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Research Article

Pullback Attractors for a Class of Semilinear Second-Order Nonautonomous Evolution Equations with Hereditary Characteristics

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In this paper, we investigate the long-time behavior for the nonautonomous semilinear second-order evolution equation $(\partial^2 u/zt^2) - \Delta u - \Delta(zu/zt) - \Delta(z^2 u/zt^2) = f(t, u(x, t-\rho(t))) + g(t, x), \text{in}(\tau, \infty) \times \Omega$ with some hereditary characteristics, where Ω is an open-bounded domain of $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$. Firstly, we establish the existence of solutions for the second-order nonautonomous evolution equation by the standard Faedo–Galerkin method, but without the uniqueness of solutions. Then by proving the pullback asymptotic compactness for the multivalued process $\{U(t,\tau)\}$ on $C_{H_0^1(\Omega), H_0^1(\Omega)}$, we obtain the existence of pullback attractors in the Banach spaces $C_{H_0^1(\Omega), H_0^1(\Omega)}$ for the multivalued process generated by a class of second-order nonautonomous evolution equations with hereditary characteristics and ill-posedness.

1. Introduction

The study of nonlinear dynamics is a fascinating question which is at the very heart of understanding of many important problems of the natural sciences. The long-time behavior of PDEs can be described in the terms of attractors of the corresponding semigroups, such as Babin and Vishik [1], Chepyzhov and Vishik [2], Chueshov and Lasiecka [3], Hale [4], Ladyzhenskaya [5], or Temam [6], and the references therein. The study of pullback attractor for infinite dimensional dynamical systems has attracted much attention and has made fast progress in recent decades [7–13].

In this paper, we consider the following nonautonomous semilinear second-order evolution equation with delays:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \dfrac{\partial u}{\partial t} - \nu \Delta \dfrac{\partial^2 u}{\partial t^2} = f(t, u(x, t-\rho(t))) + g(t,x), & \text{in}(\tau, \infty) \times \Omega, \\ u(t,x) = \phi(t-\tau, x), & t \in [\tau - h, \tau], \quad x \in \Omega, \\ \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial \phi(t-\tau, x)}{\partial t}, & t \in [\tau - h, \tau], \quad x \in \Omega, \\ u(t,x) = 0, & \text{on}(\tau, \infty) \times \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open-bounded domain of $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, τ is the initial time, and ϕ is the initial data on the interval $[\tau - h, \tau]$ with $h > 0$.

The nonlinearity $f(\cdot)$ and the external force $g(t, x)$ satisfy the following conditions, respectively.

In (A_1), there exist positive constants α_1 and α_2 such that the functions $f \in \mathscr{C}(\mathbb{R} \times \mathbb{R}^N; \mathbb{R})$ and $\rho \in \mathscr{C}(\mathbb{R}; [0, h])$ satisfy

$$|f(t, v)|^2 \leq \alpha_1^2 + \alpha_2^2 |v|^2, \quad \forall t \in \mathbb{R}, v \in \mathbb{R}^N, \tag{2}$$

$$|\rho\prime(t)| \leq \rho_* < 1, \quad \forall t \in \mathbb{R}. \tag{3}$$

In (A_2), the external force $g(t, x)$ belongs to the space $L^2_{loc}(\mathbb{R}, L^2(\Omega))$ such that

$$\int_{-\infty}^{\tau} e^{\lambda r} |g(r, \cdot)|^2_{L^2(\Omega)} \mathrm{d}r < \infty, \quad \forall \tau \in \mathbb{R}, \lambda > 0. \tag{4}$$

When $\nu = 0$ and without variable delays, Equation (1) becomes the usual strongly damped wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \frac{\partial u}{\partial t} = f(u) + g(t, x). \tag{5}$$

Its asymptotic behavior has been studied extensively in terms of attractors [1,14–20] . The long-time behavior for the strongly damped wave equation with delays has been investigated in Refs. [7,12].

For each fixed $\nu > 0$ and without variable delays, Equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \frac{\partial u}{\partial t} - \nu \Delta \frac{\partial^2 u}{\partial t^2} = f(u) + g(t, x). \tag{6}$$

It is a special form of the so-called improved Boussinesq equation (see [21–24]) with damped term $-\Delta u_t$, which was used to describe ion-sound waves in plasma by Makhankov [22,25] and also known to represent other sorts of "propagation problems" of, for example, lengthways waves in nonlinear elastic rods and ion-sonic waves of space transformations by a weak nonlinear effect [21]. Carvalho and Cholewa [26] presented systematic results including the existence-uniqueness and long-time behavior of Equation (6) by using the semigroup approach. The long-time behavior of, especially the global attractor, exponential attractors has been extensively studied by several authors [26–29] . For the nonautonomous semilinear second-order evolution (6) with the memory term, we get

$$u_{tt} - \Delta u - \Delta u_t - \int_0^{\infty} k_{\varepsilon}(s) \Delta u_t(t - s) \mathrm{d}s - \nu \Delta u_{tt} + f(u) = g(x, t). \tag{7}$$

Zhang et al. in Ref. [30] constructed the existence of robust family of exponential attractors while the nonlinearity is critical and the time-dependent external forcing term is assumed to be only translation-bounded.

Indeed, for Equation (6), in all above results, we require the solution operator given as follows:

$$S(t)\colon u_0 \mapsto u(t). \tag{8}$$

To be well-defined and continuous in a proper phase space. However, for many interesting problems, the well-posedness of the solution operator $S(t)$ is not known or does not hold true [11–13, 31–33] .

