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Research Article

Existence of Covers and Envelopes of a Left Orthogonal Class and Its Right Orthogonal Class of Modules

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In this paper, we investigate the notions of $\mathscr{X}^{\perp}$-projective, $\mathscr{X}$-injective, and $\mathscr{X}$-flat modules and give some characterizations of these modules, where $\mathscr{X}$ is a class of left modules. We prove that the class of all $\mathscr{X}^{\perp}$-projective modules is Kaplansky. Further, if the class of all $\mathscr{X}$-injective R-modules is contained in the class of all pure projective modules, we show the existence of $\mathscr{X}^{\perp}$-projective covers and $\mathscr{X}$-injective envelopes over a $\mathscr{X}^{\perp}$-hereditary ring. Further, we show that a ring R is Noetherian if and only if $\mathscr{W}$-injective R-modules coincide with the injective R-modules. Finally, we prove that if $\mathscr{W} \subseteq \mathscr{S}$, every module has a $\mathscr{W}$-injective precover over a coherent ring, where $\mathscr{W}$ is the class of all pure projective R-modules and $\mathscr{S}$ is the class of all $fp-\Omega^1$-modules.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and all R-modules, if not specified otherwise, are left R-modules. R-Mod denotes the category of left R-modules.

The notion of fp-injective modules over arbitrary rings was first introduced by Stenström in [1]. An R-module M is called fp-injective (absolutely pure) if $\operatorname{Ext}_R^1(N, M) = 0$ for all finitely presented R-modules N. Let $\mathscr{X}$ be a class of left R-modules. Mao and Ding in [2] introduced the concept of $\mathscr{X}$-injective modules (see Definition 6). Selvaraj et al. developed $\mathscr{X}$-injective and $\mathscr{X}$-flat R-modules and studied covers and envelopes of modules with Goresntein properties in [3–5].

A pair $(\mathscr{A}, \mathscr{B})$ is a cotorsion theory (see Definition 4). In our article, $(\mathscr{A}, \mathscr{B})$ is a cotorsion theory generated by the class $\mathscr{X}$ [6], that is, $\mathscr{A} = {}^{\perp}(\mathscr{X}^{\perp})$ and $\mathscr{B} = \mathscr{X}^{\perp}$. If R is a hereditary ring (that is, every ideal is projective), then $\mathscr{A}$ is closed under submodules and containing all injective R-modules in the cotorsion theory $(\mathscr{A}, \mathscr{B})$. In this case, results are trivial. For this reason, we introduced $\mathscr{X}$-hereditary ring (see Definition 16), and we restrict the setting to $\mathscr{X}^{\perp}$-hereditary rings.

The notions of (pre)covers and (pre)envelopes of modules were introduced by Enochs in [7] and, independently, by Auslander and Smalø in [8]. Since then, the existence and the properties of (pre)covers and (pre)envelopes relative to certain submodule categories have been studied widely. The theory of (pre)covers and (pre)envelopes, which play an important role in homological algebra and representation theory of algebras, is now one of the main research topics in relative homological algebra.

Salce introduced the notion of a cotorsion theory in [9]. Enochs showed the important fact that closed and complete cotorsion pairs provide minimal versions of covers and envelopes. Eklof and Trlifaj [10] proved that a cotorsion pair $(\mathscr{A}, \mathscr{B})$ is complete when it is cogenerated by a set. Consequently, many classical cotorsion pairs are complete. In this way, Bican et al. [11] showed that every module has a flat cover over an arbitrary ring. These results motivate us to define $\mathscr{X}$-projective R-modules (see Definition 14) and $^{\perp}(\mathscr{X}^{\perp})$-pure projective R-modules (see Definition 21), and we prove the existence of $\mathscr{X}^{\perp}$-projective cover and $\mathscr{X}$-injective envelope. In particular, we denote $\mathscr{P}\mathscr{I}$ by the class of all pure injective R-modules, and we prove the following main result.

Theorem 1. *Let R be a $\mathscr{X}^{\perp}$ -hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. Then, every R-module M has a $\mathscr{X}^{\perp}$-projective cover and an $\mathscr{X}$-injective envelope.*

Self-injective rings were introduced by Johnson and Wong in [12]. A ring R is said to be self injective if R over itself is an injective R-module. In this paper, we introduce self-$\mathscr{X}$-injective ring (see Definition 27). Recall that an R-module M is called *reduced* [13] if it has no nonzero injective submodules. An R-module M is said to be *coreduced* [14] if it has no nonzero projective quotient modules. Mao and Ding [15] proved that an FI-injective R-module decomposes into an injective and a reduced FI-injective R-module over a coherent ring. Similarly, we prove the following result.

Theorem 2. *Let R be a self $\mathscr{X}$-injective and $\mathscr{X}^{\perp}$-hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. Then, an R -module M is $\mathscr{X}^{\perp}$-projective if and only if M is a direct sum of a projective R-module and a coreduced $\mathscr{X}^{\perp}$-projective R-module.*

Pinzon [16] proved that every module has an fp-injective cover over a coherent ring. We prove the following result that provides the existence of $\mathscr{W}$-injective cover, where $\mathscr{W}$ is the class of all pure projective R-modules and $\mathscr{S}$ the class of all small $-\Omega^1$-R-modules (see Definition 41).

Theorem 3. *Let R be a coherent ring and $\mathscr{W} \subseteq \mathscr{S}$. Then, every R-module has a $\mathscr{W}$-injective cover.*

This paper is organized as follows: In Section 2, we recall some notions that are necessary for our proofs of the main results of this paper.

In Section 3, we investigate the notions of $\mathscr{X}$-injective and $\mathscr{X}$-flat modules and give some characterizations of these classes of modules.

In Section 4, we introduce $\mathscr{X}^{\perp}$-hereditary ring. Further, we investigate $\mathscr{X}^{\perp}$-projective module and give some characterizations. Further, we prove Theorems 1 and 2.

In Section 5, we prove that if M is a submodule of an $\mathscr{X}$-injective R-module A, then, $i : M \longrightarrow A$ is a special $\mathscr{X}$-injective envelope of M if and only if A is an $\mathscr{X}^{\perp}$-projective essential extension of M.

In the last section, we assume that $\mathscr{W}$ is the class of all pure projective R-modules and we prove that every R-module has a $\mathscr{W}$-injective preenvelope. Finally, we prove the main theorem of this section Theorem 3.

2. Preliminaries

In this section, we recall some known definitions and some terminology that will be used in the rest of the paper.

Given a class $\mathscr{C}$ of left R-modules, we write

$$\mathscr{C}^{\perp} = \left\{N \in R - \text{Mod} \middle| \text{Ext}_R^1(M, N) = 0 \quad \forall \quad M \in \mathscr{C}\right\},$$
$$^{\perp}\mathscr{C} = \left\{N \in R - \text{Mod} \middle| \text{Ext}_R^1(N, M) = 0 \quad \forall \quad M \in \mathscr{C}\right\}. \tag{1}$$

Following [7], we say that a map $f \in \text{Hom}_R(C, M)$ with $C \in \mathscr{C}$ is a $\mathscr{C}$-precover of M, if the group homomorphism $\text{Hom}_R(C', f)$: $\text{Hom}_R(C', C) \longrightarrow \text{Hom}_R(C', M)$ is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $f \in \text{Hom}_R(C, M)$ of M is called a $\mathscr{C}$-cover of M if f is right minimal, that is, if $fg = f$ implies that g is an automorphism for each $g \in \text{End}_R(C)$. $\mathscr{C} \subseteq R - \text{Mod}$ is a precovering class (resp., covering class) provided that each module has a $\mathscr{C}$-precover (resp., $\mathscr{C}$-cover). Dually, we have the definition of $\mathscr{C}$-preenvelope (resp., $\mathscr{C}$-envelope).

A $\mathscr{C}$-precover f of M is said to be *special* [6] if f is an epimorphism and $\ker f \in \mathscr{C}^{\perp}$.

A $\mathscr{C}$-preenvelope f of M is said to be *special* [6] if f is a monomorphism and $\text{coker} f \in {}^{\perp}\mathscr{C}$.

A $\mathscr{C}$-envelope $\phi : M \longrightarrow C$ is said to have the *unique mapping property* [17] if for any homomorphism $f : M \longrightarrow C'$ with $C' \in \mathscr{C}$, there is a unique homomorphism $g : C \longrightarrow C'$ such that $g\phi = f$.

A module is said to be *pure projective* [18] if it is projective with respect to pure exact sequence.

An R-module M is called super finitely presented ([19]) if there exist a projective resolution with each projective R-module is finitely generated.

