Hindawi
Journal of Mathematics
Volume 2022, Article ID 8203921, 11 pages
https://doi.org/10.1155/2022/8203921

Research Article

Convolution Results and Fekete–Szegö Inequalities for Certain Classes of Symmetric q-Starlike and Symmetric q-Convex Functions

Tamer M. Seoudy [1,2]

[1]Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

Correspondence should be addressed to Tamer M. Seoudy; tmsaman@uqu.edu.sa

Received 10 January 2022; Accepted 16 May 2022; Published 20 June 2022

Academic Editor: A. Ghareeb

In this paper, by using the concept of the symmetric q-difference operator, we introduce certain classes of symmetric q-starlike and symmetric q-convex functions. Convolution results, coefficient estimates, and Fekete–Szegö inequalities for the analytic functions belonging to these classes are obtained.

1. Introduction

Let us represent the class of analytic (or holomorphic or regular) functions in $\Delta = \{\xi \in \mathbb{C}: |\xi| < 1\}$ by $\mathscr{H}(\Delta)$ and suppose that $\mathscr{A}$ is the subclass of $\mathscr{H}(\Delta)$ defined as follows:

$$\mathscr{A} = \left\{ g \in \mathscr{H}(\Delta): g(\xi) = \xi + \sum_{j=2}^{\infty} b_j \xi^j \right\}. \quad (1)$$

Further, let $\mathscr{S}(\alpha)$ and $\mathscr{C}(\alpha)$ denote the subclasses of $\mathscr{A}$ that consist, respectively, of starlike of order α and convex of order $\alpha (0 \le \alpha < 1)$ in Δ (see [1]). For two given regular functions $h_1, h_2 \in \mathscr{H}(\Delta)$, we say that $h_1(\xi)$ is subordinate to $h_2(\xi)$ or $h_2(\xi)$ is superordinate to $h_1(\xi)$, written symbolically as $h_1(\xi) \prec h_2(\xi)$ if there exists a Schwarz function ω, which (by definition) is regular in Δ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ for all $\xi \in \Delta$, such that

$$h_1(\xi) = h_2(\omega(\xi)), \quad (\xi \in \Delta). \quad (2)$$

Moreover, if $h_1(\xi)$ is univalent function in Δ, then the following equivalence holds true (see [2, 3]):

$$h_1(\xi) \prec h_2(\xi) \Leftrightarrow h_1(0) = h_2(0), \quad h_1(\Delta) \subset h_2(\Delta). \quad (3)$$

For functions g given by (1) and h given by

$$h(\xi) = \xi + \sum_{j=2}^{\infty} c_j \xi^j, \quad (4)$$

the convolution or Hadamard product of g and h is defined by

$$(g * h)(\xi) = \xi + \sum_{j=2}^{\infty} b_j c_j \xi^j = (h * g)(\xi). \quad (5)$$

Let $\mathscr{S}[L, M]$ and $\mathscr{C}[L, M]$ denote the subclasses of $\mathscr{A}$ for $-1 \le M < L \le 1$ which are defined by (see [4–9])

$$\mathscr{S}[L, M] = \left\{ g(\xi) \in \mathscr{A}: \frac{\xi g'(\xi)}{g(\xi)} \prec \frac{1 + L\xi}{1 + M\xi} \right\},$$
$$\mathscr{C}[L, M] = \left\{ g(\xi) \in \mathscr{A}: \frac{(\xi g'(\xi))'}{g'(\xi)} \prec \frac{1 + L\xi}{1 + M\xi} \right\}. \quad (6)$$

We note that $\mathcal{S}[1-2\alpha,-1]=\mathcal{S}(\alpha)$ and $\mathcal{C}[1-2\alpha,-1]=\mathcal{C}(\alpha)\,(0\le\alpha<1)$.

For $0<q<1$, the symmetric q-difference of a function g is defined as follows (see [10]):

$$\widetilde{D}_q g(\xi)=\frac{g(q\xi)-g(q^{-1}\xi)}{(q-q^{-1})\xi},\quad (\xi\neq 0), \tag{7}$$

and $\widetilde{D}_q g(0)=g'(0)$ provided that $g(\xi)$ is differentiable at 0. From (1) and (7), we deduce that

$$\widetilde{D}_q g(\xi)=1+\sum_{j=2}^{\infty}\widetilde{[j]}_q b_j \xi^{j-1}, \tag{8}$$

where $\widetilde{[j]}_q$ is the q-number given by

$$\widetilde{[j]}_q=\frac{q^j-q^{-j}}{q-q^{-1}}. \tag{9}$$

Furthermore, if $g(\xi)$ and $h(\xi)$ are the two functions, then

$$\widetilde{D}_q[\gamma_1 g(\xi)\pm\gamma_2 h(\xi)]=\gamma_1\widetilde{D}_q g(\xi)\pm\gamma_2\widetilde{D}_q h(\xi), \tag{10}$$

where γ_1 and γ_2 are constants.

$$\begin{aligned}\widetilde{D}_q[g(\xi)h(\xi)]&=h(q\xi)\widetilde{D}_q g(\xi)+g(q^{-1}\xi)\widetilde{D}_q h(\xi),\\ \widetilde{D}_q\left[\frac{g(\xi)}{h(\xi)}\right]&=\frac{h(q\xi)\widetilde{D}_q g(\xi)-g(q\xi)\widetilde{D}_q h(\xi)}{h(q\xi)h(q^{-1}\xi)}.\end{aligned} \tag{11}$$