To the best of our knowledge, the long-time dynamics of Equation (1) with hereditary characteristics has not been considered by predecessors. There are some barriers encountered. On the one hand, Equation (1) contains the term $-\Delta u_{tt}$, and it is essentially different from the usual wave equation in Refs. [1, 7, 12, 14–20]. For example, the wave equation has some smoothing effect; for example, although the initial data only belongs to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, for Equation (1), if the initial data $\phi \in C_{H_0^1(\Omega), H_0^1(\Omega)}$, then the solution $u_t(\cdot)$ is always in $C_{H_0^1(\Omega), H_0^1(\Omega)}$ and has no higher regularity because of $-\Delta u_{tt}$, and it will cause some difficulties [26–29]. On the other hand, suppose that $(A_1) - (A_2)$ hold true, then $g \in L^2_{loc}(\mathbb{R}; H)$ and $\phi \in C_{H_0^1(\Omega), H_0^1(\Omega)}$, then the uniqueness of the weak solutions for Equation (1) are lost; that is, we need to overcome some difficulties brought by ill-posedness. In addition, the delay term also causes some difficulties to obtain the pullback attractors.

This paper is organized as follows. In Section 1, we have expounded on research progress regarding our research problem and have given some assumptions. In Section 2, we introduce some notations and functions spaces, and we recall some useful results on nonautonomous multivalued dynamical systems and pullback attractors. In Section 3, we prove the existence of solutions for Equation (1) in $\phi \in C_{H_0^1(\Omega), H_0^1(\Omega)}$. The existence of pullback attractor for the multivalued process $\{U(t, \tau)\}$ corresponding to Equation (1) is proved in Section 4.

2. Preliminaries

Next, we iterate some definitions and abstract results concerning the multivalued dynamical systems and the pullback attractor, which is necessary to obtain our main results [7–13,34] .

Let X be a complete metric space with metric $d_X(\cdot, \cdot)$, and let $\mathscr{P}(X)$ be the class of nonempty subsets of X. Denoted by $H_X^{\text{semi}}(\cdot, \cdot)$ the Hausdorff semidistance between two nonempty subsets of a complete metric space X can be defined as

$$H_X^{\text{semi}}(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b). \tag{9}$$

Definition 1. A family of mappings $U(t, \tau)\colon X \longrightarrow \mathscr{P}(X)$ and $t \geq \tau$, $\tau \in \mathbb{R}$ is called to be a multivalued process if

$$U(\tau, \tau)x = x, \quad \forall \tau \in \mathbb{R}, x \in X, \tag{10}$$

$$U(t, r)U(r, \tau)x = U(t, \tau)x \text{for all} \tau \leq r \leq t, \quad x \in X. \tag{11}$$

Let $\mathfrak{D}$ be a nonempty class of parametrized sets $\mathscr{D} = \{D(t)\}_{t \in \mathbb{R}} \subset \mathscr{P}(X)$.

Definition 2. A collection $\mathfrak{D}$ of some families of nonempty closed subsets of X is said to be inclusion-closed if for each $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$, we have

$$\{\widetilde{D}(t)\colon \widetilde{D}(t) \text{is a non empty subset of} D(t), \quad \forall t \in \mathbb{R}\}. \quad (12)$$

which also belongs to $\mathfrak{D}$.

Definition 3. Let $\{U(t,\tau)\}$ be a multivalued process on X, then we get those as follows:

(1) $\mathscr{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ is called a $\mathfrak{D}$-pullback absorbing set for $\{U(t,\tau)\}$ if for any $\mathscr{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ and each $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathscr{B}, t) \in \mathbb{R}^+$ such that

$$U(t,t-s)B(t-s) \subset Q(t), \quad \forall s \ge t_0 \quad (13)$$

(2) $\{U(t,\tau)\}$ is said to be $\mathfrak{D}$-pullback asymptotically upper-semicompact in X with respect to $\mathscr{B}$ if for each fixed $t \in \mathbb{R}$, any sequence $y_n \in U(t,t-s_n)x_n$ has a convergent subsequence in X whenever $s_n \longrightarrow +\infty (n \longrightarrow \infty)$ and $x_n \in B(t-s_n)$ with $\mathscr{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$

Theorem 1. *A family of nonempty compact subsets $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ of X is called to be a $\mathfrak{D}$-pullback attractor for the multivalued process $\{U(t,\tau)\}$ if*

(1) $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ is an invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t), \quad \forall t \ge \tau, \tau \in \mathbb{R}. \quad (14)$$

(2) $\mathscr{A}$ attracts every member of $\mathfrak{D}$, i.e., for every $\mathscr{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ and any fixed $t \in \mathbb{R}$, we get

$$\lim_{s\longrightarrow+\infty} H_X^{\text{semi}}(U(t,t-s)B, A(t)) = 0. \quad (15)$$

Theorem 2. *Let $\{U(t,\tau)\}$ be a multivalued process on Banach space X, and let $\mathscr{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ be a $\mathfrak{D}-$ pullback absorbing set for $\{U(t,\tau)\}$ in $\mathfrak{D}$. Suppose that U can be written as*

$$U(t,\tau) = U_1(t,\tau) + U_2(t,\tau), \quad \forall t \ge \tau, \quad (16)$$

and for any fixed $t \in \mathbb{R}$, then we get those as follows:

(1) $\lim_{s\longrightarrow\infty}\|U_1(t,t-s)Q(t-s)\|_X = 0$

(2) For any fixed $s > 0$, every sequence $y_n \in U_2(t,t-s)Q(t-s)$ is a Cauchy sequence in X

Then $\{U(t,\tau)\}$ is $\mathfrak{D}$-pullback asymptotically upper-semicompact in X.

Theorem 3. *Let $\mathfrak{D}$ be an inclusion-closed collection of some families of nonempty closed subsets of X and $\{U(t,\tau)\}$ be a multivalued process on X. Also, U has a closed values and let $U(t,\tau)x$ be upper semicontinuous in x for fixed $t \ge \tau, \tau \in \mathbb{R}$. Suppose that $\{U(t,\tau)\}$ is $\mathfrak{D}$-pullback asymptotical upper-semicompact in X, then $\{U(t,\tau)\}$ has a $\mathfrak{D}$-pullback absorbing set $\mathscr{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$, and $Q(t)$ is closed for every $t \in \mathbb{R}$. Then, the $\mathfrak{D}$-pullback attractor $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ is unique for each $t \in \mathbb{R}$ and is given by*

$$\mathscr{A}(t) = \bigcap_{T\in\mathbb{R}^+} \overline{\bigcup_{s\ge T} U(t,t-s)Q(t-s)}. \quad (17)$$

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, which are Hilbert spaces for the usual inner products and associated norms. Let $Au = -\Delta u$ for any $u \in D(A)$, where $D(A) = \{u \in V\colon Au \in H\} = H_0^1(\Omega) \cap H^2(\Omega)$. Note that $D(A)$ is also a Hilbert space for the norm $\|u\|_{D(A)} = |Au|$, $u \in D(A)$.