Following [20, 21], an R-module M is called *weak injective* if $\text{Ext}_R^1(N, M) = 0$ for every super finitely presented R-module N. A right R-module M is called *weak flat* if $\text{Tor}_1^R(M, N) = 0$ for every super finitely presented R-module N.

A class $\mathscr{C}$ of left R-modules is said to be *injectively resolving* [6] if $\mathscr{C}$ contains all injective modules and if given an exact sequence of left R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0. \tag{2}$$

$C \in \mathscr{C}$ whenever $A, B \in \mathscr{C}$.

Definition 4. A pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ of classes of modules is called a *cotorsion theory* [6] if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

(1) A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is said to be *perfect* [22] if every module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope

Definition 5. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is said to have enough injectives [13] if for every R-module M there is an exact sequence $0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$.

For an R-module M, $fd(M)$ denotes the flat dimension of M and $id(M)$ denotes the injective dimension of M.

The $\mathcal{X}^{\perp}$-coresolution dimension of M, denoted by $\text{cores.dim}_{\mathcal{X}^{\perp}}(M)$, is defined to be the smallest nonnegative integer n such that $\text{Ext}_R^{n+1}(A, M) = 0$ for all R-modules $A \in \mathcal{X}$ (if no such n exists, set $\text{cores.dim}_{\mathcal{X}^{\perp}}(M) = \infty$), and $\text{cores.dim}_{\mathcal{X}^{\perp}}(R)$ is defined as $\sup\{\text{cores.dim}_{\mathcal{X}^{\perp}}(M) | M \in R - \text{Mod}\}$.

We denote by $\mathbb{Z}$ the ring of all integers and by $\mathbb{Q}$ the field of all rational numbers. For a left R-module M, we denote by $M^{+} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the *character module* of M.$\mathcal{I}_0$ denotes the class of all injective left R-modules.

For unexplained terminology, we refer to [23, 24].

3. $\mathcal{X}$-Injective and $\mathcal{X}$-Flat Modules

In this section, we assume that $\mathcal{X}$ is any class in $R - \text{Mod}$. We begin with the following definition:

Definition 6 (see [2]). A left R-module M is called $\mathcal{X}$-injective if $\text{Ext}_R^1(X, M) = 0$ for all left R-modules $X \in \mathcal{X}$. A right R-module N is said to be $\mathcal{X}$-flat if $\text{Tor}_1^R(N, X) = 0$ for all left R-modules $X \in \mathcal{X}$.

We denote by $\mathcal{X}^{\perp}$ the class of all $\mathcal{X}$-injective modules and ${}^{\perp}(\mathcal{X}^{\perp})$ the class of all $\mathcal{X}^{\perp}$-projective R-modules and, further, $\mathcal{X}^{T}$ the class of all $\mathcal{X}$-flat R-modules.

Example 7.

(1) If $\mathcal{X} = R - \text{Mod}$, then, $\mathcal{X}^{\perp}$ is the class of all injective R-modules and $\mathcal{X}^{T}$ is the class of all flat right R-modules.

(2) If $\mathcal{X}$ is the class of all finitely presented R-modules, then, $\mathcal{X}^{\perp}$ is the class of all fp-injective R-modules.

(3) If $\mathcal{X}$ is the class of all left super finitely presented R-module, then, $\mathcal{X}^{\perp}$ is the class of all weak injective R-modules and $\mathcal{X}^{T}$ is the class of all weak flat right R-modules.

Proposition 8. *Let M be an R-module. Then, the following are true:*

(1) Let $\mathcal{I}_0 \subseteq \mathcal{X}$. Then, M is injective if and only if M is $\mathcal{X}$-injective and $id(M) \leq 1$.

(2) M is injective if and only if M is $\mathcal{X}^{\perp}$-injective and $\text{cores.dim}_{\mathcal{X}^{\perp}}(M) \leq 1$.

Proof.

(1) The direct implication is clear. Conversely, let M be $\mathcal{X}$-injective and $id(M) \leq 1$. For any R-module N, consider an exact sequence $0 \longrightarrow N \longrightarrow E(N) \longrightarrow L \longrightarrow 0$ with $E(N)$ an injective envelope of N. We have an exact sequence

$$\cdots \longrightarrow \text{Ext}_R^1(E(N), M) \longrightarrow \text{Ext}_R^1(N, M) \longrightarrow \text{Ext}_R^2(L, M) \longrightarrow \cdots \quad (3)$$

Since $id(M) \leq 1$, $\text{Ext}_R^2(L, M) = 0$, and hence, M is injective.

(2) The direct implication is clear by the definition of an injective module. Conversely, let M be $\mathcal{X}^{\perp}$-injective and $\text{cores.dim}_{\mathcal{X}^{\perp}}(M) \leq 1$. Consider an exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow L \longrightarrow 0$ with $E(M)$ an injective envelope of M. For any R-module $X \in \mathcal{X}$, we have an exact sequence $\cdots \longrightarrow \text{Ext}_R^1(X, E(M)) \longrightarrow \text{Ext}_R^1(X, L) \longrightarrow \text{Ext}_R^2(X, M) \longrightarrow \cdots$. Since $\text{cores.dim}_{\mathcal{X}^{\perp}}(M) \leq 1$, $\text{Ext}_R^2(X, M) = 0$, and hence, L is $\mathcal{X}$-injective. Therefore, $\text{Ext}_R^1(L, M) = 0$, so that the exact sequence is split. It follows that M is a direct summand of E, as desired.

□

We now give some of the characterizations of $\mathcal{X}$-injective module:

Proposition 9. *Let $\mathcal{I}_0 \subseteq \mathcal{X}$. The following are equivalent for a left R-module M :*

(1) M is $\mathcal{X}$-injective

(2) For every exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$, with $E \in \mathcal{X}$,$E \longrightarrow L$ is an $\mathcal{X}$-precover of L

(3) M is a kernel of an $\mathcal{X}$-precover $f : A \longrightarrow B$ with A an injective module

(4) M is injective with respect to every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, with $C \in \mathcal{X}$.

Proof. $(1) \Rightarrow (2)$. Consider an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0, \quad (4)$$

where$E \in \mathcal{X}$. Then, by hypothesis, $\text{Hom}_R(E', E) \longrightarrow \text{Hom}_R(E', L)$ is surjective for all left R-modules $E' \in \mathcal{X}$, as desired.

$(2) \Rightarrow (3)$. Let $E(M)$ be an injective hull of M and consider the exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow^{f} E(M)/M$

$\longrightarrow 0$. Since $E(M)$ is injective, it belongs to $\mathscr{X}$. So assertions (3)holds.

(3) $\Rightarrow$ (1). Let M be a kernel of an $\mathscr{X}$-precover $f : A \longrightarrow B$ with A an injective module. Then, we have an exact sequence $0 \longrightarrow M \longrightarrow A \longrightarrow A/M \longrightarrow 0$. Therefore, for any left R-module $N \in \mathscr{X}$, the sequence $\mathrm{Hom}_R(N, A) \longrightarrow \mathrm{Hom}_R(N, A/M) \longrightarrow \mathrm{Ext}^1_R(N, M) \longrightarrow 0$ is exact. By hypothesis, $\mathrm{Hom}_R(N, A) \longrightarrow \mathrm{Hom}_R(N, A/M)$ is surjective. Thus, $\mathrm{Ext}^1_R(N, M) = 0$, and hence (1) follows

(1) $\Rightarrow$ (4). Consider an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0, \tag{5}$$

where $C \in \mathscr{X}$. Then, $\mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective, as desired.

(4) $\Rightarrow$ (1). For each left R-module $N \in \mathscr{X}$, there exists a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P a projective R-module, which induces an exact sequence $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(K, M) \longrightarrow \mathrm{Ext}^1_R(N, M) \longrightarrow 0$. By hypothesis, $\mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(K, M) \longrightarrow 0$ is exact. Thus, $\mathrm{Ext}^1_R(N, M) = 0$, and hence (1) follows. □

The following note is useful for understanding the notations in Examples 11 and 15.

Note 10. From Introduction, $\mathscr{B} = \mathscr{X}^{\perp}$. We get a new cotorsion theory $({}^{\perp}(\mathscr{B}^{\perp}), \mathscr{B}^{\perp})$ generated by the class $\mathscr{B}$.

Example 11. Let $(R, \mathfrak{m})$ be a commutative Noetherian and complete local domain. Assume that the depth$R \geq 2$ and $\mathrm{cores.dim}_{\mathscr{B}}(R) \leq 1$. Then, $R/\mathfrak{m} \oplus E(R)$ is an $\mathscr{B}^{\perp}$-injective R-module, where $\mathscr{B}$ is the class of all $\mathscr{X}$-injective R-modules.