Making use of the symmetric q-difference $\widetilde{D}_q g(\xi)$ given by (7), we introduce the subclasses $\widetilde{\mathcal{S}}_q[L,M]$ and $\widetilde{\mathcal{C}}_q[L,M]$ of $\mathcal{A}$ for $0<q<1$ and $-1\le M<L\le 1$ as follows:

$$\widetilde{\mathcal{S}}_q[L,M]=\left\{g(\xi)\in\mathcal{A}\colon \frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}\prec\frac{1+L\xi}{1+M\xi}\right\}, \tag{12}$$

$$\widetilde{\mathcal{C}}_q[L,M]=\left\{g(\xi)\in\mathcal{A}\colon \frac{\widetilde{D}_q(\xi\widetilde{D}_q g(\xi))}{\widetilde{D}_q g(\xi)}\prec\frac{1+L\xi}{1+M\xi}\right\}. \tag{13}$$

From (12) and (13), we have

$$g(\xi)\in\widetilde{\mathcal{C}}_q[L,M]\Leftrightarrow\xi\widetilde{D}_q g(\xi)\in\widetilde{\mathcal{S}}_q[L,M]. \tag{14}$$

We also note that

(i) $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{S}}_q[L,M]=\mathcal{S}[L,M]$ and $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{C}}_q[L,M]=\mathcal{C}[L,M]$ (see [6, 7, 9, 11, 12]);

(ii) $\widetilde{\mathcal{S}}_q[1-2\alpha,-1]=\widetilde{\mathcal{S}}_q(\alpha)\,(0\le\alpha<1)$,

$$\widetilde{\mathcal{S}}_q(\alpha)=\left\{g(\xi)\in\mathcal{A}\colon \Re\left\{\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)}\right\}>\alpha,\xi\in\Delta\right\}, \tag{15}$$

and $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{S}}_q(\alpha)=\mathcal{S}(\alpha)$ (see [1, 13, 14]).

(iii) $\widetilde{\mathcal{C}}_q[1-2\alpha,-1]=\widetilde{\mathcal{C}}_q(\alpha)\,(0\le\alpha<1)$,

$$\widetilde{\mathcal{C}}_q(\alpha)=\left\{g(\xi)\in\mathcal{A}\colon \Re\left\{\frac{\widetilde{D}_q(\xi\widetilde{D}_q g(\xi))}{\widetilde{D}_q g(\xi)}\right\}>\alpha,\xi\in\Delta\right\}, \tag{16}$$

and $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{C}}_q(\alpha)=\mathcal{C}(\alpha)$ (see [1, 8, 15]).

(iv) $\widetilde{\mathcal{S}}_q[(1-2\alpha)\beta,-\beta]=\widetilde{\mathcal{S}}_q(\alpha,\beta)\,(0\le\alpha<1,0<\beta\le 1)$,

$$\widetilde{\mathcal{S}}_q(\alpha,\beta)=\left\{g(\xi)\in\mathcal{A}\colon \left|\frac{(\xi\widetilde{D}_q g(\xi)/g(\xi))-1}{(\xi\widetilde{D}_q g(\xi)/g(\xi))+1-2\alpha}\right|<\beta,\xi\in\Delta\right\}, \tag{17}$$

and $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{S}}_q(\alpha,\beta)=\mathcal{S}(\alpha,\beta)$ (see [16]).

(v) $\widetilde{\mathcal{C}}_q[(1-2\alpha)\beta,-\beta]=\widetilde{\mathcal{C}}_q(\alpha,\beta)\,(0\le\alpha<1,0<\beta\le 1)$,

$$\widetilde{\mathcal{C}}_q(\alpha,\beta)=\left\{g(\xi)\in\mathcal{A}\colon \left|\frac{(\widetilde{D}_q(\xi\widetilde{D}_q g(\xi))/\widetilde{D}_q g(\xi))-1}{(\widetilde{D}_q(\xi\widetilde{D}_q g(\xi))/\widetilde{D}_q g(\xi))+1-2\alpha}\right|<\beta,\xi\in\Delta\right\}, \tag{18}$$

and $\lim_{q\longrightarrow 1^-}\widetilde{\mathcal{C}}_q(\alpha,\beta)=\mathcal{C}(\alpha,\beta)$ (see [16]).

In the present article, our aim is to investigate convolution properties, coefficient estimates, and Fekete–Szegö inequalities for the classes $\widetilde{\mathcal{S}}_q[L,M]$ and $\widetilde{\mathcal{C}}_q[L,M]$. The motivation of this article is to generalize and improve previously known results.

2. Convolution Results and Coefficient Estimates

Unless otherwise mentioned, we assume throughout this investigation that

$$\begin{aligned}&0 \le \theta < 2\pi,\\&-1 \le M < L \le 1,\\&0 < q < 1,\\&\xi \in \Delta.\end{aligned} \tag{19}$$

Theorem 1. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{S}}_q[L, M]$ if and only if*

$$\frac{1}{\xi}\left[g(\xi) * \frac{\xi - \left[1 + R\left(q - 1 + q^{-1}\right)\right]\xi^2 + R\xi^3}{(1 - q\xi)(1 - \xi)\left(1 - q^{-1}\xi\right)}\right] \ne 0, \tag{20}$$

where R is given by

$$R = R(\theta, L, M) = \frac{e^{-i\theta} + L}{L - M}. \tag{21}$$

Proof. It is easy to check that

$$\begin{aligned}g(\xi) * \frac{\xi}{1 - \xi} &= g(\xi),\\ g(\xi) * \frac{\xi}{(1 - q\xi)\left(1 - q^{-1}\xi\right)} &= \xi\widetilde{D}_q g(\xi).\end{aligned} \tag{22}$$

