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Research Article

New Generalized Riemann–Liouville Fractional Integral Versions of Hadamard and Fejér–Hadamard Inequalities

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In this paper, a new class of functions, namely, exponentially $(\alpha, h-m)-p$-convex functions is introduced to unify various classes of functions already defined in the subject of convex analysis. By using this class of functions, generalized versions of well known fractional integral inequalities of Hadamard and Fejér–Hadamard type are obtained. The results of this paper generate fractional integral inequalities of Hadamard and Fejér–Hadamard type for various types of convex and exponentially convex functions simultaneously.

1. Introduction and Preliminary Results

Inequalities are important tools for mathematical modeling of problems that occur in the diverse fields of science and engineering. Convex functions are very useful in establishing new and generalized inequalities. For example, Jensen's inequality for convex functions is one of the most celebrating inequalities in the literature. Many classical inequalities are direct consequences of Jensen's inequality. Motivated from the properties and representations of convex functions, researchers have published a lot of new definitions of functions which are usually utilized for extensions, refinements, and generalizations of well known inequalities. In recent decades, it becomes a fashion for authors to generalize the classical concepts related to ordinary derivatives and integrals via fractional integral/derivative operators. These techniques are used frequently in generalizing the classical mathematical inequalities. For a detailed study, we refer the readers to [1–13].

The goal of this paper is to establish general Riemann–Liouville fractional integral inequalities of Hadamard and Fejér–Hadamard type by defining a new class of functions which will concurrently hold for many kinds of convex and exponentially convex functions. Next, we give definitions of Riemann–Liouville fractional integrals which we will utilize to establish main results. After that we will give definition of convex function with a detailed discussion on related definitions.

Definition 1 (see [14]). Let $f \in L_1[a,b]$. Then, the left- and right-sided Riemann–Liouville fractional integrals of f of order $\tau \in \mathbb{R}$ $(\tau > 0)$ are given as follows:

$$I_{a^+}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_a^x (x-t)^{\tau-1} f(t)\mathrm{d}t, \quad x > a, \tag{1}$$

$$I_{b^-}^{\tau} f(x) = \frac{1}{\Gamma(\tau)} \int_x^b (t-x)^{\tau-1} f(t)\mathrm{d}t, \quad x < b, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [15]). A real-valued function $f\colon [a,b] \longrightarrow \mathbb{R}$ is called convex if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \in [a,b], t \in [0,1]. \tag{3}$$

There are many kinds of functions which have been defined inspiring by inequality (3). For example, functions, namely, p-convex [16], h-convex [17], m-convex [15], s-convex [18], harmonically convex [6], and many others are defined just by convenient possible modifications in the inequality (3). Moreover, (s,m)-convex [19], (α,m)-convex [20], $(h-m)$-convex [21], $(\alpha,h-m)$-convex [22], and (p,h)-convex [3] functions have been defined elegantly after the definition of convex function. Further, in [23], the notion of $(\alpha,h-m)-p$-convex function is defined which unifies all the aforementioned convexities.

There also exists a class of exponentially convex functions stated as follows.

Definition 3 (see [24]). A real-valued function $f\colon J \subset \mathbb{R} \longrightarrow \mathbb{R}_+$ is called exponentially convex on J if the following inequality holds:

$$f(tx + (1-t)y) \leq \frac{tf(x)}{e^{\eta x}} + \frac{(1-t)f(y)}{e^{\eta y}}, \quad t \in [0,1], \forall x, y \in J, \eta \in \mathbb{R}. \tag{4}$$

The term exponentially convex function is used likewise to convex function, and notions of exponentially p-convex [25], exponentially h-convex [26], exponentially s-convex [25] have been introduced. Also, definitions of exponentially (s,m)-convex [27], exponentially (α,m)-convex [26], exponentially $(h-m)$-convex [26], exponentially $(\alpha,h-m)$-convex [28], and exponentially (p,h)-convex [29] functions have been published.

The exponentially $(\alpha,h-m)$-convex function is defined as follows.

Definition 4 (see [28]). Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$, and let $h\colon J \longrightarrow \mathbb{R}$ be a nonnegative function. Then, a function $f\colon I \longrightarrow \mathbb{R}$ on an interval of real line is said to be exponentially $(\alpha,h-m)$-convex, if for all $x, y \in I$, $t \in (0,1)$, $\alpha, m \in [0,1]$, and $\eta \in \mathbb{R}$, the following inequality holds:

$$f(tx + m(1-t)y) \leq \frac{h(t^\alpha)f(x)}{e^{\eta u_1}} + \frac{mh(1-t^\alpha)f(y)}{e^{\eta y}}. \tag{5}$$

The following example is important to distinguish an exponentially convex function from convex function.

Example 1 (see [30]). The function $f(x) = x\exp(-x)$ is exponentially $(1, I_d - 1)$-convex function but not $(1, I_d - 1)$-convex function. More precisely the function f is exponentially convex function on $[0,\infty)$ but not a convex function on this domain.

All the aforementioned definitions have been used to derive Hadamard and Fejér–Hadamard type inequalities. We are motivated to combine all types of convexities and exponential convexities in a single definition. We will define exponentially $(\alpha,h-m)-p$-convex function and prove Hadamard and Fejér–Hadamard type inequalities which will unify a plenty of classical inequalities.

The paper is organized as follows: In Section 2, a new class of functions will be called exponentially $(\alpha,h-m)-p$-convex function. Some new definitions will be deduced in connection with existing definitions in the literature of mathematical inequalities. In Section 3, we will present the Hadamard and Fejér–Hadamard inequalities for newly defined functions via Riemann–Liouville fractional integrals. We will identify a number of implications of the results established in this section.

2. Exponentially $(\alpha,h-m)-p$-Convex Function and Deduced Definitions

We define exponentially $(\alpha,h-m)-p$-convex function as follows.

Definition 5. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$, and let $h\colon J \longrightarrow \mathbb{R}$ be a nonnegative function. Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f\colon I \longrightarrow \mathbb{R}$ is called exponentially $(\alpha,h-m)-p$-convex if for $t \in (0,1)$, $\eta \in \mathbb{R}$ and $(\alpha,m) \in [0,1]^2$, the following inequality holds:

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \leq \frac{h(t^\alpha)f(a)}{e^{\eta a}} + \frac{mh(1-t^\alpha)f(b)}{e^{\eta b}}, \tag{6}$$

where $a, b \in I$ provided $(ta^p + m(1-t)b^p)^{1/p} \in I$.

Remark 1. The following convex functions are reproduced from above definition:

(i) In Definition 5, if we put $p = -1$, $m = \alpha = 1$, and $\eta = 0$, we have harmonically h-convex function reproduced (see Definition 2.10 in [31]).