Let X be a Banach space with norm $\|\cdot\|_X$. Let $h > 0$ be a given positive number, which will denote the delay time, and let C_X denote the Banach space $C^0([-h,0];X)$ with the sup-norm, then we get

$$\|\psi\|_{C_X} \coloneqq = \sup_{s\in[-h,0]} \|\psi(s)\|_X, \quad \text{for} \psi \in C_X. \quad (18)$$

We can denote by $C_{X,X}$ the Banach spaces $C_X \cap C^1([-h,0];X)$ with the norm $\|\cdot\|_{C_{X,X}}$ that is defined by

$$\|\psi\|_{C_{X,X}}^2 \coloneqq = \|\psi\|_{C_X}^2 + \|\psi'\|_{C_X}^2, \quad \text{for} \psi \in C_{X,X}. \quad (19)$$

Given $\tau \in \mathbb{R}, T > \tau$ and $u\colon [\tau - h, T \longrightarrow)X$, for each $t \in [\tau, T)$, we can denote as

$$u_t\colon [-h,0] \longrightarrow X. \quad (20)$$

Denote the function defined on by

$$u_t(s) = u(t+s), \quad s \in [-h,0]. \quad (21)$$

Without the loss of generality, we assume that $\nu = 1$ in the following discussion.

3. Existence of Solutions

In this section, we want to prove the existence of solutions which can be obtained by the standard Faedo–Galerkin methods (see [1,6,35]), and the multivalued evolution processes corresponding to Equation (1) will be constructed. We only give the sketch of proof, and the details similar to the proof of Theorem 4 in Ref. [2], Sec. XV.3 and the arguments in [6] Sec. IV. 4.4.

Theorem 4. *Suppose that $(A_1) - (A_2)$ holds true, $g \in L_{loc}^2(\mathbb{R}; H)$ and $\phi \in C_{V,V}$, then there exist solutions $u(t)$ of Equation (1) such that*

$$u \in \mathscr{C}^1([\tau - h, T]; V),$$
$$\frac{\partial u}{\partial t} \in \mathscr{C}^1([\tau - h; V), \quad \forall T > \tau. \quad (22)$$

Proof. (Sketch)

Let $Au = -\Delta u$ for any $u \in D(A)$, where $D(A) = \{u \in V\colon Au \in H\} = H_0^1(\Omega) \cap H^2(\Omega)$. Since A is self-adjoint, positive operator and has a compact inverse, and there exists a complete set of eigenvectors $\{\omega_i\}_{i=1}^{\infty}$ in H, and the corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ satisfy

$$A\omega_i = \lambda_i\omega_i, \quad 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_i \le \ldots \longrightarrow +\infty, i \longrightarrow +\infty. \tag{23}$$

Setting $V_m = \text{span}\{\omega_1, \omega_2, \ldots, \omega_m\}$ and P_m is the orthogonal projection onto H_m, then we get

$$P_m u = \sum_{i=1}^{m} (u, \omega_i)\omega_i, u \in H. \tag{24}$$

We consider the approximate solutions of Equation (1) in the form

$$u_m(t) = \sum_{j=1}^{m} \alpha_{jm}(t) w_j. \tag{25}$$

Then $u_m(t)$ satisfies

$$\begin{cases} \dfrac{\partial^2 u_m}{\partial t^2} - \Delta u_m - \Delta\dfrac{\partial u_m}{\partial t} - \nu\Delta\dfrac{\partial^2 u_m}{\partial t^2} \\ = f(t, u_m(x, t-\rho(t))) + P_m g(t, x), \\ u_m(t, x) = P_m\phi(t-\tau, x), \quad t \in [\tau - h, \tau], \\ \dfrac{\partial u_m(t, x)}{\partial t} = \dfrac{\partial P_m\phi(t-\tau, x)}{\partial t}, \quad t \in [\tau - h, \tau]. \end{cases} \tag{26}$$

Let $v_m = u_m' + \eta u_m\ (0 < \eta < 1)$, and we write Equation (26) as

$$\begin{aligned} &\frac{d}{dt}v_m - \eta v_m + \eta^2 u_m - \left(1 - \eta + \eta^2\right)\Delta u_m \\ &- \Delta\frac{d}{dt}v_m - (1-\eta)\Delta v_m \\ &= f(t, u_m(t-\rho(t))) + P_m g(t). \end{aligned} \tag{27}$$

Multiplying Equation (27) by v_m in $L^2(\Omega)$, we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v_m\|^2 + \|\nabla v_m\|^2 + \eta^2\|u_m\|^2 + \left(1-\eta+\eta^2\right)\|\nabla u_m\|^2\right) \\ &- \eta\|v_m\|^2 + \eta^3\|u_m\|^2 + \eta\left(1-\eta+\eta^2\right)\|\nabla u_m\|^2 + (1-\eta)\|\nabla v_m\|^2 \\ &= (f(t, u_m(t-\rho(t))), v_m) + (g(t), v_m). \end{aligned} \tag{28}$$

Noting (2), using Young's inequality, we get that

$$\begin{aligned} &|(f(t, u_m(t-\rho(t))), v_m)| \le \frac{\eta}{2}\|v_m\|^2 \\ &+ \frac{\alpha_2^2}{2\eta}\|u_m(t-\rho(t))\|^2 + \frac{\alpha_1^2|\Omega|}{2\eta}, \end{aligned} \tag{29}$$

and

$$|(g(t), v_m)| \le \frac{\eta}{2}\|v_m\|^2 + \frac{1}{2\eta}\|g(t)\|^2. \tag{30}$$