In order to prove that equation (20) holds, we will write (12) by using the definition of the subordination as follows:

$$\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)} = \frac{1 + Lw(\xi)}{1 + Mw(\xi)}, \tag{23}$$

where $w(\xi)$ is Schwarz function; hence, we have

$$\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)} \ne \frac{1 + Le^{i\theta}}{1 + Me^{i\theta}}, \tag{24}$$

which is equivalent to

$$\frac{1}{\xi}\left[\left(1 + Me^{i\theta}\right)\xi\widetilde{D}_q g(\xi) - \left(1 + Le^{i\theta}\right)g(\xi)\right] \ne 0. \tag{25}$$

By using (22) in (25), we obtain

$$\begin{aligned}&\frac{1}{\xi}\left[\left(1 + Me^{i\theta}\right)\left(g(\xi) * \frac{\xi}{(1 - q\xi)\left(1 - q^{-1}\xi\right)}\right) - \left(1 + Le^{i\theta}\right)\left(g(\xi) * \frac{\xi}{1 - \xi}\right)\right]\\&= \frac{(M - L)e^{i\theta}}{\xi}\left\{g(\xi) * \frac{\xi - \left[1 + \left(\left(e^{-i\theta} + L\right)/(L - M)\right)\left(q - 1 + q^{-1}\right)\right]\xi^2 + \left(\left(e^{-i\theta} + L\right)/(L - M)\right)\xi^3}{(1 - q\xi)(1 - \xi)\left(1 - q^{-1}\xi\right)}\right\} \ne 0,\end{aligned} \tag{26}$$

which proves the necessary condition (20) for Theorem 1.

Reversely, suppose that $g(\xi) \in \mathscr{A}$ satisfies condition (20). Since it was shown in the first part of the proof that assumption (20) is equivalent to (25), we obtain that

$$\frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)} \ne \frac{1 + Le^{i\theta}}{1 + Me^{i\theta}}, \quad (0 \le \theta < 2\pi). \tag{27}$$

If we denote

$$\begin{aligned}\varphi(\xi) &= \frac{\xi\widetilde{D}_q g(\xi)}{g(\xi)},\\ \psi(\xi) &= \frac{1 + L\xi}{1 + M\xi},\end{aligned} \tag{28}$$

relation (27) means that

$$\varphi(\Delta) \cap \psi(\partial\Delta) = \varnothing. \tag{29}$$

Thus, the simply connected domain $\varphi(\Delta)$ is included in a connected component of $\mathbb{C}\backslash\psi(\partial\Delta)$. Therefore, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function $\psi(\xi)$, it follows that $\varphi(\xi) \prec \psi(\xi)$, which implies that $g(\xi) \in \widetilde{\mathscr{S}}_q[L, M]$. This completes the proof of Theorem 1.

Putting $L = 1 - 2\alpha\,(0 \le \alpha < 1)$ and $M = -1$ in Theorem 1, we get the following convolution result for $\widetilde{\mathscr{S}}_q(\alpha)$. □

Corollary 1. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{S}}_q(\alpha)$ if and only if*

$$\frac{1}{\xi}\left[g(\xi) * \frac{\xi - \left[1 + T\left(q - 1 + q^{-1}\right)\right]\xi^2 + T\xi^3}{(1 - q\xi)(1 - \xi)\left(1 - q^{-1}\xi\right)}\right] \ne 0, \tag{30}$$

where T is given by

$$T = T(\theta, \alpha) = \frac{e^{-i\theta} + 1 - 2\alpha}{2(1 - \alpha)}, \quad (0 \le \alpha < 1). \tag{31}$$

Theorem 2. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{C}}_q[L, M]$ if and only if*

$$\frac{1}{\xi}\left[g(\xi)*\frac{\xi-\left[R\left(q+q^{-1}\right)+1\right]\left(q-1+q^{-1}\right)\xi^2+\left[R\left(q+2+q^{-1}\right)-1\right]\left(q-1+q^{-1}\right)\xi^3-(2R-1)\xi^4}{\left(1-q^2\xi\right)(1-q\xi)(1-\xi)\left(1-q^{-1}\xi\right)\left(1-q^{-2}\xi\right)}\right]\neq 0, \tag{32}$$

where R is given by (21).

Proof. From relation (14), we have

$$g(\xi)\in\widetilde{\mathscr{C}}_q[L,M]\Leftrightarrow\xi\widetilde{D}_q g(\xi)\in\widetilde{\mathscr{S}}_q[L,M]. \tag{33}$$

Then, according to Theorem 1, we obtain

$$g(\xi)\in\widetilde{\mathscr{C}}_q[L,M]\Leftrightarrow\frac{1}{\xi}\left[\xi\widetilde{D}_q g(\xi)*\phi(\xi)\right]\neq 0, \tag{34}$$

where $\phi(\xi)$ is given by

$$\phi(\xi)=\frac{\xi-\left[1+R\left(q-1+q^{-1}\right)\right]\xi^2+R\xi^3}{(1-q\xi)(1-\xi)\left(1-q^{-1}\xi\right)}, \tag{35}$$

and we note that

$$\xi\widetilde{D}_q\phi(\xi)=\frac{\xi-\left[R\left(q+q^{-1}\right)+1\right]\left(q-1+q^{-1}\right)\xi^2+\left[R\left(q+2+q^{-1}\right)-1\right]\left(q-1+q^{-1}\right)\xi^3-(2R-1)\xi^4}{\left(1-q^2\xi\right)(1-q\xi)(1-\xi)\left(1-q^{-1}\xi\right)\left(1-q^{-2}\xi\right)}. \tag{36}$$

