1. Introduction

In this paper, $M$ is a finite and undirected simple graph. Let $V(M)$ and $E(M)$ be sets of vertices and edges of $M$, respectively. Then, we put $n = |V(M)|$ and $m = |E(G)|$. If $(a, b) \subseteq V(M)$, then the length of a shortest path connecting $a$ and $b$ in $M$ is the distance between $a$ and $b$ in $M$ and denoted by $d_M(a, b)$. Let $x$ be a vertex of $M$, and let $r$ be a positive integer. Then, the open $r$-neighborhood of $x$ in $M$, $N_r(x)$, is the set of all vertices at distance $r$ from $x$; that is, $N_r(x) = \{v \in V(M): d_M(v, x) = r\}$. The $r$-distance degree of a vertex $x$ in $M$ is the size of the open $r$-neighborhood of $x$ in $M$, and it is denoted by $d_r(x/M)$ or simply $d_r(x)$ if no misunderstanding is possible; that is, $d_r(x/M) = d_r(x) = |N_r(x)|$. It is clear that $d_1(x/M)$ is the degree of vertex $x$ in $M$, and we denoted it by $d_M(x)$ or simply $d(x)$. Also, the eccentricity of a vertex $x$ in $M$, $e(x)$, is defined as $e(x) = \max\{d_M(v, u): v \in V(M)\}$, and the diameter and radius of graph $M$ are defined as $\text{diam}(M) = \max\{e(v): v \in V(M)\}$ and $\text{rad}(M) = \min\{e(v): v \in V(M)\}$, respectively.

The subdivision graph $S(M)$ of a simple graph $M$ is the graph obtained from $M$ by inserting an additional vertex into each edge of $M$, or equivalently, by replacing each of its edges with a path of length 2 [1].

The wheel graph $W_{1,q}$ of order $q + 1$ is the join of $K_1$ and $C_q$ in which $K_1$ is the complete graph with one vertex, and $C_q$ is the $q$-vertex cycle graph. Clearly, $|V(W_{1,q})| = q + 1$ and $|E(W_{1,q})| = 2q$. The apex vertex of the wheel is the vertex corresponding to $K_1$, and the rim vertices of the wheel are the vertices corresponding to $C_q$ [2]. Note that all notions and notations not defined here can be obtained from the book of Harary [2].

In chemical graph theory, a numerical parameter of a given graph that is applicable in some chemical problems is called a topological index. The Zagreb group indices are two degree-based topological indices that were defined by Gutman and Trinajstic [3] in 1972 and elaborated in [4]. These indices are defined as

$$
M_1(M) = \sum_{x \in V(M)} d_M(x)^2,
$$

$$
M_2(M) = \sum_{ab \in E(M)} d_M(a)d_M(b).
$$

For the main properties of these two indices, we refer the interested readers to [3–7].

In 2017, Naji et al. [8] introduced three topological indices depending on the second degree of vertices. These invariants are so-called leap Zagreb topological indices and can be defined as follows:
In [9], the first leap Zagreb topological index of some graph operations is computed, and in [10], some formulas for the leap Zagreb indices of generalized rts point line transformation graphs \( T^{rts}(M) \), when \( s = 1 \), are obtained. We refer to [8–14] for more details on the leap Zagreb indices of graphs. In [15], Sharma et al. introduced the eccentric connectivity index of the graph \( M \) as
\[
\xi_c(M) = \sum_{v \in V(M)} d(v)e(v).
\]
For mathematical properties, the interested readers can consult [15–17].

Recently, authors found in [18] introduced the leap eccentric connectivity index for the Cartesian product, composition, corona product, symmetric difference, and disjunctions. The eccentric connectivity index of subdivision graph of some well-known graphs is a \([C_3, C_4]\)-free graph. According to Theorem 1, for a \( (C_3, C_4) \)-free graph \( M \), we have
\[
\sum_{v \in V(G)} d_2(v) = M_1 - 2m.
\]

2. Main Results

The aim of this paper is to present the exact values of leap eccentric connectivity index of subdivision graph of some standard graphs.