Proof. Consider the residue field $k = R/\mathfrak{m}$ and an exact sequence $0 \longrightarrow k \longrightarrow E(k) \longrightarrow^{\phi} E(k)/k \longrightarrow 0$. If G is an R-module, the sequence $\mathrm{Hom}_R(G, E(k)) \longrightarrow \mathrm{Hom}_R(G, E(k)/k) \longrightarrow \mathrm{Ext}^1_R(G, k) \longrightarrow 0$ is exact. By ([25], p 43), ϕ is an injective cover of $E(k)/k$. Since $\mathrm{cores.dim}_{\mathscr{B}}(R) \leq 1$, then, the class of all injective R-modules and the class of all $\mathscr{B}$-injective R-modules are equal, that is, $\mathscr{I}_0 = \mathscr{B}^{\perp}$. Clearly, ϕ is an $\mathscr{B}$-injective cover of $E(k)/k$. Thus, $\mathrm{Hom}_R(G', E(k)) \longrightarrow \mathrm{Hom}_R(G', E(k)/k)$ is surjective for every $\mathscr{B}$-injective R-module G'. We get $\mathrm{Ext}^1_R(G', k) = 0$ for every $\mathscr{B}$-injective R-module G', and hence, k is $\mathscr{B}^{\perp}$-injective. On the other hand, $E(R)$ is injective and so $\mathscr{B}^{\perp}$-injective. Therefore, $k \oplus E(R)$ is $\mathscr{B}^{\perp}$-injective. □

We now give some characterizations of $\mathscr{X}$-flat module:

Proposition 12. *The following are equivalent for a right R-module M:*

(1) *M is $\mathscr{X}$-flat*

(2) *For every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $C \in \mathscr{X}$, the functor $M \otimes_R -$ preserves the exactness*

(3) *$\mathrm{Ext}^1_R(M, G^+) = 0$ for all $G \in \mathscr{X}$*

(4) *M^+ is $\mathscr{X}$-injective*

Proof. (1) $\Rightarrow$ (2). Consider an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $C \in \mathscr{X}$. Since M is $\mathscr{X}$-flat, $\mathrm{Tor}^R_1(M, C) = 0$. Hence, the functor $M \otimes -$ preserves the exactness.

(2) $\Rightarrow$ (1). Let $G \in \mathscr{X}$. Then, there exists a short sequence $0 \longrightarrow K \longrightarrow F \longrightarrow G \longrightarrow 0$ with F a projective module, which induces an exact sequence $0 \longrightarrow \mathrm{Tor}^R_1(M, G) \longrightarrow M \otimes K \longrightarrow M \otimes F \longrightarrow M \otimes G \longrightarrow 0$. By hypothesis, $\mathrm{Tor}^R_1(M, G) = 0$. Thus, M is $\mathscr{X}$-flat.

(1) $\Leftrightarrow$ (3).It follows from the natural isomorphism ([26], p 34) $\mathrm{Tor}^R_1(M, G)^+ \cong \mathrm{Ext}^1_R(M, G^+)$.

(1) $\Leftrightarrow$ (4).It follows from the natural isomorphism ([27], VI 5.1) $\mathrm{Ext}^1_R(G, M^+) \cong \mathrm{Tor}^R_1(M, G)^+$. □

Proposition 13. *Let R be a coherent ring. Then, a right R-module N is flat if and only if N is $\mathscr{X}$-flat and $fd(N) \leq 1$.*

Proof. The "only if" part is trivial. Conversely, suppose that N is $\mathscr{X}$-flat. By Proposition 12, N^+ is $\mathscr{X}$-injective. By ([28], Theorem 2.1), $fd(N) = id(N^+)$. Then $\mathrm{cores.dim}_{\mathscr{X}^{\perp}}(N^+) \leq 1$ since $\mathrm{cores.dim}_{\mathscr{X}^{\perp}}(N^+) \leq id(N^+)$. By Proposition 8, N^+ is injective and hence N is flat. □

4. $\mathscr{X}^{\perp}$-Projective Cover and $\mathscr{X}$-Injective Envelope

Now, we introduce $\mathscr{X}$-projective modules.

Definition 14. An R-module M is called $\mathscr{X}$-projective if $\mathrm{Ext}^1_R(M, X) = 0$ for all R-modules $X \in \mathscr{X}$.

Example 15. Let $(R, \mathfrak{m})$ be a complete local ring and $\mathscr{B}$ be the class of all $\mathscr{X}$-injective R-modules. Assume that the $\mathrm{cores.dim}_{\mathscr{B}}(R) \leq 1$. Then $E(k)/k$ is $\mathscr{B}^{\perp}$-projective, where $k = R/\mathfrak{m}$ is the residue field and $E(M)$ is an $\mathscr{B}$-injective envelope of k.

Proof. By Note 10, we get a cotorsion theory $({}^{\perp}(\mathscr{B}^{\perp}), \mathscr{B}^{\perp})$ generated by the class $\mathscr{B}$. Consider an exact sequence $0 \longrightarrow k \longrightarrow E(k) \longrightarrow E(k)/k \longrightarrow 0$, where $E(k)$ is an injective envelope of k. Since $\mathrm{cores.dim}_{\mathscr{B}}(R) \leq 1$, then, the class of all injective R-modules and the class of all $\mathscr{B}$-injective R-modules coincide, that is, $\mathscr{I}_0 = \mathscr{B}^{\perp}$. It follows that $E(k)$ is an $\mathscr{B}$-injective envelope of k. Since the class of all $\mathscr{B}^{\perp}$-projective modules is closed under extensions, then by Lemma 2.1.2 in [25], $E(k)/k$ is $\mathscr{B}^{\perp}$-projective. □

We now introduce Definition 16.

Definition 16. A ring R is called left $\mathscr{X}$-hereditary if every left ideal of R is $\mathscr{X}$-projective.

Remark 17. If $\mathscr{X}$ is the class of injective left R-modules, then, every ring is $\mathscr{X}$-hereditary. It is also easy to see that a ring is left hereditary if and only if R is $\mathscr{X}$-hereditary for every class $\mathscr{X}$ of left R-modules.

Given a class $\mathscr{X}$ of left R-modules, we denote by $\mathscr{X}^{\perp_2}$ the class

$$\{N \in R-\mathrm{Mod} \mid \mathrm{Ext}_R^2(X, N) = 0, \quad \text{for every } X \in \mathscr{X}\}. \tag{6}$$

Example 18. Let R be a commutative Noetherian ring. If $\mathscr{X} = \{R/\mathfrak{p} : \mathfrak{p} \in \mathrm{Spec}R\}$, then R is a $\mathscr{X}^{\perp_2}$-hereditary ring.

Proof. Let I be an ideal of R. We claim that I is $\mathscr{X}^{\perp_2}$-projective. By hypothesis, $\mathrm{Ext}_R^2(R/\mathfrak{p}, G) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}R$ and for all $G \in \mathscr{X}^{\perp_2}$. It follows that $id(G) \leq 1$. Thus, $\mathrm{Ext}_R^2(R/I, G) = 0$ for all ideals I of R and for all $G \in \mathscr{X}^{\perp_2}$. Consider an exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. So $\mathrm{Ext}_R^1(I, G) = 0$ for all $G \in \mathscr{X}^{\perp_2}$. Hence, I is $\mathscr{X}^{\perp_2}$-projective, as desired.

Note that an R-module M is $\mathscr{X}^{\perp}$-projective if $\mathrm{Ext}_R^1(M, U) = 0$ for all R-modules $U \in \mathscr{X}^{\perp}$. Clearly, $({}^{\perp}(\mathscr{X}^{\perp}), \mathscr{X}^{\perp})$ is a cotorsion theory. □

Proposition 19. *A ring R is left $\mathscr{X}^{\perp}$-hereditary if and only if every submodule of a $\mathscr{X}^{\perp}$-projective left R-module is $\mathscr{X}^{\perp}$-projective.*

Proof. Let R be left $\mathscr{X}^{\perp}$-hereditary and I be a left ideal of R. Then, there is an exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. By hypothesis, $\mathrm{Ext}_R^2(R/I, U) = 0$ for any $\mathscr{X}$-injective left R-module U. Thus, $id(U) \leq 1$. Let G be a submodule of an $\mathscr{X}^{\perp}$-projective left R-module H. Then, for any $\mathscr{X}$-injective left R-module U, the sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^1(H, U) \longrightarrow \mathrm{Ext}_R^1(G, U) \longrightarrow \mathrm{Ext}_R^2\left(\frac{H}{G}, U\right) \longrightarrow \cdots \tag{7}$$

is exact. Thus, $\mathrm{Ext}_R^1(G, U) = 0$ since H is $\mathscr{X}^{\perp}$-projective and $id(U) \leq 1$. The reverse implication is clear.