(ii) In Definition 5, for $p = 1$ and $\eta = 0$, we have $(\alpha,h-m)$-convex function reproduced (see Definition 4.5 in [20]).

(iii) In Definition 5, for $\alpha = m = 1$ and $\eta = 0$, we have (p,h)-convex function reproduced (see [3]).

(iv) In Definition 5, for $p = 1$, exponentially $(\alpha,h-m)$-convex function is reproduced (see Definition 1 in [26]). For further deduced functions, see Remark 1 in [26].

(v) In Definition 5, for $\alpha = p = 1$, exponentially $(h-m)$-convex function is reproduced (see Definition 2 in [26]).

(vi) In Definition 5, for $p = 1$ and $h(t) = t$, exponentially (α, m)-convex function is reproduced (see Definition 3 in [26]).

(vii) In Definition 5, for $p = -1$, $\alpha = m = 1$, $h(t) = t^s$, and $\eta = 0$, we have harmonic s-convex function in second sense reproduced (see Remark 1 in [32]).

(viii) In Definition 5, for $p = -1$, $\alpha = m = 1$, $h(t) = t$, and $\eta = 0$, we have harmonic convex function reproduced (see [33]).

(ix) In Definition 5, for $p = 1$, $\alpha = 1$, $h(t) = t^s$, and $\eta = 0$, we have (s, m)-convex function in second sense reproduced (see Definition 1.2 in [19]).

(x) In Definition 5, for $p = -1$, $\alpha = 1$, $h(t) = t$, and $\eta = 0$, we have m-HA-convex function reproduced (see Definition 2 in [34]).

(xi) In Definition 5, for $p = -1$, $h(t) = t$, and $\eta = 0$, (α, m)-HA-convex function is reproduced (see Definition 2.1 in [35]).

For $\alpha = 1$ in (6), we get exponentially $(h - m) - p$-convex function as follows:

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \leq \frac{h(t)f(a)}{e^{\eta a}} + \frac{mh(1-t)f(b)}{e^{\eta b}}. \tag{7}$$

For $h(t) = t$ in (6), we get exponentially $(\alpha, m) - p$-convex function as follows:

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \leq \frac{t^\alpha f(a)}{e^{\eta a}} + \frac{m\left(1-t^\alpha\right)f(b)}{e^{\eta b}}. \tag{8}$$

For $m = 1$ in (6), we get exponentially $(\alpha, h) - p$-convex function as follows:

$$f\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right) \leq \frac{h\left(t^\alpha\right)f(a)}{e^{\eta a}} + \frac{h\left(1-t^\alpha\right)f(b)}{e^{\eta b}}. \tag{9}$$

For $\alpha = 1$ and $h(t) = t^s$ in (6), we get exponentially $(s, m) - p$-convex function as follows:

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \leq \frac{t^s f(a)}{e^{\eta a}} + \frac{m(1-t)^s f(b)}{e^{\eta b}}. \tag{10}$$

For $h(t) = t^s$ in (6), we get exponentially $(s, m) - p$-Godunova–Levin function of second kind as follows:

$$f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right) \leq \frac{1}{t^s}\frac{f(a)}{e^{\eta a}} + \frac{m}{(1-t)^s}\frac{f(b)}{e^{\eta b}}. \tag{11}$$

For $m = \alpha = 1$ and $h(t) = 1$ in (6), we get exponentially (p, P)-convex function as follows:

$$f\left(\left(ta^p + (1-t)b^p\right)^{1/p}\right) \leq \frac{f(a)}{e^{\eta a}} + \frac{f(b)}{e^{\eta b}}. \tag{12}$$

For $\alpha = m = 1$, $p = -1$, and $h(t) = (1/t)$ in (6), we get exponentially Godunova–Levin type exponentially harmonic convex function as follows:

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq \frac{1}{t}\frac{f(a)}{e^{\eta a}} + \frac{1}{1-t}\frac{f(b)}{e^{\eta b}}. \tag{13}$$

For $\alpha = m = 1$, $p = -1$, and $h(t) = (1/t^s)$ in (6), we get exponentially harmonic convex function as follows:

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq \frac{1}{t^s}\frac{f(a)}{e^{\eta a}} + \frac{1}{(1-t)^s}\frac{f(b)}{e^{\eta b}}. \tag{14}$$

For $p = -1$ in (6), we get exponentially $(\alpha, h - m)$-HA-convex function as follows:

$$f\left(\frac{ab}{tb + m(1-t)a}\right) \leq \frac{h\left(t^\alpha\right)f(a)}{e^{\eta a}} + \frac{mh\left(1-t^\alpha\right)f(b)}{e^{\eta b}}. \tag{15}$$

For $p = -1$ and $m = 1$ in (6), we get exponentially (α, h)-HA-convex function as follows:

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq \frac{h\left(t^\alpha\right)f(a)}{e^{\eta a}} + \frac{h\left(1-t^\alpha\right)f(b)}{e^{\eta b}}. \tag{16}$$

For $p = -1$, $m = \alpha = 1$, and $h(t) = t$ in (6), we get exponentially HA-convex function as follows:

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq \frac{tf(a)}{e^{\eta a}} + \frac{(1-t)f(b)}{e^{\eta b}}. \tag{17}$$

For $p = -1$ and $h(t) = t$ in (6), we get exponentially (α, m)-HA-convex function as follows:

$$f\left(\frac{ab}{tb + m(1-t)a}\right) \leq \frac{t^\alpha f(a)}{e^{\eta a}} + \frac{m\left(1-t^\alpha\right)f(b)}{e^{\eta b}}. \tag{18}$$

From now to onward, we will use the notation $E_p(\alpha, h - m)$ for exponentially $(\alpha, h - m) - p$-convex function.

3. Inequalities of Hadamard Type for $E_p(\alpha, h - m)$ Function

Theorem 1. *Let $f\colon I \subset (0, \infty) \longrightarrow \mathbb{R}$ be an $E_p(\alpha, h - m)$ positive function as defined in Definition 5 and $f \in L_1[a, b]$, $a, b \in I, a < b$. Then, for $(\alpha, m) \in (0, 1]^2$, one can have fractional integral inequalities for operators (1) and (2) as follows.*

(i) For $p > 0$, we have

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\leq\frac{\Gamma(\tau+1)}{(mb^p-a^p)^{\tau}}\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^{\alpha}}\right)(I^{\tau}_{a^{p}+}f^{\circ}\xi)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(I^{\tau}_{b^{p}-}f^{\circ}\xi)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h(t^{\alpha})\mathrm{d}t\right.$$
$$\left.+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h(1-t^{\alpha})\mathrm{d}t\right\}, \tag{19}$$

where $\xi(z)=z^{1/p}$, $z\in[a^p,mb^p]$, $\mathfrak{D}_1(\eta)=e^{-\eta bm^{1/p}}$ for $\eta<0$, $\mathfrak{D}_1(\eta)=e^{-\eta a}$ for $\eta\geq 0$, $\mathfrak{D}_2(\eta)=e^{-\eta(a/(m^{(1/p)}))}$ for $\eta>0$, and $\mathfrak{D}_2(\eta)=e^{-\eta b}$ for $\eta\leq 0$.