Using relation (36) and the following identity:

$$\xi\widetilde{D}_q g(\xi)*\phi(\xi)=g(\xi)*\xi\widetilde{D}_q\phi(\xi),\quad (g,\phi\in\mathscr{A}), \tag{37}$$

it is easy to check that (34) is equivalent to (32). Thus, the proof of Theorem 2 is completed.

Putting $L=1-2\alpha\,(0\leq\alpha<1)$ and $M=-1$ in Theorem 2, we obtain the following result for $\widetilde{\mathscr{C}}_q(\alpha)$. □

Corollary 2. *If $g(\xi)\in\mathscr{A}$ has the series form (1), then $g(\xi)\in\widetilde{\mathscr{C}}_q(\alpha)\,(0\leq\alpha<1)$ if and only if*

$$\frac{1}{\xi}\left[g(\xi)*\frac{\xi-\left[T\left(q+q^{-1}\right)+1\right]\left(q-1+q^{-1}\right)\xi^2+\left[T\left(q+2+q^{-1}\right)-1\right]\left(q-1+q^{-1}\right)\xi^3-(2T-1)\xi^4}{\left(1-q^2\xi\right)(1-q\xi)(1-\xi)\left(1-q^{-1}\xi\right)\left(1-q^{-2}\xi\right)}\right]\neq 0, \tag{38}$$

where T is given by (31).

Theorem 3. *If $g(\xi)\in\mathscr{A}$ has the series form (1), then $g(\xi)\in\widetilde{\mathscr{S}}_q[L,M]$ if and only if*

$$1-\sum_{j=2}^{\infty}\frac{\left(\widetilde{[j]}_q-1\right)\left(e^{-i\theta}+M\right)-L+M}{L-M}b_j\xi^{j-1}\neq 0. \tag{39}$$

Proof. From Theorem 1, we find that $g(\xi)\in\widetilde{\mathscr{S}}_q[L,M]$ if and only if

$$\frac{1}{\xi}\left[g(\xi)*\frac{\xi-\left[1+R\left(q-1+q^{-1}\right)\right]\xi^2+R\xi^3}{(1-q\xi)(1-\xi)\left(1-q^{-1}\xi\right)}\right]\neq 0, \tag{40}$$

for all R given by (21). The left hand side of (38) can be written as

$$\begin{aligned}&\frac{1}{\xi}\left[g(\xi)*\left(\frac{R\xi}{(1-\xi)}-\frac{(R-1)\xi}{(1-q\xi)\left(1-q^{-1}\xi\right)}\right)\right]\\&=\frac{1}{\xi}\left[Rg(\xi)-(R-1)\xi\widetilde{D}_q g(\xi)\right]\\&=1-\sum_{j=2}^{\infty}\left(\widetilde{[j]}_q(R-1)-R\right)b_j\xi^{j-1}\\&=1-\sum_{j=2}^{\infty}\frac{\left(\widetilde{[j]}_q-1\right)\left(e^{-i\theta}+M\right)-L+M}{L-M}b_j\xi^{j-1}.\end{aligned} \tag{41}$$

Thus, the proof of Theorem 3 is completed.

Putting $L=1-2\alpha\,(0\leq\alpha<1)$ and $M=-1$ in Theorem 3, we obtain the following result for $\widetilde{\mathscr{S}}_q(\alpha)$. □

Corollary 3. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{S}}_q(\alpha)$ if and only if*

$$1 - \sum_{j=2}^{\infty} \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} - 1\right) - 2(1-\alpha)}{2(1-\alpha)} b_j \xi^{j-1} \neq 0. \quad (42)$$

Theorem 4. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{C}}_q[L, M]$ if and only if*

$$1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} + M\right) - L + M}{L - M} b_j \xi^{j-1} \neq 0\,(\xi \in \Delta). \quad (43)$$

Proof. From Theorem 1, we find that $g(\xi) \in \widetilde{\mathscr{C}}_q[L, M]$ if and only if

$$\frac{1}{\xi}\left[g(\xi) * \frac{\xi - \left[R\left(q + q^{-1}\right) + 1\right]\left(q - 1 + q^{-1}\right)\xi^2 + \left[R\left(q + 2 + q^{-1}\right) - 1\right]\left(q - 1 + q^{-1}\right)\xi^3 - (2R-1)\xi^4}{\left(1 - q^2\xi\right)(1 - q\xi)(1-\xi)\left(1 - q^{-1}\xi\right)\left(1 - q^{-2}\xi\right)}\right] \neq 0, \quad (44)$$

for all R given by (21). The left hand side of (44) can be written as

$$\begin{aligned}
&\frac{1}{\xi}\left[g(\xi) * \left(\frac{R\xi}{(1 - q\xi)\left(1 - q^{-1}\xi\right)} - \frac{(R-1)\left(\xi + \xi^2\right)}{\left(1 - q^2\xi\right)(1-\xi)\left(1 - q^{-2}\xi\right)(1-\xi)}\right)\right] \\
&= \frac{1}{\xi}\left[R\xi\widetilde{D}_q g(\xi) - (R-1)\xi\widetilde{D}_q\left(\xi\widetilde{D}_q g(\xi)\right)\right] \\
&= 1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q\left(\widetilde{[j]}_q (R-1) - R\right) b_j \xi^{j-1} \\
&= 1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} + M\right) - L + M}{L - M} b_j \xi^{j-1},
\end{aligned} \quad (45)$$

and this proves Theorem 4.