In general, ${}^{\perp}(\mathscr{X}^{\perp})$ is not closed under pure submodules; for example, if $\mathscr{F}$ is the class of flat modules, then ${}^{\perp}(\mathscr{F}^{\perp}) = \mathscr{F}$ and this class is not closed under pure submodules in general. As an easy consequence of Proposition 19, we have that the class ${}^{\perp}(\mathscr{X}^{\perp})$ is closed under pure submodules over a $\mathscr{X}^{\perp}$-hereditary ring. □

Definition 20 (see [22]). Let $\mathscr{K}$ be a class of R-modules. Then, $\mathscr{K}$ is said to be Kaplansky class if there exists a cardinal $\aleph$ such that for every $M \in \mathscr{K}$ and for each $x \in M$, there exists a submodule K of M such that $x \in K \subseteq M$ and $K, M/K \in \mathscr{K}$, and $\mathrm{Card}(K) \leq \aleph$.

Definition 21. An R-module G is called ${}^{\perp}(\mathscr{X}^{\perp})$-pure injective if $f : A' \longrightarrow G$ extends to a homomorphism $g : A \longrightarrow G$ for all $A', A \in {}^{\perp}(\mathscr{X}^{\perp})$, where $A' \subseteq_{*} A$ (that is, A' is a pure submodule of A). We denote by $\mathscr{B}'$ the class of all ${}^{\perp}(\mathscr{X}^{\perp})$-pure projective modules.

Remark 22. The class of all pure injective R-modules is contained in the class of all ${}^{\perp}(\mathscr{X}^{\perp})$-pure injective R-modules.

Example 23.

(1) Clearly, $(R-\mathrm{Mod}, \mathscr{I}_0)$ is a cotorsion theory. Then, the class of all $R-Mod$-pure injective R-modules is the class of all pure injective R-modules.

(2) If $\mathscr{F}$ is the class of all flat R-modules and $\mathscr{C}$ is the class of all cotorsion R-modules, then, $(\mathscr{F}, \mathscr{C})$ is a cotorsion theory. Then, the class of all $\mathscr{F}$-pure injective R-modules is the class of all cotorsion R-modules.

If R is Noetherian and $R \in \mathscr{I}_0$, then, the class of all $\mathscr{I}_0$-pure injective R-modules is $\mathscr{I}_0^{\perp}$.

Proof.

(1) Straightforward

(2) Let C be a $\mathscr{F}$-pure injective R-module. We show that C is a cotorsion R-module. For any flat R-module F, there is a pure exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow F \longrightarrow 0$, where P is a projective R-module. Then, we get the following exact sequence:

$$\mathrm{Hom}_R(P, C) \longrightarrow \mathrm{Hom}_R(K, C) \longrightarrow \mathrm{Ext}_R^1(F, C) \longrightarrow 0 \tag{8}$$

It follows that $\mathrm{Ext}_R^1(F, C) = 0$ because $\mathrm{Hom}_R(P, C) \longrightarrow \mathrm{Hom}_R(K, C)$ is surjective. Hence, C is cotorsion. Conversely, let C be a cotorsion R-module. Suppose the following sequence:

$$0 \longrightarrow A \longrightarrow F \longrightarrow \frac{A}{F} \longrightarrow 0, \tag{9}$$

where $A, F \in \mathscr{F}$ is pure exact. Then, we get the exact sequence $\mathrm{Hom}_R(F, C) \longrightarrow \mathrm{Hom}_R(A, C) \longrightarrow \mathrm{Ext}_R^1(F/A, C) \longrightarrow 0$. It follows that $\mathrm{Ext}_R^1(F/A, C) = 0$ since F/A is flat. Hence C, is $\mathscr{F}$-pure injective.

(3) By Exercises 6 in [13], $(\mathscr{I}_0, \mathscr{I}_0^{\perp})$ is a cotorsion theory. Let G be an $\mathscr{I}_0$-pure injective R-module. We show that $G \in \mathscr{I}_0^{\perp}$. For any R-module $I_0 \in \mathscr{I}_0$, consider the following exact sequence:

$$\mathbf{E}_{\bullet} : 0 \longrightarrow K \longrightarrow P \longrightarrow I_0 \longrightarrow 0, \tag{10}$$

where P is projective. By hypothesis, P is injective. Since the cotorsion theory $(\mathscr{I}_0, \mathscr{I}_0^{\perp})$ has enough injectives, K is injective, that is $K \in \mathscr{I}_0$. It follows that the above sequence $\mathbf{E}_{\bullet}$ is pure exact. Then, we get the exact sequence $\mathrm{Hom}_R(P, G) \longrightarrow \mathrm{Hom}_R(K, G) \longrightarrow \mathrm{Ext}_R^1(I_0, G) \longrightarrow 0$. Thus, $\mathrm{Ext}_R^1(I_0, G) = 0$ since G is $\mathscr{I}_0$-pure injective. Hence, $G \in \mathscr{I}_0^{\perp}$. Conversely, Let G be an $\mathscr{I}_0$-injective R-module. Suppose the following exact sequence:

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \frac{E_2}{E_1} \longrightarrow 0, \tag{11}$$

where $E_1, E_2 \in \mathcal{I}_0$ is pure exact. It follows that $E_2/E_1 \in \mathcal{I}_0$ since the cotrosion theory $(\mathcal{I}_0, \mathcal{I}_0^{\perp})$ has enough injectives. Then, $\mathrm{Hom}_R(E_2, G) \longrightarrow \mathrm{Hom}_R(E_1, G)$ is surjective. Hence, G is $\mathcal{I}_0$-pure injective. □

Proposition 24. *Let be a $\mathcal{X}^{\perp}$-hereditary ring and $\mathcal{X}^{\perp} \subseteq \mathcal{B}'$. Then, ${}^{\perp}(\mathcal{X}^{\perp})$ is a Kaplansky class.*

Proof. Let $M \in {}^{\perp}(\mathcal{X}^{\perp})$ and $x \in M$. Consider the inclusion $< x > \longrightarrow M$, and we get by Lemma 5.3.12 in [13], a cardinal $\aleph_\alpha$ and a pure submodule $F \subseteq M$ such that $<x> \subseteq F$ and $\mathrm{Card}(F) \leq \aleph_\alpha$. We get the pure exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow \frac{M}{F} \longrightarrow 0. \tag{12}$$

By Proposition 19, F is $\mathcal{X}^{\perp}$-projective. It follows that

$$\mathrm{Hom}_R(M, G) \longrightarrow \mathrm{Hom}_R(F, G) \longrightarrow \mathrm{Ext}_R^1\left(\frac{M}{F}, G\right) \longrightarrow 0, \tag{13}$$

for any $\mathcal{X}$-injective R-module G. This implies that $\mathrm{Hom}_R(M, G) \longrightarrow \mathrm{Hom}_R(F, G)$ is surjective by hypothesis. Thus, $\mathrm{Ext}_R^1(M/F, G) = 0$. Hence, we proved the proposition. □

Theorem 25. *Let R be a $\mathcal{X}^{\perp}$-hereditary ring and $\mathcal{X}^{\perp} \subseteq \mathcal{B}'$. If the class of all $\mathcal{X}$-projective R -modules is closed under direct limits, then, every R-module M has a $\mathcal{X}^{\perp}$-projective cover and an $\mathcal{X}$-injective envelope.*

Proof. By Proposition 24, ${}^{\perp}(\mathcal{X}^{\perp})$ is a Kaplansky class. Since all projective modules are $\mathcal{X}^{\perp}$-projective, ${}^{\perp}(\mathcal{X}^{\perp})$ contains the projective modules. Clearly, ${}^{\perp}(\mathcal{X}^{\perp})$ is closed under extensions. By hypothesis, ${}^{\perp}(\mathcal{X}^{\perp})$ is closed under direct limits. Then by Theorem 2.9 in [22], $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$ is a perfect cotorsion theory. Hence, by Definition 4, every module has a ${}^{\perp}(\mathcal{X}^{\perp})$-cover and a $\mathcal{X}^{\perp}$-envelope. □

Note that if $\mathcal{X}^{\perp} \subseteq \mathcal{P}\mathcal{I}$, then the class of all $\mathcal{X}^{\perp}$-projective R-modules is closed under direct limits by Lemma 3.3.4 in [6]. Then, by Theorem 25 and Remark 22, we have the following.