(ii) For $p<0$, we have

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\leq\frac{\Gamma(\tau+1)}{(a^p-mb^p)^{\tau}}\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^{\alpha}}\right)(I^{\tau}_{a^{p}-}f^{\circ}\xi)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(I^{\tau}_{b^{p}+}f^{\circ}\xi)\left(\frac{a^p}{m}\right)\right)$$
$$\leq\tau\left\{\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_4(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h(t^{\alpha})\mathrm{d}t\right.$$
$$\left.+m\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_4(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h(1-t^{\alpha})\mathrm{d}t\right\}, \tag{20}$$

where $\xi(z)=z^{1/p}$, $z\in[mb^p,a^p]$, $\mathfrak{D}_3(\eta)=e^{-\eta bm^{1/p}}$ for $\eta<0$, $\mathfrak{D}_3(\eta)=e^{-\eta a}$ for $\eta\geq 0$, $\mathfrak{D}_4(\eta)=e^{-\eta(a/m^{1/p})}$ for $\eta<0$, and $\mathfrak{D}_4(\eta)=e^{-\eta b}$ for $\eta\geq 0$.

Proof. (i) By using (6), one can have the following inequality:

$$f\left(\left(\frac{x^p+my^p}{2}\right)^{1/p}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{f(x)}{e^{\eta x}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(y)}{e^{\eta y}}. \tag{21}$$

For $x=(ta^p+m(1-t)b^p)^{1/p}$ and $y=(tb^p+(1-t)(a^p/m))^{1/p}$ in (21), we get

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)\frac{f\left((ta^p+m(1-t)b^p)^{1/p}\right)}{e^{\eta\left((ta^p+m(1-t)b^p)^{1/p}\right)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left((tb^p+(1-t)(a^p/m))^{1/p}\right)}{e^{\eta\left((tb^p+(1-t)(a^p/m))^{1/p}\right)}}. \tag{22}$$

Multiplying the above inequality with $t^{\tau-1}$ on both sides and integrating over $[0,1]$, we have

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}\mathrm{d}t\leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\frac{t^{\tau-1}f\left((ta^p+m(1-t)b^p)^{1/p}\right)}{e^{\eta\left((ta^p+m(1-t)b^p)^{1/p}\right)}}\mathrm{d}t$$
$$+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\frac{t^{\tau-1}f\left((tb^p+(1-t)(a^p/m))^{1/p}\right)}{e^{\eta\left((tb^p+(1-t)(a^p/m))^{1/p}\right)}}\mathrm{d}t. \tag{23}$$

Set $ta^p+m(1-t)b^p=x$, that is, $t=(mb^p-x)/(mb^p-a^p)$ and $tb^p+(1-t)(a^p/m)=y$, that is, $t=(y-(a^p/m))/(b^p-(a^p/m))$ in right hand side of the above inequality. Then, after some calculations, one can obtain the first inequality of (19).

On the other hand, by using (6) on the right hand side of (22), one can obtain the inequality as follows:

$$\begin{aligned} &h\left(\frac{1}{2^{\alpha}}\right)\frac{f\left(\left(ta^{p}+m(1-t)b^{p}\right)^{1/p}\right)}{e^{\eta\left(\left(ta^{p}+m(1-t)b^{p}\right)^{1/p}\right)}}+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(\left(tb^{p}+(1-t)\left(a^{p}/m\right)\right)^{1/p}\right)}{e^{\eta\left(\left(tb^{p}+(1-t)\left(a^{p}/m\right)\right)^{1/p}\right)}}\\ &\leq\frac{h\left(1/2^{\alpha}\right)}{e^{\eta\left(\left(ta^{p}+m(1-t)b^{p}\right)^{1/p}\right)}}\left(h\left(t^{\alpha}\right)\frac{f(a)}{e^{\eta a}}+\frac{mh\left(1-t^{\alpha}\right)f(b)}{e^{\eta b}}\right)\\ &+\frac{mh\left(\left(2^{\alpha}-1\right)/2^{\alpha}\right)}{e^{\eta\left(\left(tb^{p}+(1-t)\left(a^{p}/m\right)\right)^{1/p}\right)}}\left(\frac{h\left(t^{\alpha}\right)f(b)}{e^{\eta b}}+\frac{mh\left(1-t^{\alpha}\right)f\left(a/m^{2}\right)}{e^{\left(\eta a/m^{2}\right)}}\right). \end{aligned} \tag{24}$$

Multiplying the above inequality with $t^{\tau-1}$, by integrating over $[0,1]$, one can get

$$\begin{aligned} &\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\int_{0}^{1}t^{\tau-1}f\left(\left(ta^{p}+m(1-t)b^{p}\right)^{1/p}\right)dt\\ &+\mathfrak{D}_{2}(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{0}^{1}t^{\tau-1}f\left(\left(tb^{p}+(1-t)\frac{a^{p}}{m}\right)^{1/p}\right)dt\\ &\leq\left(\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_{2}(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}\right)\int_{0}^{1}t^{\tau-1}h\left(t^{\alpha}\right)dt\\ &+m\left(\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_{2}(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(a/m^{2}\right)}{e^{\left(\eta a/m^{2}\right)}}\right)\int_{0}^{1}t^{\tau-1}h\left(1-t^{\alpha}\right)dt. \end{aligned} \tag{25}$$

Set $ta^{p}+m(1-t)b^{p}=x$, that is. $t=(mb^{p}-x)/(mb^{p}-a^{p})$ and $tb^{p}+(1-t)(a^{p}/m)=y$, that is, $t=(y-(a^{p}/m))/(b^{p}-(a^{p}/m))$ in (25). Then, after some calculations, the second inequality of (19) is obtained.

(ii) Proof is similar as (i). □

Remark 2.

(i) In Theorem 1 (i), if we put $\alpha=m=1$, $h(t)=t$, $\eta=0$, and $p=1$, then Theorem 2 in [12] is reproduced.