By the Poincáre inequality $\lambda_1\|u\|^2 \le \|\nabla u\|^2$, we get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v_m\|^2 + \|\nabla v_m\|^2 + \eta^2\|u_m\|^2 + \left(1-\eta+\eta^2\right)\|\nabla u_m\|^2\right) \\ &+ \left(\frac{\lambda_1(1-\eta)}{2} - 2\eta\right)\|v_m\|^2 + \frac{1-\eta}{2}\|\nabla v_m\|^2 + \eta^3\|u_m\|^2 \\ &+ \eta\left(1-\eta+\eta^2\right)\|\nabla u_m\|^2 \\ &\le \frac{\alpha_2^2}{2\eta}\|u_m(t-\rho(t))\|^2 + \frac{1}{2\eta}\|g(t)\|^2 + \frac{\alpha_1^2|\Omega|}{2\eta}. \end{aligned} \tag{31}$$

Choosing $\eta = \min\{1/3, \lambda_1/\lambda_1 + 6\}$, we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v_m\|^2 + \|\nabla v_m\|^2 + \eta^2\|u_m\|^2 + \left(1-\eta+\eta^2\right)\|\nabla u_m\|^2\right) \\ &+ \eta\left(\|v_m\|^2 + \|\nabla v_m\|^2 + \eta^2\|u_m\|^2 + \left(1-\eta+\eta^2\right)\|\nabla u_m\|^2\right) \\ &\le \frac{\alpha_2^2}{2\eta}\|u_m(t-\rho(t))\|^2 + \frac{1}{2\eta}\|g(t)\|^2 + \frac{\alpha_1^2|\Omega|}{2\eta}. \end{aligned} \tag{32}$$

Note that for any $\eta > 0$, we have

$$\begin{aligned} &\|v_m\|^2 + \|\nabla v_m\|^2 + \eta^2\|u_m\|^2 + \left(1-\eta+\eta^2\right)\|\nabla u_m\|^2 \\ &\sim \|v_m\|^2 + \|\nabla v_m\|^2 + \|u_m\|^2 + \|\nabla u_m\|^2 \\ &\sim \|\nabla v_m\|^2 + \|\nabla u_m\|^2. \end{aligned} \tag{33}$$

Now integrating (32) from τ to t, we get

$$\begin{aligned} &\|\nabla v_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \eta\int_\tau^t\left(\|\nabla v_m(s)\|^2 + \|\nabla u_m(s)\|^2\right)ds \\ &\le \|\nabla v_m(\tau)\|^2 + \|\nabla u_m(\tau)\|^2 + \frac{\alpha_2^2}{\eta}\int_\tau^t\|u_m(s-\rho(s))\|^2 ds \\ &+ \frac{1}{\eta}\int_\tau^t\|g(s)\|^2 ds + \frac{\alpha_1^2|\Omega|}{\eta}(t-\tau). \end{aligned} \tag{34}$$

In view of $\rho(s) \in [0, h]$ and the fact

$$\frac{1}{1-\rho\prime(s)} \le \frac{1}{1-\rho^*}, \tag{35}$$

for all $s \in \mathbb{R}$, setting $r = s - \rho(s)$, we arrive at

$$\begin{aligned}
&\frac{\alpha_2^2}{\eta}\int_\tau^t \|u_m(s-\rho(s))\|^2 \mathrm{d}s\\
&\le \frac{\alpha_2^2}{\eta(1-\rho_*)}\left(\int_{\tau-h}^{\tau}\|u_m(s)\|^2\mathrm{d}s+\int_\tau^t\|u_m(s)\|^2\mathrm{d}s\right)\\
&\le \frac{\alpha_2^2}{\eta\lambda_1(1-\rho_*)}\|\phi\|_{V,V}^2\\
&\quad+\frac{\alpha_2^2}{\eta\lambda_1(1-\rho_*)}\int_\tau^t\left(\|\nabla v_m(s)\|^2+\|\nabla u_m(s)\|^2\right)\mathrm{d}s.
\end{aligned} \tag{36}$$

Thus, we obtain that

$$\begin{aligned}
&\|\nabla v_m(t)\|^2+\|\nabla u_m(t)\|^2\\
&\le \|\nabla v_m(\tau)\|^2+\|\nabla u_m(\tau)\|^2+\frac{\alpha_2^2}{\eta\lambda_1(1-\rho_*)}\|\phi\|_{V,V}^2\\
&\quad+\frac{1}{\eta}\int_\tau^t\|g(s)\|^2\mathrm{d}s\\
&\quad+\frac{\alpha_1^2|\Omega|}{\eta}(t-\tau)+\frac{\alpha_2^2}{\eta\lambda_1(1-\rho_*)}\\
&\quad\int_\tau^t\left(\|\nabla v_m(s)\|^2+\|\nabla u_m(s)\|^2\right)\mathrm{d}s.
\end{aligned} \tag{37}$$

By the integral form of Gronwall lemma, we infer that

$$\begin{aligned}
&\|\nabla v_m(t)\|^2+\|\nabla u_m(t)\|^2\\
&\le Ce^{C(t-\tau)}\left(\|\phi\|_{V,V}^2+(t-\tau)+\int_\tau^t\|g(s)\|^2\mathrm{d}s\right).
\end{aligned} \tag{38}$$

Then,

$$\{u_m,u_m'\}\text{is a bounded set of} L^\infty(\tau-h,T;V\times V)\text{as} m\longrightarrow\infty. \tag{39}$$

Thus, we can extract a subsequence, still denoted as m, such that

$$u_m\longrightarrow u \text{in} L^\infty(\tau-h,T;V)\text{weak}-\text{star, as} m\longrightarrow\infty, \tag{40}$$

and

$$u_m'\longrightarrow u \text{in} L^\infty(\tau-h,T;V)\text{weak}-\text{star, as} m\longrightarrow\infty. \tag{41}$$