Putting $L = 1 - 2\alpha\,(0 \le \alpha < 1)$ and $M = -1$ in Theorem 4, we obtain the following convolution result for $\widetilde{\mathscr{C}}_q(\alpha)$. ☐

Corollary 4. *If $g(\xi) \in \mathscr{A}$ has the series form (1), then $g(\xi) \in \widetilde{\mathscr{S}}_q(\alpha)$ if and only if*

$$1 - \sum_{j=2}^{\infty} \widetilde{[j]}_q \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} - 1\right) - 2(1-\alpha)}{2(1-\alpha)} b_j \xi^{j-1} \neq 0. \quad (46)$$

Theorem 5. *If the function $g(\xi) \in \mathscr{A}$ has the series form (1) satisfying the inequality*

$$\sum_{j=2}^{\infty} \left\{\left(\widetilde{[j]}_q - 1\right)(1 - M) + L - M\right\}\left|b_j\right| \le L - M, \quad (47)$$

then $g(\xi) \in \widetilde{\mathscr{S}}_q[L, M]$.

Proof. Hence,

$$\begin{aligned}
&\left|1 - \sum_{j=2}^{\infty} \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} + M\right) + M - L}{L - M} b_j \xi^{j-1}\right| \\
&\ge 1 - \left|\sum_{j=2}^{\infty} \frac{\left(\widetilde{[j]}_q - 1\right)\left(e^{-i\theta} + M\right) + M - L}{L - M} b_j \xi^{j-1}\right| \\
&\ge 1 - \sum_{j=2}^{\infty} \frac{\left(\widetilde{[j]}_q - 1\right)(1 - M) + L - M}{L - M}\left|b_j\right| > 0.
\end{aligned} \quad (48)$$

Thus, inequality (47) holds, and our result follows from Theorem 3.

Putting $L = 1 - 2\alpha\,(0 \le \alpha < 1)$ and $M = -1$ in Theorem 5, we obtain the following result for $\widetilde{\mathscr{S}}_q(\alpha)$. ☐

Corollary 5. *If the function $g(\xi) \in \mathscr{A}$ has the series form (1) satisfying the inequality*

$$\sum_{j=2}^{\infty} \left(\widetilde{[j]}_q - \alpha\right)\left|b_j\right| \le 1 - \alpha, \quad (49)$$

then $g(\xi) \in \widetilde{\mathcal{S}}_q(\alpha)$.

By using arguments and analysis to those in the proof of Theorem 5, we can derive the following theorem.

Theorem 6. *If the function* $g(\xi) \in \mathcal{A}$ *has the series form (1) satisfying the inequality*

$$\sum_{j=2}^{\infty} \widetilde{[j]}_q \left\{ \left(\widetilde{[j]}_q - 1 \right)(1-M) + L - M \right\} \left| b_j \right| \leq L - M, \tag{50}$$

then $g(\xi) \in \widetilde{\mathcal{C}}_q[L, M]$.

Putting $L = 1 - 2\alpha\,(0 \leq \alpha < 1)$ and $M = -1$ in Theorem 6, we obtain the following result for $\widetilde{\mathcal{C}}_q(\alpha)$.

Corollary 6. *If the function* $g(\xi) \in \mathcal{A}$ *has the series form (1) satisfying the inequality*

$$\sum_{j=2}^{\infty} \widetilde{[j]}_q \left(\widetilde{[j]}_q - \alpha \right) \left| b_j \right| \leq 1 - \alpha, \tag{51}$$

then $g(\xi) \in \widetilde{\mathcal{C}}_q(\alpha)$.

3. Fekete–Szegö Inequalities

In this section, we obtain the Fekete–Szegö inequalities for the classes $\widetilde{\mathcal{S}}_q[L, M]$ and $\widetilde{\mathcal{C}}_q[L, M]$. In order to establish our results, we need the following lemmas.

Lemma 1 (see [17]). *If* $p(\xi) = 1 + \delta_1\xi + \delta_2\xi^2 + \cdots$ *is a function with positive real part in* Δ *and* v *is a complex number, then*

$$\left| \delta_2 - v\delta_1^2 \right| \leq 2 \max\{1; |2v - 1|\}. \tag{52}$$

The result is sharp for the functions given by

$$\begin{aligned} p(\xi) &= \frac{1 + \xi^2}{1 - \xi^2}, \\ p(\xi) &= \frac{1 + \xi}{1 - \xi}. \end{aligned} \tag{53}$$

Lemma 2 (see [17]). *If* $p(\xi) = 1 + \delta_1\xi + \delta_2\xi^2 + \cdots$ *is an analytic function with a positive real part in* Δ*, then*

$$\left| \delta_2 - \nu\delta_1^2 \right| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases} \tag{54}$$