Theorem 3. Suppose \( n \geq 3 \). Then,
\[
\xi_c(S(K_n)) = \begin{cases} 36, & \text{if } n = 3, \\ n(n-1)(4n-5), & \text{otherwise}. \end{cases}
\]

Proof. Let \( a_1, a_2, \ldots, a_n \) be the vertices of \( K_n \), and let \( b_1, b_2, \ldots, b_m \) be the new vertices added to \( K_n \) to obtain \( S(K_n) \), where \( m \) is the size of \( K_n \). Then, \( d_2(a_i) = n - 1 \), \( d_2(b_j) = 2n - 4 \), \( e(a_i) = 3 \), and \( e(b_j) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{otherwise}. \end{cases} \)

By definition, we have two following cases:

Case 1. If \( n = 3 \), then
\[
\xi_c(S(K_3)) = \sum_{i=1}^{6} (2)(3) = 6(6) = 36.
\]

Case 2. If \( n \geq 4 \), then
\[
\xi_c(S(K_4)) = \sum_{w \in V(S(K_n))} d_2(w)e(w)
\]
\[
= \sum_{i=1}^{n} d_2(a_i)e(a_i) + \sum_{j=1}^{m} d_2(b_j)e(b_j)
\]
\[
= \sum_{i=1}^{n} (n-1)(3) + m(2n-4)(4)
\]
\[
= 3n(n-1) + 8m(n-2).
\]

Since for the complete graph \( K_n \), \( m = n(n-1)/2 \), it follows that
Theorem 4. For \( r \geq s \geq 2 \), let \( K_{rs} \) be the complete bipartite graph. Then,

\[
L^c(S(K_{rs})) = rs(3r + 3s + 2).
\]

Proof. Suppose \( r \geq s \geq 2 \) and \((V_1, V_2)\) is a partition of the vertex set, where \( V_1 = \{v_1, v_2, v_3, \ldots, v_r\} \), \( V_2 = \{u_1, u_2, u_3, \ldots, u_s\} \) and let \( W = \{w_1, w_2, w_3, \ldots, w_s\} \) be the set of new vertices in \( S(K_{rs}) \). Then, \( d_2(v_i) = s \), \( d_2(u_i) = r \), \( d_2(w_k) = r + s - 2 \), \( e(v_i) = 4 \), \( e(u_i) = 4 \), and \( e(w_k) = 3 \). By definition,

\[
L^c(S(K_{rs})) = \sum_{v \in V_1} d_2(v) \cdot e(v) + \sum_{u \in V_2} d_2(u) \cdot e(u)
\]

\[
= \sum_{v \in V_1} \sum_{w \in W} d_2(w) \cdot e(w)
\]

\[
= \sum_{i=1}^{s} (s \cdot 4) + \sum_{j=1}^{r} (r \cdot 4) + \sum_{k=1}^{r \cdot s} (r + s - 2) \cdot 3
\]

\[
= 4rs + 4rs + 3rs(r + s - 2)
\]

\[
= rs(3r + 3s + 2).
\]

\[
\square
\]

Theorem 5. Let \( K_{1,n-1} \) be the star graph of order \( n \geq 3 \). Then,

\[
L^c(S(K_{1,n-1})) = 3n(n - 1).
\]

Proof. Let \( v_0 \in K_{1,n-1} \), with \( d(v_0) = n - 1 \), be the central vertex, \( v_1, v_2, \ldots, v_n-1 \) are the pendent vertices of \( K_{1,n-1} \), and \( u_1, u_2, \ldots, u_{n-1} \) are the new vertices added to \( K_{1,n-1} \) to obtain \( S(K_{1,n-1}) \). If \( i = 1, 2, \ldots, n-1 \), then \( d_2(v_i) = n - 1 \), \( d_2(u_i) = n - 2 \), \( e(v_0) = 2 \), \( e(v_i) = 4 \), and \( e(u_i) = 3 \). By definition,

\[
L^c(S(K_{1,n-1})) = d_2(v_0) \cdot e(v_0) + \sum_{i=1}^{n-1} d_2(v_i) \cdot e(v_i) + \sum_{j=1}^{n-1} d_2(u_j) \cdot e(u_j)
\]

\[
= (n - 1) \cdot 2 + \sum_{i=1}^{n-1} (1 \cdot 4) + \sum_{j=1}^{n-1} (n - 2) \cdot 3
\]

\[
= 3n(n - 1).
\]

\[
\square
\]