Furthermore,

$$u_m\longrightarrow u \text{in} L^2(\Omega\times[\tau-h,T])\text{strongly}, \tag{42}$$

and

$$u_m\longrightarrow u \text{for almost every} (t,x)\in[\tau-h,T]\times\Omega. \tag{43}$$

Note that $f\in\mathscr{C}(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$, then

$$f(u_m)\longrightarrow f(u)\text{in} L^2(\tau-h,T;V)\text{weakly}. \tag{44}$$

We then pass the limit in Equation (26), and we can find that u is a solution of Equation (1) such that

$$u\in L^\infty(\tau-h,T;V)\text{and} u\prime\in L^\infty(\tau-h,T;V). \tag{45}$$

The continuity properties

$$\begin{aligned}
&u\in\mathscr{C}^1([\tau-h,T];V),\\
&\frac{\partial u}{\partial t}\in\mathscr{C}^1([\tau-h,T];V),\quad \forall T>\tau,
\end{aligned} \tag{46}$$

can be established with the methods indicated in Section II.3 and II.4 in the research by Temam [6] (e.g., Theorem 3.1 and 3.2).

This completes the proof. □

Remark 1. According to Theorem 4, we can define a family of multivalued mappings $\{U(t,\tau)\}$ on $C_{V,V}$ as

$$U(t,\tau)\colon C_{V,V}\longrightarrow C_{V,V}, \tag{47}$$

corresponding to Equation (1) by

$$U(t,\tau)\phi=\left\{u_t(\cdot;\tau,\phi)|u(\cdot)\text{are solutions of Equation (1)with}\phi\in C_{V,V}\right\}. \tag{48}$$

Then, $\{U(t,\tau)\}$ is multivalued process on $C_{V,V}$.

4. Pullback Attractors in $C_{V,V}$

We denote by $\mathfrak{R}$ the set of all functions $r\colon\mathbb{R}\longrightarrow(0,+\infty)$ such that

$$\lim_{t\longrightarrow-\infty}e^{(\delta/3)t}r^2(t)=0, \tag{49}$$

where $\delta>0$ is defined in (50) and is denoted by $\mathfrak{D}_{C_{V,V}}$ in the class of all families $\mathscr{D}=\{D(t)\}_{t\in\mathbb{R}}\subset\mathscr{P}(C_{V,V})$ such that $D(t)\subset\overline{\mathscr{N}}(0,r_{\mathscr{D}}(t))$, for some $r_{\mathscr{D}}\in\mathfrak{R}$, where $\mathscr{P}(C_{V,V})$ denotes the family of all nonempty subsets of $C_{V,V}$, and $\overline{\mathscr{N}}(0,r_{\mathscr{D}}(t))$ denotes the closed ball in $C_{V,V}$ centered at zero with radius $r_{\mathscr{D}}(t)$.

Lemma 1. *(Existence of $\mathfrak{D}$-pullback absorbing set) Suppose that $(A_1)-(A_2)$ holds true, then $g\in L^2_{loc}(\mathbb{R};H)$ and $\phi\in C_{V,V}$, and there exists a constant δ satisfying*

$$0<\delta_*<\delta<\delta^*<1, \tag{50}$$

where δ_* satisfies

$$\delta_*^4=\frac{9\alpha_2^2e^{\delta_*h/3}}{5(1-\rho_*)}. \tag{51}$$

$$\delta^*=\min\left\{\frac{3}{4},\frac{\lambda_1}{\lambda_1+3}\right\}. \tag{52}$$

λ_1 is the positive constant in the Poincáre inequality. Then the multivalued process $\{U(t,\tau)\}$ possesses a $\mathfrak{D}_{C_{V,V}}$-pullback absorbing set $\mathscr{Q}_{C_{V,V}}$ in $\mathfrak{D}_{C_{V,V}}$.

Proof. Let $v=u'+\delta u$, and we write Equation (1) as

$$\frac{\mathrm{d}}{\mathrm{d}t}v - \delta v + \delta^2 u - \left(1-\delta+\delta^2\right)\Delta u - \Delta\frac{\mathrm{d}}{\mathrm{d}t}v - (1-\delta)\Delta v = f(t, u(t-\rho(t))) + g(t). \tag{53}$$

Multiplying Equation (1) by v in $L^2(\Omega)$, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &- \delta\|v\|^2 + \delta^3\|u\|^2 + \delta\left(1-\delta+\delta^2\right)\|\nabla u\|^2\\ &+ (1-\delta)\|\nabla v\|^2\\ &= (f(t, u(t-\rho(t))), v) + (g(t), v).\end{aligned} \tag{54}$$

Noting (2), using Young's inequality, for $\epsilon_1, \epsilon_2 > 0$, we infer that

$$|(f(t, u(t-\rho(t))), v)| \le \frac{\varepsilon_1}{2}\|v\|^2 + \frac{\alpha_2^2}{2\varepsilon_1}\|u(t-\rho(t))\|^2 + \frac{\alpha_1^2|\Omega|}{2\varepsilon_1}, \tag{55}$$

and

$$|(g(t), v)| \le \frac{\varepsilon_2}{2}\|v\|^2 + \frac{1}{2\varepsilon_2}\|g(t)\|^2. \tag{56}$$

Applying the Poincáre inequality, we get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &+\left(\frac{\lambda_1(1-\delta)}{2} - \delta - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2}\right)\|v\|^2 + \frac{1-\delta}{2}\|\nabla v\|^2 + \delta^3\|u\|^2\\ &+ \delta\left(1-\delta+\delta^2\right)\|\nabla u\|^2\\ &\le \frac{\alpha_2^2}{2\varepsilon_1}\|u(t-\rho(t))\|^2 + \frac{\alpha_1^2|\Omega|}{2\varepsilon_1} + \frac{1}{2\varepsilon_2}\|g(t)\|^2.\end{aligned} \tag{57}$$