When $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $p(\xi)$ is $((1 + \xi)/(1 - \xi))$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(\xi)$ is $((1 + \xi^2)/(1 - \xi^2))$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(\xi) = \left(\frac{1 + \lambda}{2} \right) \frac{1 + \xi}{1 - \xi} + \left(\frac{1 - \lambda}{2} \right) \frac{1 - \xi}{1 + \xi}, \quad (0 \leq \lambda \leq 1), \tag{55}$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if $p(\xi)$ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\begin{aligned} &\left| \delta_2 - \nu\delta_1^2 \right| + \nu\left| \delta_1 \right|^2 \leq 2, \quad \left(0 \leq \nu \leq \frac{1}{2} \right), \\ &\left| \delta_2 - \nu\delta_1^2 \right| + (1 - \nu)\left| \delta_1 \right|^2 \leq 2, \quad \left(\frac{1}{2} \leq \nu \leq 1 \right). \end{aligned} \tag{56}$$

Theorem 7. *If* $g(\xi)$ *defined by (1) belongs to the class* $\widetilde{\mathcal{S}}_q[L, M]$*, then*

$$\left| b_3 - \mu b_2^2 \right| \leq \frac{L - M}{\widetilde{[3]}_q - 1} \max\left\{ 1; \left| M - \frac{L - M}{\widetilde{[2]}_q - 1} \left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1}\mu \right) \right| \right\}. \tag{57}$$

Proof. If $g(\xi) \in \widetilde{\mathcal{S}}_q[L, M]$, then there is a Schwarz function w in Δ such that

$$\frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)} = \frac{1 + Lw(\xi)}{1 + Mw(\xi)}. \tag{58}$$

Define the function $p(\xi)$ by

$$p(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + \delta_1\xi + \delta_2\xi^2 + \cdots. \tag{59}$$

Since $w(\xi)$ is a Schwarz function, we see that $\mathfrak{R}\{p(\xi)\} > 0$ and $p(0) = 1$. Now, by substituting (59) in (58), we have

$$\frac{\xi \widetilde{D}_q g(\xi)}{g(\xi)} = 1 + \frac{(L - M)}{2}\delta_1\xi + \frac{(L - M)}{2}\left[\delta_2 - \frac{(1 + M)}{2}\delta_1^2 \right]\xi^2 + \cdots. \tag{60}$$

From the above equation, we obtain

$$\begin{aligned} \left(\widetilde{[2]}_q - 1 \right) b_2 &= \frac{(L - M)}{2}\delta_1, \\ \left(\widetilde{[3]}_q - 1 \right) b_3 - \left(\widetilde{[2]}_q - 1 \right) b_2^2 &= \frac{(L - M)}{2}\left[\delta_2 - \frac{(1 + M)}{2}\delta_1^2 \right], \end{aligned} \tag{61}$$

or, equivalently,

$$\begin{aligned} b_2 &= \frac{(L - M)}{2\left(\widetilde{[2]}_q - 1 \right)}\delta_1, \\ b_3 &= \frac{(L - M)}{2\left(\widetilde{[3]}_q - 1 \right)}\left[\delta_2 - \frac{1}{2}\left(1 + M - \frac{(L - M)}{\widetilde{[2]}_q - 1} \right)\delta_1^2 \right]. \end{aligned} \tag{62}$$

Therefore, we have

$$b_3 - \mu b_2^2 = \frac{L-M}{2\left(\widetilde{[3]}_q - 1\right)}\left\{\delta_2 - \nu\delta_1^2\right\}, \tag{63}$$

where

$$\nu = \frac{1}{2}\left[1 + M - \frac{L-M}{\widetilde{[2]}_q - 1}\left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1}\mu\right)\right]. \tag{64}$$

Our result now follows from Lemma 1. This completes the proof of Theorem 1.

Similarly, we can prove the following theorem for the class $\widetilde{\mathcal{C}}_q[L, M]$. □

Theorem 8. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{C}}_q[L, M]$, then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q - 1\right)}\max\left\{1; \left|M - \frac{L-M}{\widetilde{[2]}_q - 1}\left(1 - \frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q - 1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q - 1\right)}\mu\right)\right|\right\}. \tag{65}$$

The result is sharp.

Putting $L = 1 - 2\alpha\,(0 \leq \alpha < 1)$ and $M = -1$ in Theorems 7 and 8, we obtain the following corollaries.

Corollary 7. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{S}}_q(\alpha)\,(0 \leq \alpha < 1)$, then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q - 1}\max\left\{1; \left|1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1}\left(1 - \frac{\widetilde{[3]}_q - 1}{\widetilde{[2]}_q - 1}\mu\right)\right|\right\}. \tag{66}$$

Corollary 8. *If $g(\xi)$ given by (1) belongs to the class $\widetilde{\mathcal{C}}_q(\alpha)\,(0 \leq \alpha < 1)$, then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q - 1\right)}\max\left\{1; \left|1 + \frac{2(1-\alpha)}{\widetilde{[2]}_q - 1}\left(1 - \frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q - 1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q - 1\right)}\mu\right)\right|\right\}. \tag{67}$$

The results are sharp.