On the other hand, \( 2(n - 2)(3n + 2) - 3n(n - 1) = n(3n - 5) - 8 > 0 \). Therefore, by (i) and (ii), \( 3n(n - 1) \leq L^c(S(K_{rs})) \leq 1/4n^2(3n + 2) \). On the left hand side, equality occurs if and only if \( K_{rs} \equiv K_{1,n-1} \), and on the right hand side, equality occurs if and only if \( K_{rs} \equiv \tilde{K}_{n/2,n/2} \).

\[
\square
\]

Proposition 1. Let \( n \) be an integer. Then,

\[
L^c(S(C_n)) = 4n^2.
\]

Proof. Since \( S(C_n) = C_{2n} \), the proof follows from Theorem 2.

\[
\square
\]

Proposition 2. Let \( n \geq 2 \) be an integer. Then,

\[
L^c(S(P_n)) = 2(3n^2 - 8n + 6).
\]

Proof. Since \( S(P_n) = P_{2n-1} \), the proof follows from Theorem 1.

\[
\square
\]
**Theorem 7.** For $n \geq 6$, $L_5^c(S(W_{1,n})) = 2n(2n + 23)$.

**Proof.** Let $v_0$ be the central vertex of $W_{1,n}$, $v_1, v_2, \ldots, v_n$ be the rim vertices of $W_{1,n}$, and let $S(W_{1,n})$ be the subdivision of $W_{1,n}$. If $w_i$ subdivides $v_0v_i$, $1 \leq i \leq n$, $u_i$ subdivides $v_iv_{j+1}$, $1 \leq j \leq n-1$ and $u_n$ subdivides $v_nv_1$. One can easily verify

$$L_5^c(S(W_{1,n})) = \sum_{v \in V(S(W_{1,n}))} d_2(v)e(v)$$

$$= d_2(v_0)e(v_0) + \sum_{i=1}^{n} d_2(v_i)e(v_i) + \sum_{i=1}^{n} d_2(u_i)e(u_i) + \sum_{i=1}^{n} d_2(w_i)e(w_i)$$

$$= (n)(3) + \sum_{i=1}^{n} (3)(5) + \sum_{i=1}^{n} (4)(6) + \sum_{i=1}^{n} (n+1)(4) = 2n(2n + 23).$$

**Theorem 8.** For natural numbers $r$ and $s$, let $D_{r,s}$ be a double star with $v_1, v_2, v_3, \ldots, v_r$ be the pendent vertices with support at $v_0$ and $u_1, u_2, u_3, \ldots, u_s$, be the pendent vertices with support at $u_0$. Then,

$$L_5^c(S(D_{r,s})) = 5(r^2 + s^2) + 13(r + s) + 8. \tag{16}$$

**Proof.** Let $x_i$ subdivides $v_0v_i$, $1 \leq i \leq r$, $y_j$ subdivides $u_0u_j$, $1 \leq j \leq s$, and $w_0$ subdivides $v_0u_0$. Then, $d_2(v_0) = r + 1$, $d_2(u_0) = s + 1$, $d_2(w_0) = r + s$, $d_2(v_i) = 1$, $d_2(u_j) = 1$, $d_2(x_i) = r$, $d_2(y_j) = s$, $e(v_0) = 4$, $e(u_0) = 4$, $e(v_i) = 3e(v_i) = 6$, $e(u_j) = 6$, $e(x_i) = 5$, and $e(y_j) = 5$. By definition, we have

$$L_5^c(S(D_{r,s})) = d_2(v_0)e(v_0) + \sum_{i=1}^{r} d_2(x_i)e(x_i) + \sum_{i=1}^{r} d_2(v_i)e(v_i) + d_2(w_0)e(w_0)$$

$$+ d_2(u_0)e(u_0) + \sum_{j=1}^{s} d_2(y_j)e(y_j) + \sum_{j=1}^{s} d_2(u_j)e(u_j)$$

$$= (r + 1)(4) + \sum_{i=1}^{r} (r)(5) + \sum_{i=1}^{r} (1)(6) + (r + s)(3) + (s + 1)(4)$$

$$+ \sum_{j=1}^{s} (s)(5) + \sum_{j=1}^{s} (1)(6)$$

$$= 5(r^2 + s^2) + 13(r + s) + 8. \tag{17}$$

**Theorem 9.** Let $n \geq 7$ be a natural number. Then,

$$L_5^c(S(D_{n-2,i})) > L_5^c(S(D_{n+3-3i})) \text{ for } i = 1, 2, \ldots, \lfloor \frac{n-2}{2} \rfloor - 1. \tag{18}$$