Let $\delta' > 0$ be determined later, then we infer that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\delta' t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\right)\\ &= \delta' e^{\delta' t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &+ e^{\delta' t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &\le \left(2\delta + \epsilon_1 + \epsilon_2 + \delta' - \lambda_1(1-\delta)\right)e^{\delta' t}\|v\|^2\\ &+ (\delta + \delta' - 1)e^{\delta' t}\|\nabla v\|^2 + \left(\delta^2\delta' - 2\delta^3\right)e^{\delta' t}\|u\|^2\\ &+ (\delta' - 2\delta)\left(1-\delta+\delta^2\right)e^{\delta' t}\|\nabla u\|^2\\ &+ \frac{\alpha_2^2}{\epsilon_1}e^{\delta' t}\|u(t-\rho(t))\|^2 + \frac{\alpha_1^2|\Omega|}{\epsilon_1}e^{\delta' t} + \frac{1}{\epsilon_2}e^{\delta' t}\|g(t)\|^2.\end{aligned} \tag{58}$$

Now integrating (58) from τ to t, we get

$$\begin{aligned}&e^{\delta' t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &\le e^{\delta' \tau}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \delta^2\|u(\tau)\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u(\tau)\|^2\right)\\ &+\left(2\delta + \epsilon_1 + \epsilon_2 + \delta' - \lambda_1(1-\delta)\right)\int_\tau^t e^{\delta' s}\|v(s)\|^2\mathrm{d}s\\ &+ (\delta + \delta' - 1)\int_\tau^t e^{\delta' s}\|\nabla v(s)\|^2\mathrm{d}s + \left(\delta^2\delta' - 2\delta^3\right)\int_\tau^t e^{\delta' s}\|u(s)\|^2\mathrm{d}s\\ &+ (\delta' - 2\delta)\left(1-\delta+\delta^2\right)\int_\tau^t e^{\delta' s}\|\nabla u(s)\|^2\mathrm{d}s\\ &+ \frac{\alpha_2^2}{\epsilon_1}\int_\tau^t e^{\delta' s}\|u(s-\rho(s))\|^2\mathrm{d}s + \frac{\alpha_1^2|\Omega|}{\epsilon_1}\int_\tau^t e^{\delta' s}\mathrm{d}s\\ &+ \frac{1}{\epsilon_2}\int_\tau^t e^{\delta' s}\|g(s)\|^2\mathrm{d}s.\end{aligned} \tag{59}$$

Note that $\rho(s) \in [0, h]$ and the fact

$$\frac{1}{1-\rho'(s)} \le \frac{1}{1-\rho^*}, \tag{60}$$

for all $s \in \mathbb{R}$.

Setting $r = s - \rho(s)$, we arrive at

$$\begin{aligned}&\frac{\alpha_2^2}{\epsilon_1}\int_\tau^t e^{\delta' s}\|u(s-\rho(s))\|^2\mathrm{d}s\\ &\le \frac{\alpha_2^2 e^{\delta' h}}{\varepsilon_1(1-\rho_*)}\left(\int_{\tau-h}^\tau e^{\delta' r}\|u(r)\|^2\mathrm{d}r + \int_\tau^t e^{\delta' r}\|u(r)\|^2\mathrm{d}r\right)\\ &\le \frac{\alpha_2^2 e^{\delta'(h+\tau)}h\|\phi\|_{V,V}^2}{\varepsilon_1\lambda_1(1-\rho_*)} + \frac{\alpha_2^2 e^{\delta' h}}{\varepsilon_1(1-\rho_*)}\int_\tau^t e^{\delta' r}\|u(r)\|^2\mathrm{d}r.\end{aligned} \tag{61}$$

Thus, we obtain

$$\begin{aligned}&e^{\delta' t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u\|^2\right)\\ &\le e^{\delta' \tau}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \delta^2\|u(\tau)\|^2 + \left(1-\delta+\delta^2\right)\|\nabla u(\tau)\|^2\right)\\ &+\left(2\delta + \epsilon_1 + \epsilon_2 + \delta' - \lambda_1(1-\delta)\right)\int_\tau^t e^{\delta' s}\|v(s)\|^2\mathrm{d}s\\ &+ (\delta + \delta' - 1)\int_\tau^t e^{\delta' s}\|\nabla v(s)\|^2\mathrm{d}s\\ &+\left(\delta^2\delta' - 2\delta^3 + \frac{\alpha_2^2 e^{\delta' h}}{\epsilon_1(1-\rho_*)}\right)\int_\tau^t e^{\delta' s}\|u(s)\|^2\mathrm{d}s\\ &+ (\delta' - 2\delta)\left(1-\delta+\delta^2\right)\int_\tau^t e^{\delta' s}\|\nabla u(s)\|^2\mathrm{d}s\\ &+ \frac{\alpha_1^2|\Omega|}{\epsilon_1\delta'}e^{\delta' t} + \frac{1}{\epsilon_2}\int_\tau^t e^{\delta' s}\|g(s)\|^2\mathrm{d}s + \frac{\alpha_2^2 e^{\delta'(h+\tau)}h\|\phi\|_{V,V}^2}{\epsilon_1\lambda_1(1-\rho_*)}.\end{aligned} \tag{62}$$

Choosing $\varepsilon_1 = \varepsilon_2 = \delta' = \delta/3$, and noting that

$$\delta^* = \min\left\{\frac{3}{4}, \frac{\lambda_1}{\lambda_1 + 3}\right\}, \quad \delta < \delta^* < 1, \tag{63}$$

we get

$$\begin{aligned} &2\delta + \varepsilon_1 + \varepsilon_2 + \delta' - \lambda_1(1-\delta) < 0, \\ &\delta + \delta' - 1 < 0. \end{aligned} \tag{64}$$

Note that δ_* satisfies $\delta_*^4 = 9\alpha_2^2 e^{(\delta_* h/3)}/5(1-\rho_*)$ if $0 < \delta_* < \delta$, then we have

$$\delta_*^4 > \frac{9\alpha_2^2 e^{(\delta_* h/3)}}{5(1-\rho_*)}. \tag{65}$$

Then we have

$$\begin{aligned} &\delta^2 \delta' - 2\delta^3 + \frac{\alpha_2^2 e^{\delta' h}}{\varepsilon_1(1-\rho_*)} < 0, \\ &(\delta' - 2\delta)(1 - \delta + \delta^2) < 0. \end{aligned} \tag{66}$$