(ii) In Theorem 1 (i), if we put $\alpha=m=1$, $p=1$, $h(t)=t$, $\eta=0$, and $\tau=1$, then classical Hadamard inequality is reproduced.

(iii) In Theorem 1 (ii), if we put $\alpha=m=1$, $h(t)=t$, $\eta=0$, and $p=-1$, then Theorem 4 in [8] is reproduced.

The other variant of the Hadamard inequality is stated and proved as follows.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then, we have the following inequalities.*

(i) For $p>0$, we have

$$\begin{aligned} &f\left(\left(\frac{a^{p}+mb^{p}}{2}\right)^{1/p}\right)\leq\Gamma(\tau+1)\left(\frac{2}{mb^{p}-a^{p}}\right)^{\tau}\\ &\times\left(\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{\left(\left(a^{p}+mb^{p}\right)/2\right)^{+}}f\circ\xi\right)\left(mb^{p}\right)+\mathfrak{D}_{2}(\eta)m^{\tau+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(I^{\tau}_{\left(\left(a^{p}+mb^{p}\right)/2m\right)^{-}}f^{\circ}\xi\right)\left(\frac{a^{p}}{m}\right)\right)\\ &\leq\tau\left\{\left(\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_{2}(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}\right)\int_{0}^{1}t^{\tau-1}h\left(\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.\\ &\left.+m\left(\mathfrak{D}_{1}(\eta)h\left(\frac{1}{2^{\alpha}}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_{2}(\eta)mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\frac{f\left(a/m^{2}\right)}{e^{\left(\eta a/m^{2}\right)}}\right)\int_{0}^{1}t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right\}, \end{aligned} \tag{26}$$

where $\xi(z) = z^{1/p}$, $z \in [a^p, mb^p]$, $\mathfrak{D}_1(\eta)$, and $\mathfrak{D}_2(\eta)$ are same as given in Theorem 1 (i).

(ii) For $p < 0$, we have

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) \leq \Gamma(\tau+1)\left(\frac{2}{a^p - mb^p}\right)^{\tau} \times \left(\mathfrak{D}_3(\eta) h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^-} f \circ \xi\right)(mb^p) + \mathfrak{D}_4(\eta) m^{\tau+1} h\left(\frac{2^\alpha - 1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2m)^+} f^{\circ}\xi\right)\left(\frac{a^p}{m}\right)\right)$$
$$\leq \tau\left\{\left(\mathfrak{D}_3(\eta) h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta) m h\left(\frac{2^\alpha - 1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1} h\left(\left(\frac{t}{2}\right)^{\alpha}\right) dt + m\left(\mathfrak{D}_3(\eta) h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_4(\eta) m h\left(\frac{2^\alpha - 1}{2^\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1} h\left(1 - \left(\frac{t}{2}\right)^{\alpha}\right) dt\right\}, \tag{27}$$

where $\xi(z) = z^{(1/p)}$, $z \in [mb^p, a^p]$, $\mathfrak{D}_3(\eta)$, and $\mathfrak{D}_4(\eta)$ are same as given in Theorem 1 (ii).

Proof. (i) For $x = ((t/2)a^p + m(1 - (t/2))b^p)^{1/p}$ and $y = ((t/2)b^p + (1 - (t/2))(a^p/m))^{1/p}$ in (21), we get

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) \leq h\left(\frac{1}{2^\alpha}\right)\frac{f\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}{e^{\eta\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}} + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\frac{f\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}{e^{\eta\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}}. \tag{28}$$

Multiplying the above inequality with $t^{\tau-1}$ on both sides and integrating over $[0, 1]$, we have

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1} dt \leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 \frac{t^{\tau-1} f\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}{e^{\eta\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}} dt + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\int_0^1 \frac{t^{\tau-1} f\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}{e^{\eta\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}} dt. \tag{29}$$

Set $(t/2)a^p + m(1 - (t/2))b^p = x$, that is, $(t/2) = (mb^p - x)/(mb^p - a^p)$ and $(1 - (t/2))(a^p/m) + (t/2)b^p = y$, that is, $(t/2) = (y - (a^p/m))/(b^p - (a^p/m))$ in right hand side of the above inequality. Then, after some calculations, one can obtain the first inequality of (26).

On the other hand, by applying the $E_p(\alpha, h - m)$ of f, from right hand side of (28), one can obtain the following inequality:

$$h\left(\frac{1}{2^\alpha}\right)\frac{f\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}{e^{\eta\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}} + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\frac{f\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}{e^{\eta\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}}$$
$$\leq \frac{h(1/2^\alpha)}{e^{\eta\left(((t/2)a^p + m(1-(t/2))b^p)^{1/p}\right)}}\left(h\left(\left(\frac{t}{2}\right)^{\alpha}\right)\frac{f(a)}{e^{\eta a}} + mh\left(1 - \left(\frac{t}{2}\right)^{\alpha}\right)\frac{f(b)}{e^{\eta b}}\right) + \frac{mh((2^\alpha - 1)/2^\alpha)}{e^{\eta\left(((t/2)b^p + (1-(t/2))(a^p/m))^{1/p}\right)}}\left(h\left(\left(\frac{t}{2}\right)^{\alpha}\right)\frac{f(b)}{e^{\eta b}} + mh\left(1 - \left(\frac{t}{2}\right)^{\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right). \tag{30}$$

Multiplying $t^{\tau-1}$ on both sides of (30), then by integrating on $[0,1]$, one can get

$$\begin{aligned}&\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt\\&+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}b^p+\left(1-\frac{t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right)dt\\&\leq\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h\left(\left(\frac{t}{2}\right)^\alpha\right)dt\\&+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt.\end{aligned}\tag{31}$$

Set $(t/2)a^p+m(1-(t/2))b^p=x$, that is, $(t/2)=(mb^p-x)/(mb^p-a^p)$ and $(1-(t/2))(a^p/m)+(t/2)b^p=y$, that is, $(t/2)=(y-(a^p/m))/(b^p-(a^p/m))$ in (31). Then, after some calculations, the second inequality of (26) is obtained.

(ii) Proof is similar as (i). □

Remark 3.

(i) In Theorem 2 (i), if we put $\alpha=1=m$, $\eta=0$, $p>0$, and $h(t)=t$, then Theorem 2.1(i) in [36] is reproduced.

(ii) In Theorem 2 (ii), if we put $\alpha=1=m$, $\eta=0$, $p<0$, and $h(t)=t$, then Theorem 2.1(ii) in [36] is reproduced.

(iii) In Theorem 2 (i), if we put $\alpha=1=m$, $p=1$, $\eta=0$, and $h(t)=t$, then Corollary 2.1 in [36] is reproduced.

