  $R$  be a ring constructed above and of characteristic  $p^5$  with  $p^4 u_i = 0, p^3 v_j = 0, p w_k = 0$ , and  $p y_l = 0$ . Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^{r-1}} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_{16}^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^4}^r)^e \times (\mathbb{Z}_{p^3}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & p \neq 2. \end{cases} \quad (45)$$

*Proof.* Since  $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$ , it suffices to determine the structure of  $1 + Z(R)$ . Let  $\xi_1, \dots, \xi_r$  be elements of  $R_0$  with  $\xi_1 = 1$  so that  $\bar{\xi}_1, \dots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$  form a basis for  $GF(p^r)$  regarded as a vector space over its prime subfield  $GF(p)$ . Then the generators with their respective orders are as indicated below.  $\square$

*Case 1.  $p$  is even;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + 2\xi_t$  of order 8;  $1 + 14\xi_t$  of order 2;  $1 + 2\xi_t u_i$  of order 8;  $1 + \xi_t u_i + \xi_t v_j$  of order 16;  $1 + \xi_t w_k$  of order 2; and  $1 + \xi_t y_l$  of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.*

*Case 2.  $p$  is odd;  $1 \leq t \leq r, 1 \leq i \leq e, 1 \leq j \leq f, 1 \leq k \leq g, 1 \leq l \leq h$ , the generators are  $1 + p\xi_t$  of order  $p^4$ ;  $1 + \xi_t u_i$  of order  $p^4$ ;  $1 + \xi_t v_j$  of order  $p^3$ ;  $1 + \xi_t w_k$  of order  $p$ ; and  $1 + \xi_t y_l$  of order  $p$ . The rest of the proof follows a similar argument and may be deduced from Proposition 11.*

## 4. Conclusions

The unit groups of some classes of five radical zero commutative completely primary finite rings whose first and second Galois ring module generators are of order  $p^k, k = 2, 3, 4$ , have been classified in this work. It is evident that the results are in piece when  $p = 2$  and  $p \geq 3$ . Since the unit groups of classes of five radical zero commutative completely primary finite rings in this work have been classified via fundamental theorem of finitely generated abelian group, the use of  $j$ -diagram technique by References [8–10] is therefore recommended for possible further study of the unit groups of such rings.

## Data Availability

The corresponding author may be consulted for all the data used within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in determining the group of units of these classes of rings and in writing this article. All have read and approved the final manuscript.

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