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Research Article

Reduction an Inverse Problem to a System of Second Kind Fredholm Integral Equations with No Singularities

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In this article, we deal with an inverse problem concerning the two-dimensional Laplace equation with local boundary conditions on a bounded region. In this problem, the goal is to reduce it into a system of Fredholm integral equations of the second kind involving kernels with weakly/or no singularities (Fredholm property) by considering some additional conditions on the parameters of the problem. Finally, the method is carried out on an example, to show the simplicity and efficiency of the method.

1. Introduction

The inverse problems arise in modeling of many physical and geophysical phenomena, such as elastography and medical imaging, seismology, potential theory, ion transport problems or chromatography, and finances (see, for example, [1–4]). In many cases of these problems which have been studied for the first time by Lavrentiev [5], the goal is to determine the coefficients or the right hand side of the differential equations for some known data about its solutions. There are many numerical methods to solve these problems. Among them, we cite the method of fundamental solutions (MFS) by Marin and Lesnic [6] and Wang et al. [7] and by Chen et al. [8] and Sun and He [9] for the two-dimensional and three-dimensional inverse problems, respectively, the boundary function method (BFM) by Wang et al. [10], the boundary particle method (BPM) by Chen and Fu [11], the variational iteration method (VIM) by Canon and Tatari [12], the globally convergent numerical method by Baysal [13], the weighted homotopy analysis method (WHAM) by Shidfar et al. [14], and the conjugate gradient method (CGM) by Lu et al. [15]. In some special cases, these problems are solved by the analytical methods by Liua and Tatar [16] or Liu [17].

In 1997, Aliev and Jahanshahi [18] proposed a new approach for reducing the direct problem concerning the mixed PDE with nonlocal boundary conditions into a system of Fredholm integral equations with weak singularities, i.e., the boundary values of the unknown function and their derivatives satisfy some of the Fredholm integral equations with weak singularities. The system obtained can be solved by some of the numerical methods, such as semiorthogonal B-spline wavelet collocation method by Sahau and Saha Ray [19], the method based on Bernstein polynomial by Basit and Khan [20], triangular functions method by Almasieh and Roodaki [21], homotopy perturbation method by Javidi and Golbabai [22], and Taylor-series expansion method by Maleknejad et al. [23].

In this article, based on the method presented in [18], we study an inverse boundary value problem for a two-dimensional Laplace equation which is a logical continuation of the paper [24] concerning the inverse problem of Cauchy-Riemann equation with nonlocal boundary conditions.

In the bounded domain $\Omega \subset \mathbb{R}^2$ (see Figure 1) where its boundary $\Gamma \coloneqq \partial\Omega$ is given as

$$\Gamma \coloneqq \Gamma_1 \cup \Gamma_2, \quad \Gamma_j : x_2 = \gamma_j(x_1), j = 1, 2, \tag{1}$$

where $\gamma_j (j = 1, 2)$ are known functions satisfying

$$\gamma_1(x_1) < \gamma_2(x_1) \text{ for all } x_1 \in [a, b], \tag{2}$$

we consider an inverse boundary value problem of the two-dimensional Laplace equation

$$\Delta u(x) = 0 \text{ in } \Omega, \tag{3}$$

subject to the boundary conditions

$$\sum_{i=1}^{2} \alpha_{ij}(x_1) \frac{\partial u}{\partial x_i} \left(x_1, \gamma_j(x_1)\right) = \phi_j(x_1), \tag{4}$$

$$u\left(x_1, \gamma_j(x_1)\right) = \psi_j(x_1), x_1 \in [a, b], j = 1, 2, \tag{5}$$

where $\alpha_{ij}, \phi_j (i, j = 1, 2)$ and ψ_1 are known C^1 functions on $[a, b]$. The interest is then to recover the unknown function ψ_2 based on a Fredholm integral equation of the second kind with respect to boundary values of the unknown function u and its first-order partial derivatives.

2. Main Results

Our first main result is stated as follows:

Theorem 1. *The solution u of Equation (3) at internal points of the domain can be expressed by Green's formula:*

$$u(\xi) = \int_\Gamma u(x) \frac{\partial G}{\partial n}(x;\xi) d\Gamma(x) - \int_\Gamma \frac{\partial u}{\partial n}(x) G(x;\xi) d\Gamma(x), \xi \in \Omega, \tag{6}$$

where n is the exterior normal to the boundary Γ and $G(x;\xi)$ is the fundamental solution of 2D Laplace operator satisfying

$$\Delta G(x;\xi) = \delta(x - \xi), \tag{7}$$

which is given by

$$G(x;\xi) = \frac{1}{2\pi} \ln |x - \xi|, \tag{8}$$

where $\delta(x - \xi)$ is the Dirac measure at point $\xi = (\xi_1, \xi_2)$ [25]. Moreover, the boundary values $u|_\Gamma$ and $(\partial u/\partial \xi_j)|_\Gamma (j = 1, 2)$ should satisfy the following boundary integral equations:

$$\frac{1}{2} u(\xi) = \int_\Gamma u(x) \frac{\partial G}{\partial n}(x;\xi) d\Gamma(x) - \int_\Gamma \frac{\partial u}{\partial n}(x) G(x;\xi) d\Gamma(x), \xi \in \Gamma, \tag{9}$$

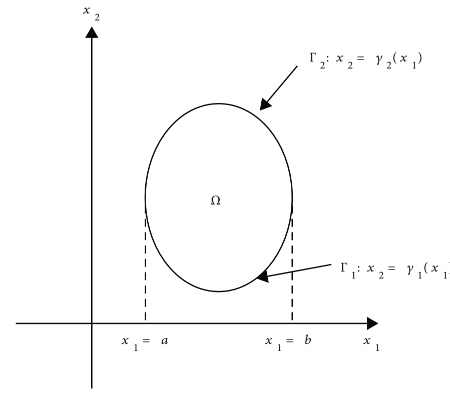

Figure 1: The domain Ω with boundaries Γ_1 and Γ_2.