Remark 4. From Theorems 1 and 2, one can deduce results for convex, exponentially convex, $E_p(1,I_d-1)$, $E_p(1,I_d-m)$, $E_p(1,h-1)$, $E_p(\alpha,I_d-m)$, $E_p(1,h-m)$, and $E_p(1,h-1)$ functions.

4. Fejér–Hadamard Type Inequalities for $E_p(\alpha,h-m)$ Function

Theorem 3. *Let $f\colon I\longrightarrow\mathbb{R}$ be an $E_p(\alpha,h-m)$ positive function as given in Definition 5 and $f((a^p+mb^p-x)/m)=f(x)$, $a,b\in I, a<b$, $m\neq 0$. If $g\colon I\longrightarrow\mathbb{R}$ is a positive function and $f,g\in L_1[a,b]$, then one can have fractional integral inequalities for operators (1) and (2) as follows.*

(i) For $p>0$, we have

$$\begin{aligned}&f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^\tau_{a^{p+}}g\circ\xi\right)(mb^p)\\&\leq\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\left(I^\tau_{a^{p+}}fg\circ\xi\right)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^\tau_{b^{p-}}fg^\circ\xi\right)\left(\frac{a^p}{m}\right)\\&\leq\frac{(mb^p-a^p)^\tau}{\Gamma(\tau)}\left\{\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\right.\\&\times\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(t^\alpha)dt\\&+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\\&\left.\times\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(1-t^\alpha)dt\right\},\end{aligned}\tag{32}$$

where $\xi(z) = z^{1/p}$, $z \in [a^p, mb^p]$, $fg \circ \xi = (f \circ \xi)(g \circ \xi)$, $\mathfrak{D}_1(\eta)$, and $\mathfrak{D}_2(\eta)$ are same as given in Theorem 1 (i).

(ii) For $p < 0$, we have

$$\begin{aligned}
&f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^p-}\, g\circ\xi\right)(mb^p)\\
&\leq \mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{a^p-}\, fg\circ\xi\right)(mb^p) + \mathfrak{D}_4(\eta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^{\tau}_{b^p+}\, fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\leq \frac{(a^p - mb^p)^\tau}{\Gamma(\tau)}\left\{\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\right.\\
&\quad\times\int_0^1 t^{\tau-1}g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)h(t^\alpha)\mathrm{d}t\\
&\quad + m\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_4(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\\
&\quad\left.\times\int_0^1 t^{\tau-1}g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)h(1-t^\alpha)\mathrm{d}t\right\},
\end{aligned}\tag{33}$$

where $\xi(z) = z^{1/p}$, $z \in [mb^p, a^p]$, $fg \circ \xi = (f \circ \xi)(g \circ \xi)$, $\mathfrak{D}_3(\eta)$, and $\mathfrak{D}_4(\eta)$ are same as given in Theorem 1 (ii).

Proof. (i) Multiplying (22) by $t^{\tau-1}g((ta^p + m(1-t)b^p)^{1/p})$, then making integration on $[0, 1]$, the following inequality is yielded:

$$\begin{aligned}
&f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t\\
&\leq \mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t\\
&\quad + \mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(tb^p + (1-t)\frac{a^p}{m}\right)^{1/p}\right)g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t.
\end{aligned}\tag{34}$$

For $ta^p + m(1-t)b^p = x$, that is, $(1-t)(a^p/m) + tb^p = ((a^p + mb^p - x)/m)$ in (34) and then utilizing condition $f(x) = f((a^p + mb^p - x)/m)$ and equations (1) and (2), the first inequality of (32) can be achieved.

Now, multiplying $t^{\tau-1}g((ta^p + m(1-t)b^p)^{1/p})$ with (24) and integrating over $[0, 1]$, we have

$$\begin{aligned}
&\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t\\
&\quad + \mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(tb^p + (1-t)\frac{a^p}{m}\right)^{1/p}\right)g\left(\left(ta^p + m(1-t)b^p\right)^{1/p}\right)\mathrm{d}t
\end{aligned}$$

$$\leq\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(t^\alpha)dt$$
$$+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(1-t^\alpha)dt. \tag{35}$$

Again, setting $ta^p+m(1-t)b^p=x$, that is, $(1-t)(a^p/m)+tb^p=((a^p+mb^p-x)/m)$ in (35) and utilizing condition $f(x)=f((a^p+mb^p-x)/m)$, then by using definitions (1) and (2), one can get second inequality of (32).

(ii) Proof is similar as (i). □

Remark 5.

(i) In Theorem 3 (i), if we put $\alpha=m=1$, $h(t)=t$, $g(x)=1$, $\eta=0$, and $p=1$ then Theorem 2 in [12] is reproduced.

(ii) In Theorem 3 (i), if we put $\alpha=m=1$, $p=1$, $h(t)=t$, $g(x)=1$, $\eta=0$, and $\tau=1$, then the Hadamard inequality is reproduced.

(iii) In Theorem 3 (i), if we put $\alpha=m=1$, $p=1$, $h(t)=t$, $\eta=0$, and $\tau=1$ then classical Fejér–Hadamard inequality is reproduced.

(iv) In Theorem 3 (ii), if we put $\alpha=m=1$, $h(t)=t$, $g(x)=1$, $\eta=0$, and $p=-1$, then Theorem 4 in [8] is reproduced.

(v) In Theorem 3 (ii), if we put $\alpha=m=1$, $h(t)=t$, $\eta=0$, and $p=-1$ then Theorem 5 in [8] is reproduced.

The second variant of the Fejér–Hadamard inequality is stated and proved as follows.

Theorem 4. *Let the assumptions of Theorem 3 hold. Then, we have the following inequalities.*

(i) For $p>0$, we have

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+}g\circ\xi\right)(mb^p)$$
$$\leq\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^+}fg\circ\xi\right)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2m)^-}fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\leq\frac{1}{\Gamma(\tau)}\left(\frac{mb^p-a^p}{2}\right)^{\tau}\left\{\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\right.$$
$$\times\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right.$$
$$\left.\left.+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\}, \tag{36}$$

where $\xi(z)=z^{1/p}$, $z\in[a^p,mb^p]$, $fg\circ\xi=(f\circ\xi)(g\circ\xi)$, $\mathfrak{D}_1(\eta)$, and $\mathfrak{D}_2(\eta)$ are same as given in Theorem 1 (i).