Theorem 9. *Let*

$$\begin{aligned}
\sigma_1 &= \frac{\left(\widetilde{[2]}_q - 1\right)(L-M) - \left(\widetilde{[2]}_q - 1\right)^2(1+M)}{\left(\widetilde{[3]}_q - 1\right)(L-M)},\\
\sigma_2 &= \frac{\left(\widetilde{[2]}_q - 1\right)(L-M) + \left(\widetilde{[2]}_q - 1\right)^2(1-M)}{\left(\widetilde{[3]}_q - 1\right)(L-M)},\\
\sigma_3 &= \frac{\left(\widetilde{[2]}_q - 1\right)(L-M) - \left(\widetilde{[2]}_q - 1\right)^2 M}{\left(\widetilde{[3]}_q - 1\right)(L-M)}.
\end{aligned} \tag{68}$$

If g given by (1) belongs to the class $\widetilde{\mathcal{S}}_q[L, M]$, then

$$\left|b_3-\mu b_2^2\right| \leq \begin{cases} -\dfrac{L-M}{\widetilde{[3]}_q-1}\left[M-\dfrac{L-M}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right], & (\mu \leq \sigma_1), \\ \dfrac{L-M}{\widetilde{[3]}_q-1}, & (\sigma_1 \leq \mu \leq \sigma_2), \\ \dfrac{L-M}{\widetilde{[3]}_q-1}\left[M-\dfrac{L-M}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right], & (\mu \geq \sigma_2). \end{cases} \tag{69}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|b_3-\mu b_2^2\right| + \frac{\left(\widetilde{[2]}_q-1\right)^2}{\left(\widetilde{[3]}_q-1\right)}\left[\frac{1+M}{L-M}-\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right]\left|b_2\right|^2 \leq \frac{L-M}{\widetilde{[3]}_q-1}. \tag{70}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|b_3-\mu b_2^2\right| + \frac{\left(\widetilde{[2]}_q-1\right)^2}{\left(\widetilde{[3]}_q-1\right)}\left[\frac{1-M}{L-M}+\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right]\left|b_2\right|^2 \leq \frac{L-M}{\widetilde{[3]}_q-1}. \tag{71}$$

Proof. Applying Lemma 2 to (63) and (64), we can obtain our results asserted by Theorem 9.

Putting $L = 1-2\alpha\,(0 \leq \alpha < 1)$ and $M = -1$ in Theorem 9, we obtain the following result. □

Corollary 9. *Let*

$$\begin{aligned} \sigma_4 &= \frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}, \\ \sigma_5 &= \frac{\left(\widetilde{[2]}_q-1\right)(1-\alpha)+\left(\widetilde{[2]}_q-1\right)^2}{\left(\widetilde{[3]}_q-1\right)(1-\alpha)}, \\ \sigma_6 &= \frac{2\left(\widetilde{[2]}_q-1\right)(1-\alpha)+\left(\widetilde{[2]}_q-1\right)^2}{2\left(\widetilde{[3]}_q-1\right)(1-\alpha)}. \end{aligned} \tag{72}$$

If g given by (1) belongs to the class $\widetilde{\mathcal{S}}_q(\alpha)\,(0 \leq \alpha < 1)$, then

$$\left|b_3-\mu b_2^2\right| \leq \begin{cases} \dfrac{2(1-\alpha)}{\widetilde{[3]}_q-1}\left[1+\dfrac{2(1-\alpha)}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right], & (\mu\leq\sigma_4), \\ \dfrac{2(1-\alpha)}{\widetilde{[3]}_q-1}, & (\sigma_4\leq\mu\leq\sigma_5), \\ -\dfrac{2(1-\alpha)}{\widetilde{[3]}_q-1}\left[1+\dfrac{2(1-\alpha)}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right], & (\mu\geq\sigma_5). \end{cases} \tag{73}$$

Further, if $\sigma_4\leq\mu\leq\sigma_6$, then

$$\left|b_3-\mu b_2^2\right| - \frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}\left(1-\frac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\left|b_2\right|^2 \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q-1}. \tag{74}$$

If $\sigma_6\leq\mu\leq\sigma_5$, then

$$\left|b_3-\mu b_2^2\right| + \frac{\left(\widetilde{[2]}_q-1\right)^2}{\left(\widetilde{[3]}_q-1\right)}\left[\frac{1}{(1-\alpha)}+\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q-1}{\widetilde{[2]}_q-1}\mu\right)\right]\left|b_2\right|^2 \leq \frac{2(1-\alpha)}{\widetilde{[3]}_q-1}. \tag{75}$$

Similarly, we can obtain the following theorem.

Theorem 10. *Let*

$$\begin{aligned} \chi_1 &= \frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\left[L-M-\left(\widetilde{[2]}_q-1\right)(1+M)\right]}{\left(\widetilde{[3]}_q-1\right)(L-M)}, \\ \chi_2 &= \frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\left[L-M+\left(\widetilde{[2]}_q-1\right)(1-M)\right]}{\left(\widetilde{[3]}_q-1\right)(L-M)}, \\ \chi_3 &= \frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\left[L-M-\left(\widetilde{[2]}_q-1\right)M\right]}{\left(\widetilde{[3]}_q-1\right)(L-M)}. \end{aligned} \tag{76}$$

If g given by (1) belongs to the class $\widetilde{\mathscr{C}}_q[L,M]$, then

$$\left|b_3-\mu b_2^2\right| = \begin{cases} -\dfrac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[M-\dfrac{L-M}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right], & (\mu\leq\chi_1), \\ \dfrac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}, & (\chi_1\leq\mu\leq\chi_2), \\ \dfrac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[M-\dfrac{L-M}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right], & (\mu\geq\chi_2). \end{cases} \tag{77}$$

Further, if $\chi_1\leq\mu\leq\chi_3$, then

$$\left|b_3-\mu b_2^2\right|+\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[\frac{1+M}{L-M}-\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right]\left|b_2\right|^2\le\frac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{78}$$

If $\chi_3\le\mu\le\chi_2$, then

$$\left|b_3-\mu b_2^2\right|+\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[\frac{1-M}{L-M}+\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right]\left|b_2\right|^2\le\frac{L-M}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{79}$$

Putting $L=1-2\alpha\,(0\le\alpha<1)$ and $M=-1$ in Theorem 10, we obtain the following result.