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_1} = \int_\Gamma \frac{\partial G}{\partial x_2}(x;\xi) \left\{ \frac{\partial u(x)}{\partial x_1} \cos(n, x_2) - \frac{\partial u(x)}{\partial x_2} \cos(n, x_1) \right\} d\Gamma(x) + \int_\Gamma \frac{\partial u}{\partial n}(x) \frac{\partial G}{\partial x_1}(x;\xi) d\Gamma(x), \xi \in \Gamma, \tag{10}$$

$$\frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_2} = \int_\Gamma \frac{\partial G}{\partial x_1}(x;\xi) \left\{ \frac{\partial u(x)}{\partial x_2} \cos(n, x_1) - \frac{\partial u(x)}{\partial x_1} \cos(n, x_2) \right\} d\Gamma(x) + \int_\Gamma \frac{\partial u}{\partial n}(x) \frac{\partial G}{\partial x_2}(x;\xi) d\Gamma(x), \xi \in \Gamma, \tag{11}$$

where (n, x_j) is the angle between the exterior normal vector n to the boundary Γ and coordinate axes x_j, $(j = 1, 2)$.

Proof. The first-order partial derivatives of Equation (8) are as follows:

$$\frac{\partial G}{\partial x_j}(x;\xi) = \frac{1}{2\pi} \times \frac{x_j - \xi_j}{|x - \xi|^2}, j = 1, 2. \tag{12}$$

Multiplying both sides of Equation (3) by (8) and (12), respectively, integrating over Ω, using the divergence theorem [25] (similar to [24]), the proof is completed. □

Let (n_i, x_j) and (τ_i, x_j) be the angles between exterior normal and unit tangent vectors on the boundaries $\Gamma_i (i = 1, 2)$ with coordinate axes $x_j (j = 1, 2)$, respectively. Then

$$\cos(n_i, x_1) = (-1)^{i+1} \sin(\tau_i, x_1), \cos(n_i, x_2) = (-1)^i \cos(\tau_i, x_1), \tag{13}$$

$$\tan(\tau_i, x_1) = \gamma_i'(x_1), \cos(\tau_i, x_1) = \frac{dx_1}{d\Gamma(x)}, i = 1, 2. \tag{14}$$

Taking into account arcs of the boundary Γ as Γ_1 and Γ_2, plugging (12)–(14) in (9)–(11), we obtain the following boundary integral equations:

$$u|_{\Gamma_j} = u\left(\xi_1, \gamma_j(\xi_1)\right) = \int_a^b k_{1j}^{(1)}(x_1,\xi_1) u|_{\Gamma_1}\, dx_1 - \int_a^b k_{2j}^{(1)}(x_1,\xi_1) u|_{\Gamma_2}\, dx_1 - 2\int_a^b \gamma_1'(x_1)\, G(x;\xi)|_{\Gamma_1\Gamma_j} \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_1} dx_1 + 2\int_a^b G(x;\xi)|_{\Gamma_1\Gamma_j} \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_1} dx_1 + 2\int_a^b \gamma_2'(x_1)\, G(x;\xi)|_{\Gamma_2\Gamma_j} \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_2} dx_1 - 2\int_a^b G(x;\xi)|_{\Gamma_2\Gamma_j} \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_2} dx_1, \; j=1,2, \tag{15}$$

$$\frac{\partial u}{\partial \xi_1}\bigg|_{\Gamma_j} = \frac{\partial u}{\partial \xi_1}\left(\xi_1, \gamma_j(\xi_1)\right) = \int_a^b k_{1j}^{(1)}(x_1,\xi_1) \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_1} dx_1 - \int_a^b k_{2j}^{(1)}(x_1,\xi_1) \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_2} dx_1 - \int_a^b k_{1j}^{(2)}(x_1,\xi_1) \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_1} dx_1 + \int_a^b k_{2j}^{(2)}(x_1,\xi_1) \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_2} dx_1, \; j=1,2, \tag{16}$$

$$\frac{\partial u}{\partial \xi_2}\bigg|_{\Gamma_j} = \frac{\partial u}{\partial \xi_2}\left(\xi_1, \gamma_j(\xi_1)\right) = \int_a^b k_{1j}^{(2)}(x_1,\xi_1) \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_1} dx_1 - \int_a^b k_{2j}^{(2)}(x_1,\xi_1) \frac{\partial u}{\partial x_1}\bigg|_{\Gamma_2} dx_1 + \int_a^b k_{1j}^{(1)}(x_1,\xi_1) \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_1} dx_1 - \int_a^b k_{2j}^{(1)}(x_1,\xi_1) \frac{\partial u}{\partial x_2}\bigg|_{\Gamma_2} dx_1, \; j=1,2, \tag{17}$$

where

$$G(x;\xi)|_{\Gamma_i\Gamma_j} = \frac{1}{4\pi} \ln\left\{(x_1-\xi_1)^2 + \left(\gamma_i(x_1)-\gamma_j(\xi_1)\right)^2\right\}, i,j=1,2, \tag{18}$$

$$k_{ij}^{(1)}(x_1,\xi_1) \coloneqq \frac{1}{\pi}\left\{\gamma'_i(x_1)\frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_i\Gamma_j} - \frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_i\Gamma_j}\right\}, \tag{19}$$

$$k_{ij}^{(2)}(x_1,\xi_1) \coloneqq \frac{1}{\pi}\left\{\gamma'_i(x_1)\frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_i\Gamma_j} + \frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_i\Gamma_j}\right\}, i,j=1,2. \tag{20}$$

As we see, the kernels $k_{ii}^{(j)}(i,j=1,2)$ in (20) contain singularities. To remove them, we state the following theorem.