Therefore,

$$\begin{aligned} &e^{(\delta/3)t}\left(\|v\|^2 + \|\nabla v\|^2 + \delta^2\|u\|^2 + (1-\delta+\delta^2)\|\nabla u\|^2\right) \\ &\le e^{(\delta/3)t}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \delta^2\|u(\tau)\|^2 + (1-\delta+\delta^2)\|\nabla u(\tau)\|^2\right) \\ &+ \frac{9\alpha_1^2|\Omega|}{\delta^2} e^{(\delta/3)t} + \frac{\delta}{3}\int_\tau^t e^{(\delta/3)s}\|g(s)\|^2 ds + \frac{3\alpha_2^2 e^{(\delta/3)(h+\tau)} h\|\phi\|_{V,V}^2}{\delta\lambda_1(1-\rho_*)}. \end{aligned} \tag{67}$$

Setting $t + \theta$ instead of t, where $\theta \in [-h, 0]$, and multiplying by $e^{-(\delta/3)(t+\theta)}$, we get

$$\begin{aligned} &\|v(t+\theta)\|^2 + \|\nabla v(t+\theta)\|^2 + \delta^2\|u(t+\theta)\|^2 \\ &\quad + (1-\delta+\delta^2)\|\nabla u(t+\theta)\|^2 \\ &\le e^{-(\delta/3)(t+\theta)} e^{(\delta/3)\tau}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2\right. \\ &\quad \left. + \delta^2\|u(\tau)\|^2 + (1-\delta+\delta^2)\|\nabla u(\tau)\|^2\right) \\ &\quad + \frac{9\alpha_1^2|\Omega|}{\delta^2} + \frac{\delta}{3} e^{-(\delta/3)(t+\theta)}\int_\tau^{t+\theta} e^{(\delta/3)s}\|g(s)\|^2 ds \\ &\quad + \frac{3\alpha_2^2 e^{-(\delta/3)t} e^{(\delta/3)(2h+\tau)} h\|\phi\|_{V,V}^2}{\delta\lambda_1(1-\rho_*)}. \end{aligned} \tag{68}$$

Note that $v = u' + \delta u$, and by (68), we infer that

$$\begin{aligned} \|u_t\|_{C_{V,V}}^2 &= \|u_t\|_{C_V}^2 + \|u_t'\|_{C_V}^2 \\ &= \sup_{s\in[-h,0]}\|\nabla u(t+\theta)\|^2 + \sup_{s\in[-h,0]}\|\nabla u'(t+\theta)\|^2 \\ &\le \sup_{s\in[-h,0]}\|\nabla u(t+\theta)\|^2 + 2\sup_{s\in[-h,0]}\|\nabla v(t+\theta)\|^2 \\ &\quad + 2\delta^2\|\nabla u(t+\theta)\|^2 \\ &\le (1+2\delta^2)\sup_{s\in[-h,0]}\|\nabla u(t+\theta)\|^2 \\ &\quad + 2\sup_{s\in[-h,0]}\|\nabla v(t+\theta)\|^2 \\ &\le C_\delta e^{-(\delta/3)t} e^{(\delta/3)(\tau+h)}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2\right. \\ &\quad \left. + \delta^2\|u(\tau)\|^2 + (1-\delta+\delta^2)\|\nabla u(\tau)\|^2\right) \\ &\quad + C_\delta\frac{9\alpha_1^2|\Omega|}{\delta^2} + C_\delta\frac{\delta}{3}e^{-(\delta/3)t}e^{(\delta/3)h}\int_\tau^{t+\theta} e^{(\delta/3)s}\|g(s)\|^2 ds \\ &\quad + \frac{3C_\delta\alpha_2^2 e^{-(\delta/3)t} e^{(\delta/3)(2h+\tau)} h\|\phi\|_{V,V}^2}{\delta\lambda_1(1-\rho_*)} \\ &\le C_1 e^{-(\delta/3)t} + C_2 + C_3 e^{-(\delta/3)t}\int_{-\infty}^t e^{(\delta/3)s}\|g(s)\|^2 ds, \end{aligned} \tag{69}$$

where

$$\begin{aligned} C_1 &= C_\delta e^{(\delta/3)(\tau+h)}\left(\|v(\tau)\|^2 + \|\nabla v(\tau)\|^2 + \delta^2\|u(\tau)\|^2\right. \\ &\quad \left. + (1-\delta+\delta^2)\|\nabla u(\tau)\|^2\right) \\ &\quad + \frac{3C_\delta\alpha_2^2 e^{(\delta/3)(2h+\tau)} h\|\phi\|_{V,V}^2}{\delta\lambda_1(1-\rho_*)}, \end{aligned} \tag{70}$$

$$C_2 = C_\delta\frac{9\alpha_1^2|\Omega|}{\delta^2}, \tag{71}$$

and

$$C_3 = C_\delta\frac{\delta}{3}e^{(\delta/3)h}. \tag{72}$$

Now, we denote by $R(t)$ the nonnegative number given for each $t \in \mathbb{R}$ by

$$R^2(t) = 2C_2 + C_3 e^{-(\delta/3)t}\int_{-\infty}^t e^{(\delta/3)s}\|g(s)\|^2 ds, \tag{73}$$

and consider the family of closed bounded balls $\mathcal{Q}_{C_{V,V}} = \{Q(t)\}_{t\in\mathbb{R}}$ in $C_{V,V}$ that is defined by

$$Q(t) = \left\{\varphi \in C_{V,V}: \|\varphi\|_{C_{V,V}} \leq R(t)\right\}. \tag{74}$$

Clearly, $\mathcal{Q}_{C_{V,V}} \in \mathfrak{D}_{C_{V,V}}$, and moreover, according to (49) and (69), the family of $\mathcal{Q}_{C_{V,V}}$ is $\mathfrak{D}_{C_{V,V}}-$ pullback absorbing for the multivalued process $\{U(t,\tau)\}$ on $C_{V,V}$.