Theorem 26. *Let R be a $\mathcal{X}^{\perp}$-hereditary ring and $\mathcal{X}^{\perp} \subseteq \mathcal{P}\mathcal{I}$. Then, every R-module M has a $\mathcal{X}^{\perp}$-projective cover and an $\mathcal{X}$ -injective envelope.*

Now we introduce a self-$\mathcal{X}$-injective ring.

Definition 27. A ring R is said to be self $\mathcal{X}$-injective if R over itself is an $\mathcal{X}$-injective module.

Example 28.

(1) If the class $\mathcal{X}$ is $R-\mathrm{Mod}$, then R is a self injective ring

(2) If $\mathcal{X}$ is the class of all finitely presented R-modules, then R is a self fp-injective ring

(3) If $\mathcal{X}$ is the class of all flat R-modules, then R is a cotorsion ring

We now give some characterizations of $\mathcal{X}^{\perp}$-projective module:

Proposition 29. *Let R be a self $\mathcal{X}$-injective ring, and let M be an R-module. Then, the following conditions are equivalent:*

(1) M is $\mathcal{X}^{\perp}$-projective

(2) M is projective with respect to every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, with A an $\mathcal{X}$-injective R -module

(3) For every exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, where P is $\mathcal{X}$-injective, $K \longrightarrow P$ is an $\mathcal{X}$-injective preenvelope of K

(4) M is cokernel of an $\mathcal{X}$-injective preenvelope $K \longrightarrow P$ with P is a projective R-module

Proof. $(1) \Rightarrow (2)$. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence, where A is $\mathcal{X}$-injective. Then, by hypothesis, $\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$ is surjective.

$(2) \Rightarrow (1)$. Let N be an $\mathcal{X}$-injective R-module. Then, there is a sequence $0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$ with E an injective envelope of N. By (2), $\mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, L)$ is surjective. Thus, $\mathrm{Ext}_R^1(M, N) = 0$, as desired.

$(1) \Rightarrow (3)$. Clearly, $\mathrm{Ext}_R^1(M, F') = 0$ for all $\mathcal{X}$-injective F'. Hence, we have an exact sequence $\mathrm{Hom}_R(P, F') \longrightarrow \mathrm{Hom}_R(K, F') \longrightarrow 0$, where P is projective.

$(3) \Rightarrow (4)$. Consider an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, where P projective. Since R is self-$\mathcal{X}$-injective, every projective module is $\mathcal{X}$-injective. Hence, P is $\mathcal{X}$ -injective. Then, by hypothesis, $K \longrightarrow P$ is an $\mathcal{X}$-injective preenvelope.

$(4) \Rightarrow (1)$. By hypothesis, there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, where $K \longrightarrow P$ is an $\mathcal{X}$-injective preenvelope with P projective. It gives rise to the exactness of $\mathrm{Hom}_R(P, N) \longrightarrow \mathrm{Hom}_R(K, N) \longrightarrow \mathrm{Ext}_R^1(M, N) \longrightarrow 0$ for each $\mathcal{X}$-injective R-module N. Since R is self $\mathcal{X}$-injective, $\mathrm{Hom}_R(P, N) \longrightarrow \mathrm{Hom}_R(K, N)$ is surjective. Hence, $\mathrm{Ext}_R^1(M, N) = 0$, as desired. □

Proposition 30. *Let R be a self $\mathscr{X}$-injective and $\mathscr{X}^{\perp}$-hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. Then the following are equivalent for an R-module M:*

(1) M is coreduced $\mathscr{X}^{\perp}$-projective

(2) M is a cokernel of an $\mathscr{X}$-injective envelope $K \longrightarrow P$ with P is a projective R-module

Proof. (1) ⇒ (2). Consider an exact sequence $0 \longrightarrow K \longrightarrow^{f} P \longrightarrow^{g} M \longrightarrow 0$ with P a projective module. Since R is self $\mathscr{X}$-injective, P is $\mathscr{X}$-injective. By Theorem 26, the natural map $f : K \longrightarrow P$ is an $\mathscr{X}$-injective preenvelope of K. By hypothesis, K has an $\mathscr{X}$-injective envelope $\alpha : K \longrightarrow P'$. Then, there exist $\beta : P' \longrightarrow P$ and $\beta' : P \longrightarrow P'$ such that $\alpha = \beta' f$ and $f = \beta\alpha$. Hence, $\alpha = (\beta\beta')\alpha$. It follows that $\beta\beta'$ is an isomorphism, $P = \mathrm{im}(\beta) \oplus \ker(\beta')$. Note that $\mathrm{im}(f) \subseteq \mathrm{im}(\beta)$, and so $P/\mathrm{im}(f) \longrightarrow P/\mathrm{im}(\beta) \longrightarrow 0$ is exact. But M is coreduced and $P/\mathrm{im}(f) \cong M$, and hence, $P/\mathrm{im}(\beta) = 0$, that is, $P = \mathrm{im}(\beta)$. So β is an isomorphism, and hence, $f : K \longrightarrow P$ is an $\mathscr{X}$-injective envelope of K.

(2) ⇒ (1). By Proposition 29, M is $\mathscr{X}^{\perp}$-projective and M is coreduced by Lemma 3.7 in [17]. □

We are now to prove the main result of this section.

Theorem 31. *Let R be a self $\mathscr{X}$-injective and $\mathscr{X}^{\perp}$-hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. Then, an R -module M is $\mathscr{X}^{\perp}$-projective if and only if M is a direct sum of a projective R-module and a coreduced $\mathscr{X}^{\perp}$-projective R-module.*

Proof. The "if" part is clear.

"Only if" part. Let M be a $\mathscr{X}^{\perp}$-projective R-module. By Proposition 29, we have an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P a projective module, where $K \longrightarrow P$ is an $\mathscr{X}$-injective preenvelope of K. By Theorem 26, K has an $\mathscr{X}$-injective envelope $f \in \mathrm{Hom}_R(K, P')$ with P' an $\mathscr{X}$-injective R-module. Then, we have the following diagram (Figure 1): □

Note that $\beta\alpha$ is an isomorphism, and so $P = \ker\beta \oplus \mathrm{im}\,\alpha$. Since $\mathrm{im}\,\alpha \cong P'$, P' and $\ker\beta$ are projective. Therefore, $P'/\mathrm{im} f$ is a coreduced $\mathscr{X}^{\perp}$-projective module by Proposition 30. By the Five Lemma, $\sigma\phi$ is an isomorphism. Hence, we have $M = \mathrm{im}\,\phi \oplus \ker\sigma$, where $\mathrm{im}\,\phi \cong P'/\mathrm{im} f$. In addition, we get the commutative diagram (Figure 2).

Hence, $\ker\sigma \cong \ker\beta$.

5. Some Relation between $\mathscr{X}^{\perp}$-Projective and $\mathscr{X}$-Injective Modules

In this section, we deals with $\mathscr{X}$-injective envelope of a module and $\mathscr{X}^{\perp}$-projective module for any class $\mathscr{X}$ in $R-\mathrm{Mod}$.

Theorem 32. *Let $\phi : M \longrightarrow A$ be an $\mathscr{X}$-injective envelope. Then, $L = A/\phi(M)$ is $\mathscr{X}^{\perp}$-projective, and hence, A is $\mathscr{X}^{\perp}$-projective whenever M is $\mathscr{X}^{\perp}$-projective.*

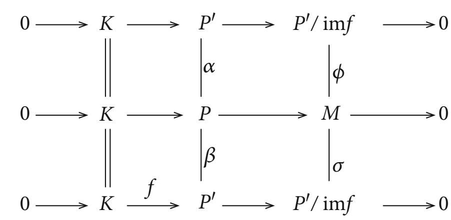

Figure 1: Commutative diagram with exact rows.

Proof. It follows from Lemma 2.1.2 in [25]. □

Theorem 33. *Let $0 \longrightarrow M \longrightarrow A \longrightarrow D \longrightarrow 0$ be a minimal generator of all $\mathscr{X}^{\perp}$-projective extensions of M. Then, A is an $\mathscr{X}$-injective envelope of M.*

Proof. It follows from Theorem 2.2.1 in [25]. □

Let M be a submodule of a module A. Then, A is called a $\mathscr{X}^{\perp}$-projective extension of a submodule M if A/M is $\mathscr{X}^{\perp}$-projective.

Recall that among all $\mathscr{X}^{\perp}$-projective extensions of M, we call one of them $0 \longrightarrow M \longrightarrow A \longrightarrow D \longrightarrow 0$ a generator for $\mathscr{E}xt({}^{\perp}(\mathscr{X}^{\perp}), M)$ (or a generator for all $\mathscr{X}^{\perp}$-projective extensions of M) if for any $\mathscr{X}^{\perp}$-projective extension $0 \longrightarrow M \longrightarrow A' \longrightarrow D' \longrightarrow 0$ of M, then there is a diagram (Figure 3).