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the boundary $\partial\Omega$ as Lyapunov curve in Cartesian coordinates $x=(x_1,x_2)$ which is convex in direction x_2 and assume that (for $i,j=1,2$), $\psi_1, \phi_j, \alpha_{ij} \in C^1[a,b]$ and $\alpha_{2j}(x_1) \neq 0$ (for all $x_1 \in [a,b]$). Then the singularities in kernels $k_{ii}^{(1)}(i=1,2)$ are regularized. Moreover, the unknown function ψ_2 in (5) can be expressed as the following boundary integral equation with no singularities*

$$\psi_2(\xi_1) = \int_a^b K_{01}^{(0)}(x_1,\xi_1)\psi_1(x_1)\,dx_1 + \int_a^b K_{02}^{(0)}(x_1,\xi_1)\psi_2(x_1)\,dx_1 + \int_a^b K_{11}^{(0)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\bigg|_{\Gamma_1} dx_1 + \int_a^b K_{12}^{(0)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\bigg|_{\Gamma_2} dx_1 + \int_a^b K_1^{(0)}(x_1,\xi_1)\phi_1(x_1)dx_1 + \int_a^b K_2^{(0)}(x_1,\xi_1)\phi_2(x_1)dx_1, \tag{21}$$

where

$$K_{01}^{(0)}(x_1,\xi_1) \coloneqq \frac{1}{\pi}\left\{\gamma_{1'}(x_1)\frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2} - \frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2}\right\}, \tag{22}$$

$$K_{02}^{(0)}(x_1,\xi_1) \coloneqq -\frac{\gamma_{2''}(x_1)}{\pi\left(1+\gamma_{2'}^2(x_1)\right)}, \tag{23}$$

$$K_{1j}^{(0)}(x_1,\xi_1) \coloneqq \frac{(-1)^j}{\pi\alpha_{2j}(x_1)}\left\{\alpha_{2j}(x_1)\gamma_{j'}(x_1) + \alpha_{1j}(x_1)\right\} \ln|x-\xi|_{\Gamma_j\Gamma_2}, \tag{24}$$

$$K_j^{(0)}(x_1,\xi_1) \coloneqq \frac{(-1)^{j+1}}{\pi\alpha_{2j}(x_1)} \ln|x-\xi|_{\Gamma_j\Gamma_2}, j=1,2. \tag{25}$$

Proof. Applying the mean value theorem twice on $\gamma_i(i=1,2)$, we get the kernels $k_{ii}^{(1)}(i=1,2)$ with no singularities,

$$k_{ii}^{(1)}(x_1,\xi_1) = \frac{1}{\pi}\left\{\frac{(x_1-\eta_1)\gamma''_i(\zeta_1)}{(x_1-\xi_1)\left(1+\gamma_i'^2(\eta_1)\right)}\right\} \simeq \frac{\gamma''_i(x_1)}{\pi\left(1+\gamma_i'^2(x_1)\right)}, i=1,2, \tag{26}$$

where η_1 and ζ_1 are between x_1, ξ_1 and x_1, η_1, respectively. □

From the boundary condition (4), we get

$$\frac{\partial u}{\partial x_2}\bigg|_{\Gamma_j} = \frac{1}{\alpha_{2j}(x_1)}\left\{\phi_j(x_1) - \alpha_{1j}(x_1)\frac{\partial u}{\partial x_1}\bigg|_{\Gamma_j}\right\}, j=1,2. \tag{27}$$

Taking into account that $\alpha_{2j}(x_1) \neq 0 (j=1,2)$, for all $x_1 \in [a,b]$, plugging the boundary condition (5) and relations (27) in Equation (15) (for $j=2$) and considering (20) and (26) with $k_{12}^{(1)}$ and $k_{22}^{(1)}$, respectively, the integral equation (21) can be resulted.

Similar to Equation (26), we obtain

$$k_{ii}^{(2)}(x_1,\xi_1)=\frac{1}{\pi}\left\{\frac{\gamma_i'(x_1)\gamma_i'(\eta_1)+1}{(x_1-\xi_1)\left(1+\gamma_i'^2(\eta_1)\right)}\right\}\simeq\frac{1}{\pi(x_1-\xi_1)},\ i=1,2, \tag{28}$$

where η_1 is between x_1 and ξ_1.

For regularization and removing singularities in $k_{ii}^{(2)}(i=1,2)$, we construct the following two linear combinations with unknown coefficients from the boundary integral equations (16)–(17):

$$\begin{aligned}A_{11}(\xi_1)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_1}+A_{21}(\xi_1)\frac{\partial u}{\partial\xi_2}\Big|_{\Gamma_1}&=-A_{11}(\xi_1)\int_a^b k_{11}^{(2)}(x_1,\xi_1)\frac{\partial u}{\partial x_2}\Big|_{\Gamma_1}dx_1\\&+A_{21}(\xi_1)\int_a^b k_{11}^{(2)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1+\cdots=\int_a^b k_{11}^{(2)}(x_1,\xi_1)A_{11}^*(x_1,\xi_1)\frac{\partial u}{\partial x_2}\Big|_{\Gamma_1}dx_1\\&-\int_a^b k_{11}^{(2)}(x_1,\xi_1)A_{11}(x_1)\frac{\partial u}{\partial x_2}\Big|_{\Gamma_1}dx_1-\int_a^b k_{11}^{(2)}(x_1,\xi_1)A_{21}^*(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1\\&+\int_a^b k_{11}^{(2)}(x_1,\xi_1)A_{21}(x_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1+\cdots,\end{aligned} \tag{29}$$