(ii) For $p<0$, we have

$$f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^-}g\circ\xi\right)(mb^p)$$
$$\leq\mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^-}fg\circ\xi\right)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(I^{\tau}_{((a^p+mb^p)/2m)^+}fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$

$$\leq \frac{1}{\Gamma(\tau)}\left(\frac{a^p - mb^p}{2}\right)^{\tau}\left\{\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\right.$$
$$\times\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt + m\left(\mathfrak{D}_3(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right. \tag{37}$$
$$\left.\left.+\mathfrak{D}_4(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},$$

where $\xi(z) = z^{1/p}$, $z \in [mb^p, a^p]$, $fg\circ\xi = (f\circ\xi)(g\circ\xi)$, $\mathfrak{D}_3(\eta)$, and $\mathfrak{D}_4(\eta)$ are same as given in Theorem 1 (ii).

Proof. (i) Multiplying (28) by $t^{\tau-1}g(((t/2)a^p + m(1-(t/2))b^p)^{1/p})$ and integrating over $[0,1]$, the following inequality is yielded:

$$f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt$$
$$\leq \mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt \tag{38}$$
$$+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}b^p + \left(1-\frac{t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt.$$

Setting $(t/2)a^p + m(1-(t/2))b^p = x$, that is $(1-(t/2))(a^p/m) + (t/2)b^p = (a^p + mb^p - x)/m$ in (38) and using condition $f(x) = f((a^p + mb^p - x)/m)$ and the definitions (1), (2), one can get first inequality of (36).

Now, multiplying $t^{\tau-1}g(((t/2)a^p + m(1-(t/2))b^p)^{1/p})$ with (30) and integrating over $[0,1]$, we have

$$\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt$$
$$+\mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_0^1 t^{\tau-1}f\left(\left(\frac{t}{2}b^p + \left(1-\frac{t}{2}\right)\frac{a^p}{m}\right)^{1/p}\right)g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)dt$$
$$\leq\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt \tag{39}$$
$$+m\left(\mathfrak{D}_1(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_2(\eta)mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\int_0^1 t^{\tau-1}g\left(\left(\frac{t}{2}a^p + m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt.$$

Again for $(t/2)a^p + m(1-(t/2))b^p = x$, that is, $(1-(t/2))(a^p/m) + (t/2)b^p = (a^p + mb^p - x)/m$ in (39) and the utilizing condition $f(x) = f((a^p + mb^p - x)/m)$ and equations (1) and (2), the second inequality of (36) can be achieved.

(ii) Proof is similar as (i). □

Remark 6.

(i) In Theorem 4 (i), if we put $\alpha = 1 = m$, $p > 0$, $g(x) = 1$, $\eta = 0$, and $h(t) = t$, then Theorem 2.1 (i) in [36] is reproduced.

(ii) In Theorem 4 (ii), if we put $\alpha = 1 = m$, $p < 0$, $g(x) = 1$, $\eta = 0$, and $h(t) = t$, then Theorem 2.1(ii) in [36] is reproduced.

(iii) In Theorem 4 (i), if we put $\alpha = 1 = m$, $p = 1$, $g(x) = 1$, $\eta = 0$, and $h(t) = t$, then Corollary 2.1 in [36] is reproduced.

Remark 7. From Theorems 3 and 4, one can deduce results for convex, exponentially convex, $E_p(1, I_d - 1)$, $E_p(1, I_d - m)$, $E_p(1, h - 1)$, $E_p(\alpha, I_d - m)$, $E_p(1, h - m)$, and $E_p(1, h - 1)$ functions.

4.1. Results for $E_p(h-m)$ Function. For $\alpha = 1$ in Theorems 1–4, one can obtain the results for $E_p(h-m)$ function:

Theorem 5. *With the same conditions of Theorem 1, for $E_p(h-m)$ functions, the following inequalities hold:*

(i) For $p>0$, we have

$$\begin{aligned}\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \frac{\Gamma(\tau+1)}{(mb^p-a^p)^\tau}\left(\mathfrak{D}_1(\eta)\left(I^\tau_{a^p+} f\circ\xi\right)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}\left(I^\tau_{b^p-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\ &\leq \tau\left\{\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)\frac{mf(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h(t)\mathrm{d}t\right.\\ &\left.+m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)m\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h(1-t)\mathrm{d}t\right\}.\end{aligned} \tag{40}$$

(ii) For $p<0$, we have

$$\begin{aligned}\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \frac{\Gamma(\tau+1)}{(a^p-mb^p)^\tau}\left(\mathfrak{D}_3(\eta)\left(I^\tau_{a^p-} f\circ\xi\right)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}\left(I^\tau_{b^p+} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\ &\leq \tau\left\{\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_4(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h(t)\mathrm{d}t\right.\\ &\left.+m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_4(\eta)m\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h(1-t)\mathrm{d}t\right\}.\end{aligned} \tag{41}$$

Theorem 6. *With the same conditions of Theorem 2, for $E_p(h-m)$ function, the following inequalities hold:*

(i) For $p>0$, we have

$$\begin{aligned}\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \Gamma(\tau+1)\left(\frac{2}{mb^p-a^p}\right)^\tau\left(\mathfrak{D}_1(\eta)\left(I^\tau_{((a^p+mb^p)/2)^+} f\circ\xi\right)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}\left(I^\tau_{((a^p+mb^p)/2m)^-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\ &\leq \tau\left\{\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h\left(\frac{t}{2}\right)\mathrm{d}t\right.\\ &\left.+m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_2(\eta)m\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h\left(1-\frac{t}{2}\right)\mathrm{d}t\right\}.\end{aligned} \tag{42}$$

(ii) For $p<0$, we have

$$\begin{aligned}\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \Gamma(\tau+1)\left(\frac{2}{a^p-mb^p}\right)^\tau\left(\mathfrak{D}_3(\eta)\left(I^\tau_{((a^p+mb^p)/2)^-} f\circ\xi\right)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}\left(I^\tau_{((a^p+mb^p)/2m)^+} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\ &\leq \tau\left\{\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_4(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1}h\left(\frac{t}{2}\right)\mathrm{d}t\right.\\ &\left.+m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_4(\eta)m\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1}h\left(1-\frac{t}{2}\right)\mathrm{d}t\right\}.\end{aligned} \tag{43}$$

Theorem 7. *With the same conditions of Theorem 3, for $E_p(h-m)$ functions, the following inequalities hold:*

(i) For $p>0$, we have

$$\begin{aligned}
&\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{a^{p+}}^{\tau} g\circ\xi\right)(mb^p) \leq \mathfrak{D}_1(\eta)\left(I_{a^{p+}}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_2(\eta)m\left(I_{b^{p-}}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\leq \frac{(mb^p-a^p)^{\tau}}{\Gamma(\tau)}\left\{\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(t)\mathrm{d}t\right.\\
&\left.+m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_2(\eta)m\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(1-t)\mathrm{d}t\right\}.
\end{aligned}\tag{44}$$