Furthermore, a generator $0 \longrightarrow M \longrightarrow A \longrightarrow D \longrightarrow 0$ is called minimal if for all the vertical maps are isomorphisms whenever A', D' are replaced by A, D, respectively.

Theorem 34. *Let R be a $\mathscr{X}^{\perp}$-hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. Then, for an R-module M, there must be a minimal generator whenever $\mathscr{E}xt({}^{\perp}(\mathscr{X}^{\perp}), M)$ has a generator.*

Proof. It follows from Theorem 2.2.2 in [25]. □

Theorem 35. *Suppose that an R-module M has an $\mathscr{X}$-injective envelope. Let M be a submodule of an $\mathscr{X}$-injective R -module L. Then, the following are equivalent:*

(1) $i : M \longrightarrow L$ is a special $\mathscr{X}$-injective envelope

L/M is $\mathscr{X}^{\perp}$-projective, and there are no direct summands L_1 of L with $L_1 \neq L$ and $M \subseteq L_1$

(2) L/M is $\mathscr{X}^{\perp}$-projective, and for any epimorphism $\alpha : L/M \longrightarrow N$ such that $\alpha\pi$ is split, $N = 0$, where $\pi : L \longrightarrow L/M$ is the canonical map

(3) L/M is $\mathscr{X}^{\perp}$-projective, and any endomorphism γ of L such that $\gamma i = i$ is a monomorphism

(4) L/M is $\mathscr{X}^{\perp}$-projective, and there is no nonzero submodule N of L such that $M \cap N = 0$ and $L = (M \oplus N)$ is $\mathscr{X}^{\perp}$-projective

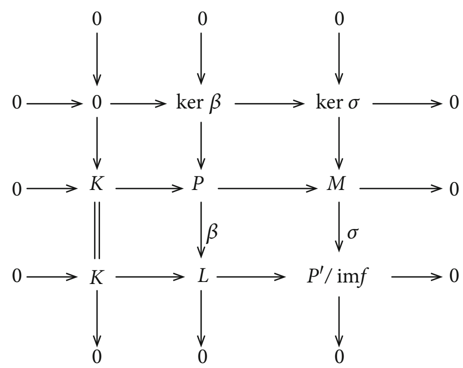

Figure 2: Pullback diagram.

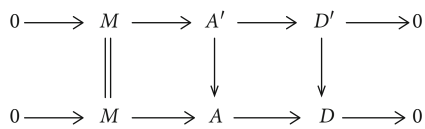

Figure 3: Commutative diagram.

Proof. (1) $\Leftrightarrow$ (2). It follows from Corollary ~1.2.3 in [25] and Theorem 32.

(2) $\Rightarrow$ (3). Since $\alpha\pi$ is split, there is a monomorphism $\beta : N \longrightarrow L$ such that $L = \ker(\alpha\pi) \oplus \beta(N)$. Note that $M \subseteq \ker(\alpha\pi)$, and so $L = \ker(\alpha\pi)$ by (2). Thus, $\beta(N) = 0$, and hence, $N = 0$.

(3) $\Rightarrow$ (2). If $L = L_1 \oplus N$ with $M \subseteq L_1$. Let $p : L \longrightarrow N$ be a canonical projection. Then, there is an epimorphism $\alpha : L/M \longrightarrow N$ such that $\alpha\pi = p$. Thus, $N = 0$ by hypothsis, and hence, $L = L_1$, as required.

(1) $\Rightarrow$ (4). By Wakamatsu's Lemma ([13], Proposition ~7.2.4), L/M is $\mathscr{X}^{\perp}$-projective. Since $\gamma i = i$ and i is monomorphism, γ is monomorphism.

(4) $\Rightarrow$ (1). Since L/M is $\mathscr{X}^{\perp}$-projective, i is a special $\mathscr{X}$-injective preenvelope. Let $\psi : M \longrightarrow \mathscr{X}^{\perp}(M)$ be an $\mathscr{X}$-injective envelope of M. Then, there exist $\mu : L \longrightarrow \mathscr{X}^{\perp}(M)$ and $\nu : \mathscr{X}^{\perp}(M) \longrightarrow L$ such that $\mu i = \psi$ and $\nu\psi = i$. Hence $\mu\nu\psi = \psi$ and $i = \nu\mu i$. Thus, $\mu\nu$ is an isomorphism, and so μ is epic. In addition, by (4), $\nu\mu$ is monic, and hence, μ is monic. Therefore, μ is an isomorphism, and hence, i is an $\mathscr{X}$-injective envelope of M.

(1) $\Rightarrow$ (5). It is obvious that L/M is $\mathscr{X}^{\perp}$-projective. Suppose there is a nonzero submodule $N \subseteq L$ such that $M \cap N = 0$ and $L = (M \oplus N)$ is $\mathscr{X}^{\perp}$-projective. Let $\pi : L \longrightarrow L/N$ be a canonical map. Since $L/(N \oplus M)$ is $\mathscr{X}^{\perp}$-projective and L is $\mathscr{X}$-injective, there is a $\beta : L/N \longrightarrow L$ in the following diagram (Figure 4) □

Hence, $\beta\pi i = i$. Note that i is an envelope, and so $\beta\pi$ is an isomorphism, whence π is an isomorphism. But this is impossible since $\pi(N) = 0$.

(5) $\Rightarrow$ (1). Let $\psi_M : M \longrightarrow \mathscr{X}^{\perp}(M)$ be an $\mathscr{X}$-injective envelope of M. Since L/M is $\mathscr{X}^{\perp}$-projective, i is a special $\mathscr{X}$-injective preenvelope. Thus, we have the following diagram shown in Figure 5.

That is, $f\psi_M = i, gi = \psi_M$. So $gf\psi_M = \psi_M$. Note that ψ_M is an $\mathscr{X}$-injective envelope, and hence, gf is an isomorphism. Without loss of generality, we may assume $gf = 1$. Write $\alpha = \phi g : L \longrightarrow Q$. It is clear that α is epic and $M \cap \ker(g) = 0$. We show that $M \oplus \ker(g) = \ker(\alpha)$. Clearly, $M \oplus \ker(g) \subseteq \ker(\alpha)$. Let $x \in \ker(\alpha)$. Then, $\alpha(x) = \phi g(x) = 0$. It follows that $g(x) = \psi_M(m)$ for some $m \in M$, and hence, $fg(x) = f\psi_M(m) = m$, $g(x) = gfg(x) = g(m)$. Thus, $x \in M \oplus \ker(g)$, and so $\ker(\alpha) \subseteq M \oplus \ker(g)$, as desired. Consequently, $L = (M \oplus \ker(g)) = L/\ker(\alpha) \cong Q$ is $\mathscr{X}^{\perp}$-projective by Wakamatsu's Lemma. Thus, $\ker(g) = 0$ by hypothesis, and hence, g is an isomorphism. So $i : M \longrightarrow L$ is an $\mathscr{X}$-injective envelope.

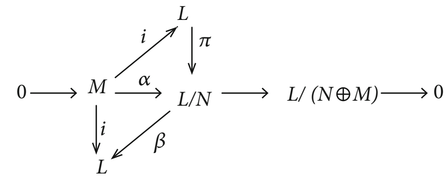

Figure 4: Commutative diagram with exact rows.

Theorem 36. *Let R be a $\mathscr{X}^{\perp}$-hereditary ring and $\mathscr{X}^{\perp} \subseteq \mathscr{P}\mathscr{I}$. If M is a submodule of an $\mathscr{X}$-injective R-module A, then, the following are equivalent:*

(1) $i : M \longrightarrow A$ is a special $\mathscr{X}$-injective envelope of M

A is a $\mathscr{X}^{\perp}$-projective essential extension of M.

Proof. (1) $\Rightarrow$ (2). It follows by Proposition 35.