$$\begin{aligned}A_{12}(\xi_1)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_2}+A_{22}(\xi_1)\frac{\partial u}{\partial\xi_2}\Big|_{\Gamma_2}&=-\int_a^b k_{22}^{(2)}(x_1,\xi_1)A_{12}^*(x_1,\xi_1)\frac{\partial u}{\partial x_2}\Big|_{\Gamma_2}dx_1\\&+\int_a^b k_{22}^{(2)}(x_1,\xi_1)A_{12}(x_1)dx_1+\int_a^b k_{22}^{(2)}(x_1,\xi_1)A_{22}^*(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_2}dx_1\\&-\int_a^b k_{22}^{(2)}(x_1,\xi_1)A_{22}(x_1)dx_1+\cdots,\end{aligned} \tag{30}$$

where $A_{ij}^*(x_1,\xi_1)\coloneqq A_{ij}(x_1)-A_{ij}(\xi_1),\ (i,j=1,2)$.

The notation " $\cdots$ " in the right-hand side of (29)–(30) stands for all integrals without singularities.

Plugging $A_{11}=-\alpha_{21},A_{21}=\alpha_{11}$ and $A_{12}=\alpha_{22},A_{22}=-\alpha_{12}$ in (29)–(30), respectively, and considering boundary conditions (4), we get

$$\begin{aligned}-\alpha_{21}(\xi_1)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_1}+\alpha_{11}(\xi_1)\frac{\partial u}{\partial\xi_2}\Big|_{\Gamma_1}&=-\int_a^b\Big\{\alpha_{21}(\xi_1)k_{11}^{(1)}(x_1,\xi_1)\\&+\alpha_{11}^*(x_1,\xi_1)k_{11}^{(2)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1\\&+\int_a^b\Big\{\alpha_{21}(\xi_1)k_{21}^{(1)}(x_1,\xi_1)-\alpha_{11}(\xi_1)k_{21}^{(2)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_1}\Big|_{\Gamma_2}dx_1\\&+\int_a^b\Big\{\alpha_{11}(\xi_1)k_{11}^{(1)}(x_1,\xi_1)-\alpha_{21}^*(x,\xi_1)k_{11}^{(2)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_2}\Big|_{\Gamma_1}dx_1\\&-\int_a^b\Big\{\alpha_{21}(\xi_1)k_{21}^{(2)}(x_1,\xi_1)+\alpha_{11}(\xi_1)k_{21}^{(1)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_2}\Big|_{\Gamma_2}dx_1\\&+\int_a^b k_{11}^{(2)}(x_1,\xi_1)\phi_1(x_1)dx_1,\end{aligned} \tag{31}$$

$$\begin{aligned}\alpha_{22}(\xi_1)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_2}-\alpha_{12}(\xi_1)\frac{\partial u}{\partial\xi_2}\Big|_{\Gamma_2}&=\int_a^b\Big\{k_{12}^{(1)}(x_1,\xi_1)\alpha_{22}(\xi_1)\\&-k_{12}^{(2)}(x_1,\xi_1)\alpha_{12}(\xi_1)\Big\}\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1\\&-\int_a^b\Big\{k_{12}^{(2)}(x_1,\xi_1)\alpha_{22}(\xi_1)+k_{12}^{(1)}(x_1,\xi_1)\alpha_{12}(\xi_1)\Big\}\frac{\partial u}{\partial x_2}\Big|_{\Gamma_1}dx_1\\&-\int_a^b\Big\{k_{22}^{(1)}(x_1,\xi_1)\alpha_{22}(\xi_1)+\alpha_{12}^*(x_1,\xi_1)k_{22}^{(2)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_1}\Big|_{\Gamma_2}dx_1\\&+\int_a^b\Big\{k_{22}^{(1)}(x_1,\xi_1)\alpha_{12}(\xi_1)-\alpha_{22}^*(x_1,\xi_1)k_{22}^{(2)}(x_1,\xi_1)\Big\}\frac{\partial u}{\partial x_2}\Big|_{\Gamma_2}dx_1.\\&+\int_a^b k_{22}^{(2)}(x_1,\xi_1)\phi_2(x_1)dx_1.\end{aligned} \tag{32}$$

Plugging (27) in (31)–(32), respectively, we obtain

$$\begin{aligned}\left(\alpha_{11}^2(\xi_1)+\alpha_{21}^2(\xi_1)\right)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_1}&=\int_a^b K_{11}^{(1)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1\\&+\int_a^b K_{12}^{(1)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_2}dx_1+\int_a^b K_1^{(1)}(x_1,\xi_1)\phi_1(x_1)dx_1\\&+\int_a^b K_2^{(1)}(x_1,\xi_1)\phi_2(x_1)dx_1+\alpha_{11}(\xi_1)\phi_1(\xi_1),\end{aligned} \tag{33}$$

$$\begin{aligned}\left(\alpha_{12}^2(\xi_1)+\alpha_{22}^2(\xi_1)\right)\frac{\partial u}{\partial\xi_1}\Big|_{\Gamma_2}&=\int_a^b K_{11}^{(2)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_1}dx_1\\&+\int_a^b K_{12}^{(2)}(x_1,\xi_1)\frac{\partial u}{\partial x_1}\Big|_{\Gamma_2}dx_1+\int_a^b K_1^{(2)}(x_1,\xi_1)\phi_1(x_1)dx_1\\&+\int_a^b K_2^{(2)}(x_1,\xi_1)\phi_2(x_1)dx_1,+\alpha_{12}(\xi_1)\phi_2(\xi_1),\end{aligned} \tag{34}$$