(ii) For $p<0$, we have

$$\begin{aligned}
&\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{a^{p-}}^{\tau} g\circ\xi\right)(mb^p) \leq \mathfrak{D}_3(\eta)\left(I_{a^{p-}}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_4(\eta)m\left(I_{b^{p+}}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\leq \frac{(a^p-mb^p)^{\tau}}{\Gamma(\tau)}\left\{\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(t)\mathrm{d}t\right.\\
&\left.+m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_4(\eta)m\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\int_0^1 t^{\tau-1} g\left(\left(ta^p+m(1-t)b^p\right)^{1/p}\right)h(1-t)\mathrm{d}t\right\}.
\end{aligned}\tag{45}$$

Theorem 8. *With the same conditions of Theorem 4, for $E_p(h-m)$ functions, the following inequalities hold:*

(i) For $p>0$, we have

$$\begin{aligned}
&\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{\left((a^p+mb^p)/2\right)^+}^{\tau} g\circ\xi\right)(mb^p)\\
&\leq \mathfrak{D}_1(\eta)\left(I_{\left((a^p+mb^p)/2\right)^+}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_2(\eta)m^{\tau+1}\left(I_{\left((a^p+mb^p)/2m\right)^-}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\leq \frac{1}{\Gamma(\tau)}\left(\frac{mb^p-a^p}{2}\right)^{\tau}\left\{\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{t}{2}\right)\mathrm{d}t\right.\\
&\left.+m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_2(\eta)m\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\frac{t}{2}\right)\mathrm{d}t\right\}.
\end{aligned}\tag{46}$$

(ii) For $p<0$, we have

$$\begin{aligned}
&\frac{1}{h(1/2)} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{((a^p+mb^p)/2)^-} g\circ\xi\right)(mb^p)\\
&\leq \mathfrak{D}_3(\eta)\left(I^{\tau}_{((a^p+mb^p)/2)^-} fg\circ\xi\right)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}\left(I^{\tau}_{((a^p+mb^p)/2m)^+} fg\circ\xi\right)\left(\frac{a^p}{m}\right)\\
&\leq \frac{1}{\Gamma(\tau)}\left(\frac{a^p-mb^p}{2}\right)^{\tau}\left\{\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_4(\eta)m\frac{f(b)}{e^{\eta b}}\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(\frac{t}{2}\right)\mathrm{d}t\right.\\
&\left.+m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}}+\mathfrak{D}_4(\eta)m\frac{f\left(a/m^2\right)}{e^{(\eta a/m^2)}}\right)\int_0^1 t^{\tau-1} g\left(\left(\frac{t}{2}a^p+m\left(1-\frac{t}{2}\right)b^p\right)^{1/p}\right)h\left(1-\frac{t}{2}\right)\mathrm{d}t\right\}.
\end{aligned} \tag{47}$$

4.2. Results for $E_p(\alpha-m)$ Functions. For $h(t)=t$ in Theorems 1–4, one can obtain the results for $E_p(\alpha-m)$ function as follows.

Theorem 9. *With the same conditions of Theorem 1, for $E_p(\alpha-m)$ functions, the following inequalities hold:*

(1) For $p>0$, we have

$$\begin{aligned}
2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \frac{\Gamma(\tau+1)}{(mb^p-a^p)^{\tau}}\left(\mathfrak{D}_1(\eta)\left(I^{\tau}_{a^{p+}} f\circ\xi\right)(mb^p)+\mathfrak{D}_2(\eta)m^{\tau+1}\left(2^{\alpha}-1\right)\left(I^{\tau}_{b^{p-}} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\
&\leq \frac{\tau}{\tau+\alpha}\left\{\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_2(\eta)m\left(2^{\alpha}-1\right)\frac{f(b)}{e^{\eta b}}\right)\right.\\
&\left.+\frac{m\alpha}{\tau+\alpha}\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}}+m\left(2^{\alpha}-1\right)\mathfrak{D}_2(\eta)\frac{f\left(a/m^2\right)}{e^{(\eta a/m^2)}}\right)\right\}.
\end{aligned} \tag{48}$$

(ii) For $p<0$, we have

$$\begin{aligned}
2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right) &\leq \frac{\Gamma(\tau+1)}{(a^p-mb^p)^{\tau}}\left(\mathfrak{D}_3(\eta)\left(I^{\tau}_{a^{p-}} f\circ\xi\right)(mb^p)+\mathfrak{D}_4(\eta)m^{\tau+1}\left(2^{\alpha}-1\right)\left(I^{\tau}_{b^{p+}} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right)\\
&\leq \frac{\tau}{\tau+\alpha}\left\{\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}}+\mathfrak{D}_4(\eta)m\left(2^{\alpha}-1\right)\frac{f(b)}{e^{\eta b}}\right)\right.\\
&\left.+\frac{m\alpha}{\tau+\alpha}\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}}+m\left(2^{\alpha}-1\right)\mathfrak{D}_4(\eta)\frac{f\left(a/m^2\right)}{e^{(\eta a/m^2)}}\right)\right\}.
\end{aligned} \tag{49}$$

Theorem 10. *With the same conditions of Theorem 2, for* $E_p(\alpha - m)$ *functions, the following inequalities hold:*

(i) For $p > 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) \le \Gamma(\tau+1)\left(\frac{2}{mb^p - a^p}\right)^{\tau} \times\left(\mathfrak{D}_1(\eta)\left(I^{\tau}_{((a^p+mb^p)/2)^+} f\circ\xi\right)(mb^p) + \mathfrak{D}_2(\eta) m^{\tau+1}(2^{\alpha}-1)\left(I^{\tau}_{((a^p+mb^p)/2m)^-} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right) \le \frac{\tau}{2^{\alpha}(\tau+\alpha)}\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta) m(2^{\alpha}-1)\frac{f(b)}{e^{\eta b}}\right) + m\left(1 - \frac{\tau}{2^{\tau}(\tau+\alpha)}\right)\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_2(\eta) m(2^{\alpha}-1)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right). \tag{50}$$

(ii) For $p < 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right) \le \Gamma(\tau+1)\left(\frac{2}{a^p - mb^p}\right)^{\tau} \times\left(\mathfrak{D}_3(\eta)\left(I^{\tau}_{((a^p+mb^p)/2)^-} f\circ\xi\right)(mb^p) + \mathfrak{D}_4(\eta) m^{\tau+1}(2^{\alpha}-1)\left(I^{\tau}_{((a^p+mb^p)/2m)^+} f\circ\xi\right)\left(\frac{a^p}{m}\right)\right) \le \frac{\tau}{2^{\alpha}(\tau+\alpha)}\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta) m(2^{\alpha}-1)\frac{f(b)}{e^{\eta b}}\right) + m\left(1 - \frac{\tau}{2^{\tau}(\tau+\alpha)}\right)\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_4(\eta) m(2^{\alpha}-1)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right). \tag{51}$$