Corollary 10. *Let*

$$\begin{aligned}\chi_4&=\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}{\widetilde{[3]}_q-1},\\ \chi_5&=\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\left[1-\alpha+\left(\widetilde{[2]}_q-1\right)\right]}{\left(\widetilde{[3]}_q-1\right)(1-\alpha)},\\ \chi_6&=\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\left[2(1-\alpha)+\left(\widetilde{[2]}_q-1\right)\right]}{2\left(\widetilde{[3]}_q-1\right)(1-\alpha)}.\end{aligned} \tag{80}$$

If g given by (1) belongs to the class $\widetilde{\mathscr{C}}_q(\alpha)\,(0\le\alpha<1)$, then

$$\left|b_3-\mu b_2^2\right|\le\begin{cases}\dfrac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[1+\dfrac{2(1-\alpha)}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right], & (\mu\le\chi_4),\\ \dfrac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}, & (\chi_4\le\mu\le\chi_5),\\ -\dfrac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[1+\dfrac{2(1-\alpha)}{\widetilde{[2]}_q-1}\left(1-\dfrac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right], & (\mu\ge\chi_5).\end{cases} \tag{81}$$

Further, if $\chi_4\le\mu\le\chi_6$, then

$$\left|b_3-\mu b_2^2\right|-\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left(1-\frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\left|b_2\right|^2\le\frac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{82}$$

If $\chi_6\le\mu\le\chi_5$, then

$$\left|b_3-\mu b_2^2\right|+\frac{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\left[\frac{1}{1-\alpha}+\frac{1}{\widetilde{[2]}_q-1}\left(1-\frac{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)}\mu\right)\right]\left|b_2\right|^2\le\frac{2(1-\alpha)}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{83}$$

4. Conclusion

In this present investigation, we have introduced two classes $\widetilde{\mathcal{S}}_q[L, M]$ and $\widetilde{\mathcal{C}}_q[L, M]$ of analytic functions by using the symmetric q-difference operator linked to an open unit disc $\Delta = \{\xi \in \mathbb{C}: |\xi| < 1\}$. We also studied convolution results, coefficient estimates, and Fekete–Szegö inequalities for the newly defined classes. We note that our results naturally include several results that are known for those subclasses, which are listed in the introduction section.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The author would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (22UQU4350561DSR01).

References

[1] M. I. S. Robertson, "On the theory of univalent functions," *The Annals of Mathematics*, vol. 37, no. 2, pp. 374–408, 1936.

[2] T. Bulboacă, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publication, Cluj-Napoca, Romania, 2005.

[3] S. S. Miller and P. T. Mocanu, *Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc, New York, NY, USA, 2000.

[4] O. P. Ahuja, "Families of analytic functions related to Ruscheweyh derivatives and subordinate to convex functions," *Yokohama Mathematical Journal*, vol. 41, pp. 39–50, 1993.

[5] R. M. Goel and B. S. Mehrok, "On the coefficients of a subclass of starlike functions," *Indian Journal of Pure and Applied Mathematics*, vol. 12, pp. 634–647, 1981.

[6] W. Janowski, "Some extremal problems for certain families of analytic functions," *Bulletin of the Polish Academy of Sciences*, vol. 21, pp. 17–25, 1973.

[7] W. Janowski, "Some extremal problems for certain families of analytic functions I," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[8] H. Silverman, E. M. Silvia, and D. Telage, "Convolution conditions for convexity, starlikeness and spiral-likeness," *Mathematische Zeitschrift*, vol. 162, no. 2, pp. 125–130, 1978.

[9] H. Silverman and E. M. Silvia, "Subclasses of starlike functions subordinate to convex functions," *Canadian Journal of Mathematics*, vol. 37, pp. 48–61, 1985.

[10] K. L. Brahim and Y. Sidomou, "On some symmetric q-special functions," *Le Matematiche*, vol. LXVIII, pp. 107–122, 2013.

[11] M. K. Aouf and T. M. Seoudy, "Classes of analytic functions related to the Dziok-Srivastava operator," *Integral Transforms and Special Functions*, vol. 22, no. 6, pp. 423–430, 2011.

[12] T. M. Seoudy and M. K. Aouf, "Convolution properties for certain classes of analytic functions defined by q-derivative operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 846719, 7 pages, 2014.

[13] H. Silverman, "Univalent functions with negative coefficients," *Proceedings of the American Mathematical Society*, vol. 51, no. 1, pp. 109–116, 1975.

[14] M. K. Aouf and T. M. Seoudy, "Convolution properties for classes of bounded analytic functions with complex order defined by q-derivative operator," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, pp. 1279–1288, 2019.

[15] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q-starlike and q-convex functions of complex order," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 135–145, 2016.

[16] V. P. Gupta and P. K. Jain, "Certain classes of univalent functions with negative coefficients II," *Bulletin of the Australian Mathematical Society*, vol. 15, no. 3, pp. 467–473, 1976.

[17] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, pp. 157–169, Cambridge, MA, USA, 1992.