where

$$\begin{aligned}K_{12}^{(1)}(x_1,\xi_1)\coloneqq\frac{\alpha_{21}(\xi_1)}{\alpha_{22}(x_1)}\{&-(\alpha_{21}(\xi_1)\alpha_{22}(x_1)\\&+\alpha_{11}(\xi_1)\alpha_{12}(x_1))k_{21}^{(1)}(x_1,\xi_1)+(\alpha_{11}(\xi_1)\alpha_{22}(x_1)\\&-\alpha_{21}(\xi_1)\alpha_{12}(x_1))k_{21}^{(2)}(x_1,\xi_1)\},\end{aligned} \tag{35}$$

$$\begin{aligned}K_1^{(1)}(x_1,\xi_1)\coloneqq\frac{\alpha_{21}(\xi_1)}{\alpha_{21}(x_1)}\Big\{&(\alpha_{21}^*(x_1,\xi_1)-\alpha_{21}(x_1))k_{11}^{(2)}(x_1,\xi_1)\\&-\alpha_{11}(\xi_1)k_{11}^{(1)}(x_1,\xi_1)\Big\},\end{aligned} \tag{36}$$

$$K_2^{(1)}(x_1,\xi_1)\coloneqq\frac{\alpha_{21}(\xi_1)}{\alpha_{22}(x_1)}\Big\{\alpha_{11}(\xi_1)k_{21}^{(1)}(x_1,\xi_1)+\alpha_{21}(\xi_1)k_{21}^{(2)}(x_1,\xi_1)\Big\}, \tag{37}$$

$$K_{11}^{(2)}(x_1,\xi_1) := \frac{\alpha_{22}(\xi_1)}{\alpha_{21}(x_1)}\left\{(\alpha_{22}(\xi_1)\alpha_{21}(x_1)+\alpha_{11}(x_1)\alpha_{12}(\xi_1))k_{12}^{(1)}(x_1,\xi_1) + (\alpha_{22}(\xi_1)\alpha_{11}(x_1)-\alpha_{12}(\xi_1)\alpha_{21}(x_1))k_{12}^{(2)}(x_1,\xi_1)\right\}, \tag{38}$$

$$K_{12}^{(2)}(x_1,\xi_1) := -\frac{\alpha_{22}(\xi_1)}{\alpha_{22}(x_1)}\left\{(\alpha_{22}(\xi_1)\alpha_{22}(x_1)+\alpha_{12}(x_1)\alpha_{12}(\xi_1))k_{22}^{(1)}(x_1,\xi_1) + (\alpha_{12}^{*}(x_1,\xi_1)\alpha_{22}(x_1)-\alpha_{12}(x_1)\alpha_{22}^{*}(x_1,\xi_1))k_{22}^{(2)}(x_1,\xi_1)\right\}, \tag{39}$$

$$K_{1}^{(2)}(x_1,\xi_1) := -\frac{\alpha_{22}(\xi_1)}{\alpha_{21}(x_1)}\left\{\alpha_{12}(\xi_1)k_{12}^{(1)}(x_1,\xi_1)+\alpha_{22}(\xi_1)k_{12}^{(2)}(x_1,\xi_1)\right\}, \tag{40}$$

$$K_{2}^{(2)}(x_1,\xi_1) := \frac{\alpha_{22}(\xi_1)}{\alpha_{22}(x_1)}\left\{\alpha_{12}(\xi_1)k_{22}^{(1)}(x_1,\xi_1)+(\alpha_{22}(x_1) - \alpha_{22}^{*}(x_1,\xi_1))k_{22}^{(2)}(x_1,\xi_1)\right\}. \tag{41}$$

Finally, plugging the kernels (20) (for $i, j = 1, 2$, $i \neq j$), (26), and (28) in kernels (37)–(41), we get

$$K_{11}^{(1)}(x_1,\xi_1) = \frac{\alpha_{21}(\xi_1)}{\pi\alpha_{21}(x_1)}\left\{(\alpha_{21}(\xi_1)\alpha_{21}(x_1)+\alpha_{11}(x_1)\alpha_{11}(\xi_1))\frac{\gamma_{1''}(x_1)}{1+\gamma_{1'}^{2}(x_1)} + \left({\alpha_{11}}'(x_1)\alpha_{21}(x_1)-\alpha_{21}'(x_1)\alpha_{11}(x_1)\right)\right\}, \tag{42}$$

$$\begin{aligned}K_{12}^{(1)}(x_1,\xi_1) = -\frac{\alpha_{21}(\xi_1)}{\pi\alpha_{22}(x_1)}\Bigg\{&\frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_2\Gamma_1}((\alpha_{12}(x_1)\alpha_{21}(\xi_1) \\ &-\alpha_{11}(\xi_1)\alpha_{22}(x_1))\gamma_{2'}(x_1)-(\alpha_{22}(x_1)\alpha_{21}(\xi_1)+\alpha_{11}(\xi_1)\alpha_{12}(x_1))) \\ &+\frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_2\Gamma_1}((\alpha_{22}(x_1)\alpha_{21}(\xi_1)+\alpha_{11}(\xi_1)\alpha_{12}(x_1))\gamma_{2'}(x_1) \\ &+(\alpha_{12}(x_1)\alpha_{21}(\xi_1)-\alpha_{11}(\xi_1)\alpha_{22}(x_1)))\Bigg\},\end{aligned} \tag{43}$$

$$K_{1}^{(1)}(x_1,\xi_1) = -\frac{\alpha_{21}(\xi_1)}{\pi\alpha_{21}(x_1)}\left\{\alpha_{11}(\xi_1)\frac{\gamma_{1''}(x_1)}{1+\gamma_{1'}^{2}(x_1)} - {\alpha_{21}}'(x_1)\right\} - \frac{\alpha_{21}(\xi_1)}{\pi(x_1-\xi_1)}, \tag{44}$$