(2) $\Rightarrow$ (1). By hypothesis, we have an exact sequence: $0 \longrightarrow M \longrightarrow A \longrightarrow L \longrightarrow 0$ with A an $\mathscr{X}$-injective module and D an $\mathscr{X}^{\perp}$-projective module. This sequence is a generator of all $\mathscr{X}^{\perp}$-projective extensions of M. By Theorems 33 and 34, we have an $\mathscr{X}^{\perp}$-projective extension sequence of M $0 \longrightarrow M \longrightarrow A' \longrightarrow L' \longrightarrow 0$ which gives an $\mathscr{X}$-injective envelope of M. Then, we have the diagram shown in Figure 6. □

It is easy to see that $A = f(A') \oplus \ker(g)$. We claim that $\ker(g) = 0$. Since $M = f\alpha'(M) \subseteq f(A')$, $\ker(g) \cap M = 0$. We define the following homomorphism $\psi : A/(M \oplus \ker(g)) \longrightarrow L'$, $a + (M \oplus \ker(g)) \mapsto \alpha' g(a)$. Obviously, ψ is well defined. By diagram chasing, we see that ψ is injective. But both g and β' are surjective, so is ψ. Therefore, $\ker(g)$ is $\mathscr{X}^{\perp}$-projective essential extension of $A/\ker(g)$. This contradicts the hypothesis that A is $\mathscr{X}^{\perp}$-projective essential extension of M. This implies that $\ker(g) = 0$ and so f is an isomorphism.

6. $\mathscr{W}$-Injective Cover

In this section, we assume $\mathscr{W}$ is the class of all pure projective R-modules and we prove that all modules have $\mathscr{W}$-injective covers.

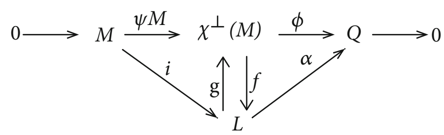

Figure 5: Commutative diagram with exact rows.

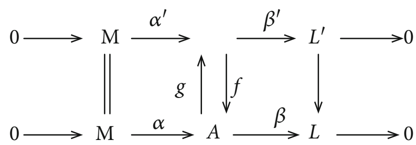

Figure 6: Commutative diagram.

Proposition 37. *The class $\mathscr{W}^{\perp}$ of all $\mathscr{W}$-injective modules is closed under pure submodules.*

Proof. Let A be a pure submodule of a $\mathscr{W}$-injective module M. Then there is a pure exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow M/A \longrightarrow 0$ and a functor $\mathrm{Hom}_R(X, -)$ preseves this sequence is exact whenever $X \in \mathscr{W}$. This implies that the sequence $0 \longrightarrow \mathrm{Hom}_R(W, A) \longrightarrow \mathrm{Hom}_R(W, M) \longrightarrow \mathrm{Hom}_R(W, M/A) \longrightarrow \mathrm{Ext}^1_R(W, A) \longrightarrow 0$ is also exact for all $W \in \mathscr{W}$. It follows that $\mathrm{Ext}^1_R(W, A) = 0$ for all $W \in \mathscr{W}$, as desired. □

Theorem 38. *Every R-module has a $\mathscr{W}$-injective preenvelope.*

Proof. Let M be an R-module. By Lemma 5.3.12 in [13], there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $\phi : M \longrightarrow G$ with G a $\mathscr{W}$-injective R-module, there exists a pure submodule A of G such that $|A| \leq \aleph_\alpha$ and $\phi(M) \subset A$. Clearly, $\mathscr{W}^{\perp}$ is closed under direct products and by Proposition 37 A is $\mathscr{W}$-injective. Hence, the theorem follows Proposition 6.2.1 in [13]. □

Proposition 39. *The class $\mathscr{W}^{\perp}$ of all $\mathscr{W}$-injective modules is injectively resolving.*

Proof. Let $0 \longrightarrow M_1 \longrightarrow^{\phi} M_2 \longrightarrow^{\psi} M_3 \longrightarrow 0$ be an exact sequence of left R-modules with $M_1, M_2 \in \mathscr{W}^{\perp}$. Let $G \in \mathscr{W}^{\perp}$. By Theorem 38, every module has a $\mathscr{W}^{\perp}$-preenvelope. By Lemma 1.9 in [29], G has a special $\mathscr{W}^{\perp}$-preenvelope. By Lemma 2.2.6 in [6], G has a special $^{\perp}(\mathscr{W}^{\perp})$-precover. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow A \longrightarrow G \longrightarrow 0$ with $A \in {}^{\perp}(\mathscr{W}^{\perp})$ and $K \in \mathscr{W}^{\perp}$. We prove that M_3 is $\mathscr{W}$-injection, i.e., to prove that $\mathrm{Ext}^1_R(G, M_3) = 0$. For this, it suffices to extend any $\alpha \in \mathrm{Hom}_R(K, M_3)$ to an element of $\mathrm{Hom}_R(A, M_3)$. Clearly, K has $^{\perp}(\mathscr{W}^{\perp})$-precover,

$$0 \longrightarrow K' \xrightarrow{f} A' \xrightarrow{g} K \longrightarrow 0, \tag{14}$$

where K, $K' \in \mathscr{W}^{\perp}$, and $A' \in {}^{\perp}(\mathscr{W}^{\perp})$. As the class $\mathscr{W}^{\perp}$ is closed under extensions, $A' \in \mathscr{W}^{\perp}$. Since $\alpha \circ g : A' \longrightarrow M_3$ with $A' \in \mathscr{W}^{\perp}$ and M_1 a $\mathscr{W}$-injective module, then there exists $\beta : A' \longrightarrow M_2$ such that $\psi \circ \beta = \alpha \circ g$. That is, we have the diagram shown in Figure 7. □

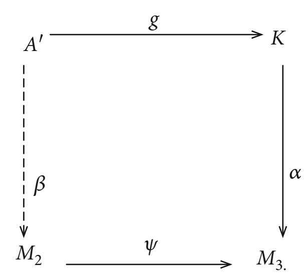

Figure 7: Commutative diagram.

Now, we define $\beta\restriction_{\mathrm{im}\phi} : A' \longrightarrow im\phi$, where $\restriction$ is a restriction map. Then, there exists $\gamma : K' \longrightarrow M_1$ such that $\beta\restriction_{im\phi} (f(K')) = \phi\gamma(K')$. Hence, we have the diagram shown in Figure 8.

The $\mathscr{W}$-injectivity of M_1 yields a homomorphism $\gamma_1 : A' \longrightarrow M_1$ such that $\gamma = \gamma_1 \circ f$. So for each $k' \in K'$, we get $(\beta \circ f)(k') = (\phi \circ \gamma)(k') = (\phi \circ (\gamma_1 \circ f))(k')$. Then, there exists a map $\beta_1 \in \mathrm{Hom}_R(K, M_2)$ such that $\beta = \beta_1 \circ g$, and we get $\alpha = \psi \circ \beta_1$. Thus, we have the diagram shown in Figure 9.

Since M_2 is $\mathscr{W}$-injective, there exists $\rho \in \mathrm{Hom}_R(A, M_2)$ such that $\beta_1 = \rho \circ f$. Thus, $\alpha = \psi \circ \beta_1 = \psi \circ (\rho \circ f)$, where $\psi \circ \rho \in \mathrm{Hom}_R(A, M_3)$. Hence, M_3 is $\mathscr{W}$-injective.

Proposition 40. *Let M be a $\mathscr{W}$-injective R-module and A be a pure submodule M. Then, an R-module M/A is $\mathscr{W}$-injective.*

Proof. By Proposition 39, $\mathscr{W}^{\perp}$ is injectively resolving. Let $M \in \mathscr{W}^{\perp}$ and A be a pure submodule of M. By Proposition 37, A is $\mathscr{W}$-injective. From the short exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow M/A \longrightarrow 0$, we get M/A is $\mathscr{W}$-injective. □

The terminology of the Definition 41 is used in [30].

Definition 41 (see [31]). An R -module M is $fp - \Omega^1$(small $- \Omega^1$) if there is a projective resolution

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{\phi} P_0 \longrightarrow M \longrightarrow 0, \tag{15}$$

such that the first syzygy $\Omega^1(\mathbf{P}_\bullet) = \mathrm{im}(\phi)$ is finitely presented (small). We denote by $\mathscr{C}$ the class of all $fp - \Omega^1 R$ -modules and $\mathscr{S}$ the class of all small $- \Omega^1$-modules.

Theorem 42 (see [31], Theorem 2.3). *Let M be an R-module.*

(1) If M is an $fp - \Omega^1$-module, then $Ext^1_R(M, -)$ commutes with direct limits

(2) If M is a small-$fp - \Omega^1$-module, then $Ext^1_R(M, -)$ commutes with direct sums

The converses of these statements are true when an R -module M is finitely generated.

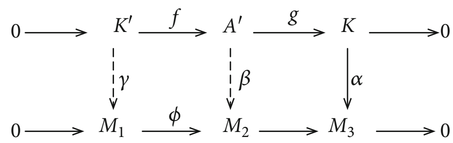

Figure 8: Commutative diagram.