**Proof.** By Theorem 8,

$$L_5^c(S(D_{n-2,i})) - L_5^c(S(D_{n+3-3i})) = 10(n - 2i - 3). \tag{19}$$

Now, if $2|n - 2$, then by (19),
\[
L^c \left( S \left( D_{i,n-2-i} \right) \right) - L^c \left( S \left( D_{i+1,n-3-i} \right) \right) \geq 10 \left( n - 2 \left( \frac{n-2}{2} - 1 \right) - 3 \right) = 10. \quad (20)
\]

And if \( 2 \mid n - 2 \), then by (19),
\[
L^c \left( S \left( D_{i,n-2-i} \right) \right) - L^c \left( S \left( D_{i+1,n-3-i} \right) \right) \geq 10 \left( n - 2 \left( \frac{n-3}{2} - 1 \right) - 3 \right) = 20. \quad (21)
\]

Therefore, \( L^c \left( S \left( D_{i,n-2-i} \right) \right) > L^c \left( S \left( D_{i+1,n-3-i} \right) \right) \) for \( i = 1, 2, \ldots, \lfloor n/2 \rfloor - 1 \).

\[\Box\]

**Corollary 1.** Let \( r, s, \) and \( n \) be three natural numbers such that \( r + s + 2 = n \geq 7 \). Then,
\[
\frac{5}{2} n^2 + 3n - 8 \leq L^c \left( S \left( D_{r,s} \right) \right) \leq 5n^2 - 17n + 32, \quad 2 \mid n - 2,
\]
\[
\frac{1}{2} (2n + 1) (n - 3) \leq L^c \left( S \left( D_{r,s} \right) \right) \leq 5n^2 - 17n + 32, \quad 2 \mid n - 2. \quad (22)
\]

On the left hand side, equalities occur if and only if \( D_{r,s} \equiv D_{\lfloor n/2 \rfloor,n-\lfloor n/2 \rfloor} \). On the right hand side, equalities occur if and only if \( D_{r,s} \equiv D_{1,n-3} \).

\[\Box\]

**Theorem 10.** Let \( M \) be an \( n \)-vertex connected graph of size \( m \) such that \( n \geq 3 \). Then,
\[
L^c \left( M \right) \leq nM_1 \left( M \right) - 2nm - LM_3 \left( M \right). \quad (23)
\]

The bound is attained for \( P_4 \).

**Proof.** Since \( e \left( v \right) \leq n - d \left( v \right) \) for every \( v \in V \left( M \right) \),
\[
L^c \left( M \right) = \sum_{v \in V \left( M \right)} d_2 \left( v \right) e \left( v \right) \leq \sum_{v \in V \left( M \right)} d_2 \left( v \right) \left( n - d \left( v \right) \right)
\]
\[
= \sum_{v \in V \left( M \right)} n d_2 \left( v \right) - \sum_{v \in V \left( M \right)} d_1 \left( v \right) d_2 \left( v \right)
\]
\[
= n \sum_{v \in V \left( M \right)} d_2 \left( v \right) - \sum_{v \in V \left( M \right)} d_1 \left( v \right) d_2 \left( v \right). \quad (24)
\]

Using definition of \( LM_3 \left( M \right) \) and Lemma 3, we get
\[
L^c \left( M \right) \leq n \sum_{v \in V \left( M \right)} \left( \sum_{uv \in E \left( M \right)} d \left( u \right) - d \left( v \right) \right) - LM_3 \left( M \right)
\]
\[
= n \sum_{v \in V \left( M \right)} d \left( v \right)^2 - 2nm - LM_3 \left( M \right)
\]
\[
= nM_1 \left( M \right) - 2nm - LM_3 \left( M \right). \quad (25)
\]
\[\Box\]