$$K_{2}^{(1)}(x_1,\xi_1) = \frac{\alpha_{21}(\xi_1)}{\pi\alpha_{22}(x_1)}\left\{\frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_2\Gamma_1}\left(\alpha_{21}(\xi_1)\gamma_2'(x_1)-\alpha_{11}(\xi_1)\right) + \frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_2\Gamma_1}(\alpha_{11}(\xi_1)\gamma_{2'}(x_1)+\alpha_{21}(\xi_1))\right\}, \tag{45}$$

$$\begin{aligned}K_{11}^{(2)}(x_1,\xi_1) = \frac{\alpha_{22}(\xi_1)}{\pi\alpha_{21}(x_1)}\Bigg\{&\frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2}((\alpha_{11}(x_1)\alpha_{22}(\xi_1) \\ &-\alpha_{21}(x_1)\alpha_{12}(\xi_1))\gamma_{1'}(x_1)-(\alpha_{21}(x_1)\alpha_{22}(\xi_1) \\ &+\alpha_{11}(x_1)\alpha_{12}(\xi_1)))+\frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2}((\alpha_{21}(x_1)\alpha_{22}(\xi_1) \\ &+\alpha_{11}(x_1)\alpha_{12}(\xi_1))\gamma_1'(x_1)+\alpha_{11}(x_1)\alpha_{22}(\xi_1) \\ &-\alpha_{21}(x_1)\alpha_{12}(\xi_1))\Bigg\},\end{aligned} \tag{46}$$

$$\begin{aligned}K_{12}^{(2)}(x_1,\xi_1) = -\frac{\alpha_{22}(\xi_1)}{\pi\alpha_{22}(x_1)}\Bigg\{&(\alpha_{22}(x_1)\alpha_{22}(\xi_1) \\ &+\alpha_{12}(\xi_1)\alpha_{12}(x_1))\frac{\gamma_{2''}(x_1)}{1+\gamma_{2'}^{2}(x_1)} \\ &+\left(\alpha_{22}(x_1)\alpha_{12}'(x_1)-\alpha_{12}(x_1)\alpha_{22}'(x_1)\right)\Bigg\},\end{aligned} \tag{47}$$

$$\begin{aligned}K_{1}^{(2)}(x_1,\xi_1) = -\frac{\alpha_{22}(\xi_1)}{\pi\alpha_{21}(x_1)}\Bigg\{&\frac{x_1-\xi_1}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2}(\alpha_{22}(\xi_1) \\ &+\alpha_{12}(\xi_1)\gamma_{1'}(x_1))+\frac{x_2-\xi_2}{|x-\xi|^2}\bigg|_{\Gamma_1\Gamma_2}(\alpha_{22}(\xi_1)\gamma_{1'}(x_1) \\ &-\alpha_{12}(\xi_1))\Bigg\},\end{aligned} \tag{48}$$

$$K_{2}^{(2)}(x_1,\xi_1) = \frac{\alpha_{22}(\xi_1)}{\pi\alpha_{22}(x_1)}\left\{\alpha_{12}(\xi_1)\frac{\gamma_{2''}(x_1)}{1+\gamma_{2'}^{2}(x_1)} - {\alpha_{22}}'(x_1)\right\} + \frac{\alpha_{22}(\xi_1)}{\pi(x_1-\xi_1)}. \tag{49}$$

According to what explained above, the following theorems can be resulted:

Theorem 3. *Under the assumptions of Theorem 2, and also*

$$\phi_i(a) = \phi_i(b) = 0, i = 1, 2, \tag{50}$$

the unknown function ψ_2 in (5) can be recovered as a system of boundary integral equations (21) and (33)–(34) subject to kernels (25) and (45)–(49), respectively, with no singularities.

Theorem 4. *Under the assumptions of Theorem 3, the main problem (3)–(5) is reduced to a system of boundary integral equations, i.e., the unknown function u can be written as the integral equation Equation (6) where the existing boundary values can be expressed as a system of boundary integral Equations (21), (33)–(34) and relations (5) and (27) with no singularities.*

The application of the method described will be illustrated in the following example.

Example 1. Consider the Laplace equation

$$\Delta u(x) = 0 \text{ in } \Omega, \tag{51}$$

subject to the following boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_j} = \phi_j(x_1), \tag{52}$$

$$u|_{\Gamma_j} = \psi_j(x_1), j = 1, 2, \tag{53}$$

where the boundary $\Gamma \coloneqq \partial\Omega$ of the elliptic domain $\Omega \subset \mathbb{R}^2$ has the following polar equation (see Example 1 in [26])

$$\gamma(\theta) = 19\sqrt{\frac{1}{361 + 39\sin^2(\theta)}}, 0 \le \theta \le 2\pi, \tag{54}$$

or in the Cartesian coordinates,

$$x^2 + \frac{y^2}{b^2} = 1, b = \frac{19}{20}, \tag{55}$$

and the boundary Γ has the following form

$$\begin{aligned} &\Gamma = \Gamma_1 \cup \Gamma_2, \\ &\Gamma_j : x_2 = \gamma_j(x_1) \coloneqq (-1)^j b\sqrt{1 - x_1^2}, j = 1, 2. \end{aligned} \tag{56}$$

Moreover, the functions $\phi_j (j = 1, 2)$ and ψ_1 in (53) are known.