Theorem 11. *Under the assumptions of Theorem 3, for* $E_p(\alpha - m)$ *functions, the following inequalities hold:*

(i) For $p > 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p + mb^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{a^{p+}} g\circ\xi\right)(mb^p) \le \mathfrak{D}_1(\eta)\left(I^{\tau}_{a^{p+}} fg\circ\xi\right)(mb^p) + \mathfrak{D}_2(\eta) m^{\tau+1}(2^{\alpha}-1)\left(I^{\tau}_{b^{p-}} fg\circ\xi\right)\left(\frac{a^p}{m}\right) \le \left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta) m(2^{\alpha}-1)\frac{f(b)}{e^{\eta b}}\right)\left(I^{\tau+\alpha}_{a^{p+}} g\circ\xi\right)(mb^p) + m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_2(\eta) m(2^{\alpha}-1)\frac{f(a/m^2)}{e^{(\eta a/m^2)}}\right)\left(\left(I^{\tau}_{a^{p+}} g\circ\xi\right)(mb^p) - \left(I^{\tau+\alpha}_{a^{p+}} g\circ\xi\right)(mb^p)\right). \tag{52}$$

(ii) For $p < 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{a^p-}^{\tau} g\circ\xi\right)(mb^p) \le \mathfrak{D}_3(\eta)\left(I_{a^p-}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_4(\eta)m^{\tau+1}\left(2^{\alpha}-1\right)\left(I_{b^p+}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le \left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta)m\left(2^{\alpha}-1\right)\frac{f(b)}{e^{\eta b}}\right)\left(I_{a^p-}^{\tau+\alpha} g\circ\xi\right)(mb^p)$$
$$+ m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}} + \mathfrak{D}_4(\eta)m\left(2^{\alpha}-1\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\left(\left(I_{a^p-}^{\tau} g\circ\xi\right)(mb^p) - \left(I_{a^p-}^{\tau+\alpha} g\circ\xi\right)(mb^p)\right). \tag{53}$$

Theorem 12. *With the same conditions of Theorem 4, for $E_p(\alpha - m)$ functions,*

(i) For $p > 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{((a^p+mb^p)/2)^+}^{\tau} g\circ\xi\right)(mb^p) \le \mathfrak{D}_1(\eta)\left(I_{((a^p+mb^p)/2)^+}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_2(\eta)m^{\tau+1}\left(2^{\alpha}-1\right)\left(I_{((a^p+mb^p)/2m)^-}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le \frac{1}{2^{\alpha}}\left(\mathfrak{D}_1(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_2(\eta)m\left(2^{\alpha}-1\right)\frac{f(b)}{e^{\eta b}}\right)\left(I_{((a^p+mb^p)/2)^+}^{\tau+\alpha} g\circ\xi\right)(mb^p) + m\left(\mathfrak{D}_1(\eta)\frac{f(b)}{e^{\eta b}}\right.$$
$$\left. + \mathfrak{D}_2(\eta)m\left(2^{\alpha}-1\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\left(\left(I_{((a^p+mb^p)/2)^+}^{\tau} g\circ\xi\right)(mb^p) - \frac{1}{2^{\alpha}}\left(I_{((a^p+mb^p)/2)^+}^{\tau+\alpha} g\circ\xi\right)(mb^p)\right). \tag{54}$$

(ii) For $p < 0$, we have

$$2^{\alpha} f\left(\left(\frac{a^p+mb^p}{2}\right)^{1/p}\right)\left(I_{((a^p+mb^p)/2)^-}^{\tau} g^{\circ}\xi\right)(mb^p)$$
$$\le \mathfrak{D}_3(\eta)\left(I_{((a^p+mb^p)/2)^-}^{\tau} fg\circ\xi\right)(mb^p) + \mathfrak{D}_4(\eta)m^{\tau+1}\left(2^{\alpha}-1\right)\left(I_{((a^p+mb^p)/2m)^+}^{\tau} fg\circ\xi\right)\left(\frac{a^p}{m}\right)$$
$$\le \frac{1}{2^{\alpha}}\left(\mathfrak{D}_3(\eta)\frac{f(a)}{e^{\eta a}} + \mathfrak{D}_4(\eta)m\left(2^{\alpha}-1\right)\frac{f(b)}{e^{\eta b}}\right)\left(I_{((a^p+mb^p)/2)^-}^{\tau+\alpha} g\circ\xi\right)(mb^p) + m\left(\mathfrak{D}_3(\eta)\frac{f(b)}{e^{\eta b}}\right.$$
$$\left. + \mathfrak{D}_4(\eta)m\left(2^{\alpha}-1\right)\frac{f\left(a/m^2\right)}{e^{\left(\eta a/m^2\right)}}\right)\left(\left(I_{((a^p+mb^p)/2)^-}^{\tau} g\circ\xi\right)(mb^p) - \frac{1}{2^{\alpha}}\left(I_{((a^p+mb^p)/2)^-}^{\tau+\alpha} g\circ\xi\right)(mb^p)\right). \tag{55}$$

Remark 8. From Theorems 1–4, one can deduce results for exponentially $(\alpha, h) - p$-convex function, exponentially $(s, m) - p$-convex function of second kind, exponentially $(s, m) - p$-Godunova–Levin-convex function of second kind, exponentially (p, P)-convex function, Godunova–Levin type exponentially harmonic convex function, s-Godunova–Levin type exponentially harmonic convex function, exponentially $(\alpha, h - m)$-HA-convex function, exponentially (α, h)-HA-convex function, exponentially HA-convex function, and exponentially (α, m)-HA-convex function.

5. Conclusion

The Hadamard and the Fejér–Hadamard inequalities for Riemann–Liouville fractional integrals are proved by applying a generalized class of functions. Two fractional versions of the Hadamard inequality lead to almost all variants of such inequalities already published by different authors using various kinds of convex functions. Hadamard type inequalities for some new classes of functions are also given. Two fractional versions of the Fejér–Hadamard inequality are also proved which appear as generalizations of the Hadamard inequalities. By using the generalized convexity defined in this paper, one can obtain extensions of other classical integral inequalities hold for convex and related functions. It is also possible to establish these inequalities for many kinds of integral operators already existing in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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