**Corollary 2.** Let \( M \) be an \( n \)-vertex connected graph of size \( m \) such that \( n \geq 3 \). Then,
\[
L^c \left( S \left( M \right) \right) \leq \left( n + m - 3 \right) M_1 \left( M \right) + 4m. \quad (26)
\]

**Proof.** For \( uv \in E \left( M \right) \), let \( v_{uv} \) be the new vertex of degree 2 on \( uv \) in \( S \left( M \right) \). By definition of \( S \left( M \right) \), \( d \left( v_{uv} / S \left( M \right) \right) = d \left( v \right) \) for \( v \in V \left( M \right) \),
\[
d_2 \left( v_{uv} / S \left( M \right) \right) = d \left( v \right) + d \left( M \right) - 2
\]
for \( uv \in E \left( M \right) \). Therefore, \( M_1 \left( S \left( M \right) \right) = M_1 \left( M \right) + 4m \) and \( LM_3 \left( S \left( M \right) \right) = M_1 \left( M \right) + 2M_1 \left( M \right) - 4m = 3M_1 \left( M \right) - 4m \). So, by Theorem 10, \( L^c \left( S \left( M \right) \right) \leq \left( n + m - 3 \right) M_1 \left( M \right) + 4m. \quad (27)
\]
\[\Box\]

**Theorem 11.** Let \( M \) be an \( n \)-vertex connected graph of size \( m \geq 2 \). Then, \( L^c \left( S \left( G \right) \right) \geq 2 \left( n + m \right) \).

**Proof.** Let \( V_0 = \{ v \in V \left( M \right) : d \left( v \right) = n - 1 \} \) and \( n_0 = \left| V_0 \right| \). Then, \( d_2 \left( v \right) = 0 \) for every \( v \in V_0 \) and for every \( u \in V \left( M \right) \), we have \( e \left( u \right) \geq 2 \) and \( d_2 \left( u \right) \geq 1 \). Hence,
\[
L^c \left( M \right) = \sum_{v \in V \left( M \right)} d_2 \left( v \right) e \left( v \right) + \sum_{v \in V \left( M \right)} d_2 \left( v \right) e \left( v \right)
\]
\[
= \sum_{v \in V \left( M \right)} \left( 0 \right) \left( 1 \right) + \sum_{v \in V \left( M \right)} \left( 1 \right) \left( 2 \right) \quad (27)
\]
\[
= 2 \left| V \left( M \right) \right|
\]
\[
= 2 \left( n + n_0 \right). \quad (27)
\]

Now, it is easy to see that the number of vertices of \( S \left( M \right) \) is \( n + m \), and the number of vertices of degree \( n - 1 \) in \( S \left( M \right) \) is zero. Therefore, by (27), we have \( L^c \left( S \left( M \right) \right) \geq 2 \left( n \left( S \left( M \right) \right) - n_0 \left( S \left( M \right) \right) \right) = 2 \left( n + m \right). \quad (28)
\]
\[\Box\]

**Theorem 12.** Let \( M \) be an \( n \)-vertex graph of size \( m \). Then,
\[
L^c \left( M \right) \leq diam \left( M \right) \left( M_1 \left( M \right) + 2m \right). \quad (28)
\]

The equality occurs if and only if \( M \) is a self-centered and \( C_3, C_4 \)-free graph.

**Proof.** By definition, for all \( v \in V \left( M \right) \), \( e \left( v \right) \leq diam \left( M \right) \), the equality holds if and only if \( M \) is a self-centered. Also, by
Lemma 3. \( \sum_{v \in V(M)} d_2(v) \leq M_1(M) - 2m \), and the equality occurs if and only if \( M \) is a \( \{C_3, C_4\} \)-free graph. Therefore,

\[
L^C(M) = \sum_{v \in V(M)} d_2(v) e(v) \leq \sum_{v \in V(M)} d_2(v) \text{diam}(M) \leq \text{diam}(M)(M_1(M) - 2m).
\]

The equalities hold if and only if \( M \) is a self-centered and \( \{C_3, C_4\} \)-free graph. \( \square \)

Corollary 3. Let \( M \) be an \( n \)-vertex graph of size \( m \). Then,

\[
L^C(S(M)) \leq \text{diam}(S(M)) M_1(M).
\]

The equality occurs if and only if \( M \equiv K_n \).

Data Availability
No data were used to support the study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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