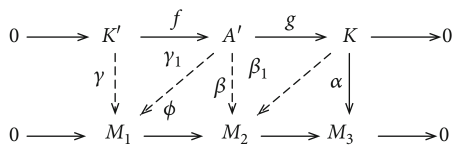

Figure 9: Commutative diagram.

We recall that an R-module M is pure projective if and only if M is a direct summand of a direct sum of finitely presented R-modules [32]. Warfield proved that every pure projective R-module is a direct sum of finitely presented modules over a Noetherian local ring in Corollary 4 in [33]. Further, Puninski and Rothmalar [18] proved that the general question of when all pure projective modules are direct sum of finitely presented R-modules. They proved this over a hereditary Noetherian ring ([18], Corollary 6.5). Thus, every pure projective R-module is finitely presented over a hereditary Noetherian ring. It follows that $\mathrm{Ext}_R^1(A,-)$ commutes with direct limits for all pure projective R-modules A. By Theorem 42, every pure projective R-module is $fp-\Omega^1$ over a hereditary Noetherian ring.

The converse part of the Proposition 43 is not necessarily true. It will be held immediately over a hereditary Noetherian ring by the argument of the previous paragraph. In this article, we proved the converse part of Proposition 43 over a Noetherian ring by Theorem 45; that is, $\mathscr{W}$-injective R-modules are injective from ([34], Theorem 3).

Proposition 43. *Every $\mathscr{W}$-injective R-module is absolutely pure.*

Proof. Every finitely presented R-module is pure projective, hence the proposition. □

Theorem 44. *Let R be a perfect ring. Then the following conditions are equivalent:*

(1) R is coherent

(2) The class of $\mathscr{W}$-injective R-modules is closed under direct limits and closed under direct sums

(3) Every absolutely pure R-module is injective

Proof. $(1)\Rightarrow(2)$. It follows by Corollary 57 (2) and (3) in [30].

$(2)\Rightarrow(3)$. By Proposition 43 and Theorem 3.2 in [1], R is coherent. A perfect coherent ring R is Noetherian. Then, we get (3) immediately by Theorem 3 in [34]. □

Let $\mathscr{FP}$ denote the class of all absolutely pure R-modules.

Theorem 45. *The following conditions are equivalent:*

(i) R is Noetherian

(ii) $\mathscr{W}$-injective R-modules coincide with the injective R-modules

(iii) Absolutely pure R-modules coincide with injective R-modules

Proof. $(i)\Rightarrow(ii)$. By Proposition 43, every $\mathscr{W}$-injective R-module is an absolutely pure R-module. It follows that every absolutely pure R-module is injective by Theorem 3 in [34]. Hence, every $\mathscr{W}$-injective R-module is injective.

$(ii)\Rightarrow(iii)$. Clearly, $\mathscr{I}_0\subseteq\mathscr{FP}\subseteq\mathscr{W}^{\perp}$. By (ii), $\mathscr{W}^{\perp}\subseteq\mathscr{I}_0$. Thus $\mathscr{I}_0=\mathscr{FP}$

$(iii)\Rightarrow(i)$. It follows Theorem 3 in [34]. □

By Theorem 45, $\mathscr{W}$-injective cover and injective cover for every R-module are coincide over a Noetherian ring. Therefore, we prove Theorem 46 and Proposition 47 without Noetherian condition.

The following result establishes an analog version of Theorem 2.6 in [16].

Theorem 46. *Suppose $\mathscr{W}\subseteq\mathscr{S}$. Then, every R-module M has a $\mathscr{W}$-injective precover.*

Proof. Let W be a set with $\mathrm{Card}(W)\le\kappa$, where κ is the cardinal in [11] (Theorem 5). Denote $\mathscr{P}(W)$ the power set of W. We find all the binary operations $*: G\times G\longrightarrow G$ for each element $G\in\mathscr{P}(W)$, and we get a new collection $\cup_{G\in\mathscr{P}(W)}\{G,*\}=\bar{\mathscr{G}}'$. From $\bar{\mathscr{G}}'$, find all the scalar multiplications, which are functions from the cross product into itself. This remains a set $\cup_{G\in\mathscr{P}(W)}\{(G,*,\cdot)\}$ which is denoted by $\bar{\mathscr{G}}$. Some collection of members of $\bar{\mathscr{G}}$ form a module, and we can get the class $\mathscr{G}$ of $\mathscr{W}$-injective modules which is contained in the class $\mathscr{W}^{\perp}$. Clearly, $\mathscr{I}_0$ is contained in the class $\mathscr{G}$. Since $\mathscr{W}\subseteq\mathscr{S}$, then by Theorem 42 (2) the direct sum of $\mathscr{W}$-injective R-modules is $\mathscr{W}$-injective. Hence $\oplus_{N\in\mathscr{I}_0}N^{(\mathrm{Hom}_R(N,M))}$ is an $\mathscr{W}$-injective R-module because every injective module is $\mathscr{W}$-injective. We prove that $\oplus_{N\in\mathscr{I}_0}N^{(\mathrm{Hom}_R(N,M))}\stackrel{\phi}{\longrightarrow}M$ is a $\mathscr{W}$-injective precover. That is, to show that if for any homomorphism $N'\longrightarrow M$ with $N'\in\mathscr{W}^{\perp}$, then we have the following diagram (Figure 10): □

Let K be the kernel of the map $N'\longrightarrow M$, where N' is sufficiently large. Then, N'/K is sufficiently small since $|N'/K|\le|M|$. By Bican et al.'s Theorem [11], K has a nonzero submodule L that is pure in N'. Therefore, L is $\mathscr{W}$-injective by Proposition 37. This implies that N'/L is $\mathscr{W}$-injective by Proposition 40. If N'/L is still sufficiently large, then repeat the process from the map $N'/L\longrightarrow M$. Since N'/K_1 is sufficiently small, K_1/L has a nonzero submodule

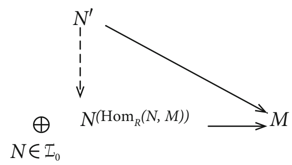

Figure 10: Commutative diagram.

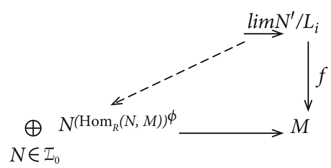

Figure 11: Commutative diagram.

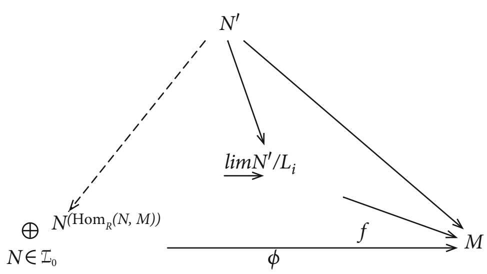

Figure 12: Commutative diagram.

L_1/K_1 that is pure in N'/L. Thus, $(N'/K_1)/(L_1/K_1) = N'/L_1$ is $\mathscr{W}$-injective. But again, this may be too large. Then by continuing this process we get $\varinjlim (N'/L_i)$ is $\mathscr{W}$-injective since $\mathscr{W}^{\perp}$ is closed under direct limits, and it is sufficiently small, namely $|\varinjlim (N'/L_i)| \leq \kappa$. Then the map $N' \longrightarrow M$ can be factored through a $\mathscr{W}$-injective module $\varinjlim (N'/L_i)$. Let $f \in \mathrm{Hom}_R(\varinjlim (N'/L_i), M)$. We define a map $\bar{f} : \varinjlim (N'/L_i) \longrightarrow \oplus N^{(\mathrm{Hom}_R(N,M))}$ such that $\bar{f}(n' + L') = (n_f, 0, 0, \cdots)$, with $n_f = n$. Clearly, $\bar{f}$ is a linear map. Then, we have the diagram shown in Figure 11.

Hence, we get the diagram shown in Figure 12.

Thus, $\bigoplus_{N\in\mathscr{I}_0} N^{(\mathrm{Hom}_R(N,M))} \longrightarrow^{\phi} M$ is a $\mathscr{W}$-injective precover.

By Theorem 42 (1), the class of all $\mathscr{W}$-injective R-modules are closed under direct limits if $\mathscr{W} \subseteq \mathscr{C}$. But we prove this result over a coherent in Proposition 47.

Proposition 47. *Let R be a coherent ring. Then, the class of all $\mathscr{W}$-injective R-modules is closed under direct limits.*

Proof. It follows Corollary 57(2) shown in [30]. □

Theorem 48 follows from Theorem 46 and Proposition 47 by Corollary 5.2.7 in [13].

Theorem 48. *Let R be a coherent ring and $\mathscr{W} \subseteq \mathscr{S}$. Then, every R-module has a $\mathscr{W}$-injective cover.*

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

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