The exact solution of (51) and also the unknown boundary value ψ_2 in (53) are given by

$$\begin{aligned} u(x_1, x_2) &= x_1(1 - 2x_1^2) - 2x_1x_2(3 + 2x_1^2) + 2x_1x_2^2(3 + 2x_2), \\ \psi_2(x_1) &= x_1 - 2x_1^3 - 2bx_1(3 + 2x_1^2)\sqrt{1 - x_1^2} \\ &\quad + 2b^2x_1(1 - x_1^2)\left(3 + 2b\sqrt{1 - x_1^2}\right). \end{aligned} \tag{57}$$

Comparing the boundary conditions (53) with (4)–(5) yields that

$$\begin{aligned} \alpha_{1j}(x_1) &= -b^2x_1, \alpha_{2j}(x_1) = (-1)^{j+1}b\sqrt{1 - x_1^2}, j = 1, 2, \\ \phi_j(x_1) &= b^2x_1\{6(3 + b)x_1^2 - 6(2 + b) - 1\} \\ &\quad + (-1)^j bx_1\sqrt{1 - x_1^2}\{4x_1^2(b^4 + 6b^2 + 1) \\ &\quad - 2b^2(2b^2 + 3) + 1\}, j = 1, 2, \\ \psi_1(x_1) &= x_1 - 2x_1^3 + 2bx_1(2x_1^2 + 3)\sqrt{1 - x_1^2} \\ &\quad + 2b^2x_1(1 - x_1^2)\left(3 - 2b\sqrt{1 - x_1^2}\right). \end{aligned} \tag{58}$$

As we see, the domain Ω as well as $\alpha_{ij}, \phi_j (i, j = 1, 2)$ and ψ_1 satisfies the hypotheses of Theorem 4. Therefore, from (25) and (45)–(49), we obtain

$$\begin{aligned} K_{01}^{(0)}(x_1, \xi_1) &= \frac{1}{\pi} \times \frac{1}{\sqrt{1 - x_1^2}} \\ &\quad \times \frac{b}{(x_1 - \xi_1)^2 + b^2\left(\sqrt{1 - x_1^2} + \sqrt{1 - \xi_1^2}\right)^2} \\ &\quad \times \left\{1 - x_1\xi_1 + \sqrt{(1 - x_1^2)(1 - \xi_1)^2}\right\}, \end{aligned} \tag{59}$$

$$K_{02}^{(0)}(x_1, \xi_1) = \frac{1}{\pi} \times \frac{b}{1 + (b^2 - 1)x_1^2} \times \frac{1}{\sqrt{1 - x_1^2}}, \tag{60}$$

$$K_{1j}^{(0)}(x_1, \xi_1) = 0, j = 1, 2, \tag{61}$$

$$\begin{aligned} K_j^{(0)}(x_1, \xi_1) &= \frac{1}{\pi} \times \frac{1}{2b\sqrt{1 - x_1^2}} \ln\left\{(x_1 - \xi_1)^2\right. \\ &\quad \left. + b^2\left(\sqrt{1 - x_1^2} + \sqrt{1 - \xi_1^2}\right)^2\right\}, j = 1, 2, \end{aligned} \tag{62}$$

$$\begin{aligned} K_{1i}^{(i)}(x_1, \xi_1) &= \frac{-1}{\pi} \times \frac{\sqrt{1 - \xi_1^2}}{1 - x_1^2} b^3 \\ &\quad \cdot \left\{1 - \frac{\sqrt{(1 - x_1^2)(1 - \xi_1)^2} + b^2x_1\xi_1}{(b^2 - 1)x_1^2 + 1}\right\}, i = 1, 2, \end{aligned} \tag{63}$$

$$\begin{aligned} K_{1j}^{(i)}(x_1, \xi_1) &= \frac{-1}{\pi} \times \frac{\sqrt{1 - \xi_1^2}}{1 - x_1^2} \\ &\quad \times \frac{b^3\left((b^2 - 1)x_1^2 + 1\right)}{(x_1 - \xi_1)^2 + b^2\left(\sqrt{1 - x_1^2} + \sqrt{1 - \xi_1^2}\right)^2} \\ &\quad \times \left\{x_1\xi_1 - \sqrt{(1 - x_1^2)(1 - \xi_1^2)} - 1\right\} i, j = 1, 2, i \ne j, \end{aligned} \tag{64}$$

$$K_i^{(i)}(x_1, \xi_1) = \frac{-1}{\pi} \times \frac{\sqrt{1 - \xi_1^2}}{1 - x_1^2} b\left\{x_1 - \frac{b^2\xi_1}{(b^2 - 1)x_1^2 + 1}\right\}, i = 1, 2, \tag{65}$$

$$\begin{aligned} K_j^{(i)}(x_1, \xi_1) &= \frac{-1}{\pi} \times \frac{\sqrt{1 - \xi_1^2}}{1 - x_1^2} \\ &\quad \times \frac{b(x_1 - \xi_1)}{(x_1 - \xi_1)^2 + b^2\left(\sqrt{1 - x_1^2} + \sqrt{1 - \xi_1^2}\right)^2} \\ &\quad \times \left\{(1 - b^2)\sqrt{(1 - x_1^2)(1 - \xi_1^2)} - b^2\right\}, i, j = 1, 2, i \ne j. \end{aligned} \tag{66}$$

Plugging the kernels (66) in (33)–(34), a system of second kind Fredholm integral equations with respect to

$(\partial u/\partial x_1)|_{\Gamma_i}(i=1,2)$ is obtained which can be solved by some of numerical methods.

In addition, after substituting the kernels (62) as well as $(\partial u/\partial x_1)|_{\Gamma_i}(i=1,2)$ into the integral equation (21), the unknown function ψ_2 can be resulted.

Finally, plugging the boundary values $(\partial u/\partial x_1)|_{\Gamma_i}(i=1,2)$ in (27) as well as ψ_2 in (6), the unknown function u in (51) is obtained.

3. Conclusion

In this paper, we studied an inverse boundary value problem including Laplace equation with nonlocal boundary conditions in a two-dimensional convex bounded domain with the boundary as Lyapunov curve. By considering some of the extra conditions, the problem is reduced to a system of second kind Fredholm integral equations with no singularities. This method can be used to any inverse boundary value problem where its fundamental solution is known.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final form of the manuscript. Also, they carried out the proofs and conceived of the study.

References

[1] A. G. Ramm, *Inverse Problems: Mathematical and Analytical Techniques with Applications to Engineering*, Springer, New York, 2005.

[2] M. Bertero, *Introduction to Inverse Problems in Imaging*, Taylor & Francis, 1992.

[3] V. B. Glasko, *Inverse Problems of Mathematical Physics*, American Institute of Physics, 1988.

[4] V. G. Romanov, *Inverse Problems in Mathematical Physics*, VNU Science Press, The Netherlands, 1987.

[5] M. M. Lavrentiev, "Inverse Problems and Integral Geometric," in *Lectures on some non-correct Mathematical Physics and Analysis Problems*, pp. 81–86, Science Publishers, 1984.

[6] L. Marin and D. Lesnic, "The method of fundamental solutions for inverse boundary value problems associated with the two-dimensional biharmonic equation," *Mathematical and Computer Modelling*, vol. 42, no. 3-4, pp. 261–278, 2005.

[7] F. Wang, C.-M. Fan, Q. Hua, and Y. Gu, "Localized MFS for the inverse Cauchy problems of two-dimensional Laplace and biharmonic equations," *Applied Mathematics and Computation*, vol. 364, p. 124658, 2020.

[8] Z. Chen, F. Wang, S. Cheng, and G. Wu, "Localized MFS for three-dimensional acoustic inverse problems on complicated domains," *International Journal of Mechanical System Dynamics*, vol. 2, no. 1, pp. 143–152, 2022.

[9] Y. Sun and S. He, "A meshless method based on the method of fundamental solution for three- dimensional inverse heat conduction problems," *International Journal of Heat and Mass Transfer*, vol. 108, pp. 945–960, 2017.

[10] F. Wang, Q. Hua, and C.-S. Liu, "Boundary function method for inverse geometry problem in two-dimensional anisotropic heat conduction equation," *Applied Mathematics Letters*, vol. 84, pp. 130–136, 2018.

[11] W. Chen and Z. J. Fu, "Boundary particle method for inverse Cauchy problem of inhomogeneouse Helmholtz equations," *Journal of Marine Science and Technology*, vol. 17, pp. 157–163, 2009.

[12] J. R. Canon and M. Tatari, "Identifying an unknown function in a parabolic equation with overspecified data via He's variational iteration method," *Chaos, Solitons and Fractals*, vol. 36, no. 1, pp. 157–166, 2008.

[13] O. Baysal, "A globally convergent numerical method for a coefficient inverse problem for a parabolic equation," *Journal of Computational and Applied Mathematics*, vol. 289, pp. 153–172, 2015.

[14] A. Shidfar, A. Babaei, and A. Molabahrami, "Solving the inverse problem of identifying an unknown source term in a parabolic equation," *Computers and Mathematics with Applications*, vol. 60, no. 5, pp. 1209–1213, 2010.

[15] T. Lu, W. W. Han, P. X. Jiang, Y. H. Zhu, J. Wu, and C. L. Liu, "A two-dimensional inverse heat conduction problem for simultaneous estimation of heat convection coefficient, fluid temperature and wall temperature on the inner wall of a pipeline," *Progress in Nuclear Energy*, vol. 81, pp. 161–168, 2015.

[16] Z. Liua and S. Tatar, "Analytical solutions of a class of inverse coefficient problems," *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2391–2395, 2012.

[17] C. S. Liu, "An analytical method for the inverse Cauchy problem of Laplace equation in a rectangular plate," *Journal of Mechanics*, vol. 27, no. 4, pp. 575–584, 2011.

[18] N. Aliev and M. Jahanshahi, "Sufficient conditions for reduction of the BVP including a mixed PDE with non-local boundary conditions to Fredholm integral equations," *International Journal of Mathematical Education in Science and Technology*, vol. 28, no. 3, pp. 419–425, 1997.

[19] P. K. Sahau and S. Saha Ray, "Numerical solutions for the system of Fredholm integral equations of second kind by a new approach involving semiorthogonal B-spline wavelet collocation method," *Applied Mathematics and Computation*, vol. 234, pp. 368–379, 2014.

[20] M. Basit and F. Khan, "An effective approach to solving the system of Fredholm integral equations based on Bernstein polynomial on any finite interval," *Alexandria Engineering Journal*, vol. 61, no. 4, pp. 2611–2623, 2022.

[21] H. Almasieh and M. Roodaki, "Triangular functions method for the solution of Fredholm integral equations system," *Ain Shams Engineering Journal*, vol. 3, no. 4, pp. 411–416, 2012.

[22] M. Javidi and A. Golbabai, "A numerical solution for solving system of Fredholm integral equations by using homotopy perturbation method," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1921–1928, 2007.

[23] K. Maleknejad, N. Aghazadeh, and M. Rabbani, "Numerical solution of second kind Fredholm integral equations system

by using a Taylor-series expansion method," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1229–1234, 2006.

[24] M. Sajjadmanesh, M. Jahanshahi, and N. Aliev, "Tikhonov-Lavrentev type inverse problem including Cauchy-Riemann equation," *Azerbaijan journal of mathematics*, vol. 3, no. 1, pp. 106–112, 2013.

[25] V. S. Vladimirov, *Equations of Mathematical Physics*, Mir Publishers, Moscow, 1981.

[26] E. Di Costanzo and A. Marasco, "Approximate analytic solution of the Dirichlet problems for Laplace's equation in planar domains by a perturbation method," *Computers and Mathematics with Applications*, vol. 63, no. 1, pp. 60–67, 2012.