This completes the proof. □

Lemma 2. *Under the assumptions of Lemma 1, the multivalued process* $\{U(t,\tau)\}$ *on* $C_{V,V}$ *is* $\mathfrak{D}_{C_{V,V}}-$ *pullback asymptotically upper-semicompact.*

Proof. Note that for any $T \geq t-s$ with $s \geq 0$, we get

$$U(T,t-s)\phi = \{u_T(\cdot;t-s,\phi)|u(\cdot)\text{is a solution of Equation (1)with}\phi \in Q(t-s)\}, \tag{75}$$

where $\mathcal{Q}_{C_{V,V}} = \{Q(t)\}_{t\in\mathbb{R}}$ is $\mathfrak{D}_{C_{V,V}}-$ pullback absorbing for the multivalued process $\{U(t,\tau)\}$ on $C_{V,V}$. Now, we decompose Equation (1) as follows:

$$u(T,x) = w(T,x) + y(T,x), \tag{76}$$

where $w(T,x)$ and $y(T,x)$ satisfy the following equations, respectively:

$$\begin{cases} \dfrac{\partial^2 w}{\partial T^2} - \Delta w - \Delta\dfrac{\partial w}{\partial T} - \Delta\dfrac{\partial^2 w}{\partial T^2} = 0, & T \geq t-s,\\ w(T,x) = \phi(T-t+s,x), & t-s-h \leq T \leq t-s,\\ \dfrac{\partial w(T,x)}{\partial T} = \dfrac{\partial\phi(T-t+s,x)}{\partial T}, & t-s-h \leq T \leq t-s,\\ w|_{\partial\Omega} = 0, & T \geq t-s, \end{cases} \tag{77}$$

$$\begin{cases} \dfrac{\partial^2 y}{\partial T^2} - \Delta y - \Delta\dfrac{\partial y}{\partial T} - \Delta\dfrac{\partial^2 y}{\partial T^2} = f(T,u(x,T-\rho(T))) + g(T,x), & T \geq t-s,\\ y(T,x) = 0, & t-s-h \leq T \leq t-s,\\ \dfrac{\partial y(t,x)}{\partial T} = 0, & t-s-h \leq T \leq t-s,\\ y|_{\partial\Omega} = 0, & T \geq t-s. \end{cases} \tag{78}$$

For Equation (77), similar to the proof of Theorem 4, we can easily obtain

$$\|w_t\|^2_{C_{V,V}} = \|w_t\|^2_{C_V} + \|w_t'\|^2_{C_V} \leq Ce^{-(\delta/3)s}\|\phi\|^2_{C_{V,V}}. \tag{79}$$

That is, we get

$$\lim_{s\longrightarrow\infty} \|U_1(t,t-s)Q(t-s)\|_{C_{V,V}} = 0. \tag{80}$$

Now, from Theorem 2, we only need to show that for any fixed $s > 0$, every sequence $y_n \in U_2(t,t-s)Q(t-s)$ is a Cauchy sequence in $C_{V,V}$.

We investigate two solutions of u_T^1 and u_T^2 for Equation (1) corresponding to the initial data ϕ^1 and ϕ^2, respectively. Let $z(T) = y^1(T) - y^2(T)$, then $z(T)$ satisfies

$$\begin{cases} \dfrac{\partial^2 z}{\partial T^2} - \Delta z - \Delta\dfrac{\partial z}{\partial T} - \Delta\dfrac{\partial^2 z}{\partial T^2} = f\left(T,u^1(x,T-\rho(T))\right) - f\left(T,u^2(x,T-\rho(T))\right), & T \geq t-s,\\ z(T,x) = 0, & t-s-h \leq T \leq t-s,\\ \dfrac{\partial z(t,x)}{\partial T} = 0, & t-s-h \leq T \leq t-s,\\ z|_{\partial\Omega} = 0, & T \geq t-s. \end{cases} \tag{81}$$

Multiplying Equation (81) by z' in $L^2(\Omega)$, we infer

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\left(\|\nabla z\prime(T)\|^2+\|z\prime(T)\|^2+\|\nabla z\|^2\right)+\|\nabla z\prime(T)\|^2 =\left(f\left(T,u^1(x,T-\rho(T))\right)-f\left(T,u^2(x,T-\rho(T))\right),z\prime(T)\right). \quad (82)$$

Integrating from $t-s$ to $t+\theta$ (where $\theta\in[-h,0]$), we have

$$\begin{aligned}&\|\nabla z\prime(t+\theta)\|^2+\|z\prime(t+\theta)\|^2+\|\nabla z(t+\theta)\|^2\\ &\leq 2\int_{t-s}^{t+\theta}\left(f\left(T,u^1(x,T-\rho(T))\right)-f\left(T,u^2(x,T-\rho(T))\right),z\prime(T)\right)\mathrm{d}T\\ &\leq 2\int_{t-s}^{t+\theta}\int_\Omega\left|f\left(T,u^1(x,T-\rho(T))\right)-f\left(T,u^2(x,T-\rho(T))\right)\|z\prime(T)\right|\mathrm{d}x\mathrm{d}T\\ &\leq 2\|z\prime(T)\|_{L^2(\Omega\times[t-s,t])}\left\|f\left(T,u^1(x,T-\rho(T))\right)-f\left(T,u^2(x,T-\rho(T)\right)\right\|_{L^2(\Omega\times[t-s,t])}.\end{aligned} \quad (83)$$

Let $u_{nT}\in U(T,t-s)\phi_n$ be with $\phi_n\in Q(t-s)$. According to (69), without of generality, we can assume that

$$u_n\longrightarrow u\text{weakly star in}L^\infty(t-s-h,t;V), \quad (84)$$

and

$$u_n'\longrightarrow u\text{weakly star in}L^\infty(t-s-h,t;V). \quad (85)$$

Then, we infer that

$$u_n\longrightarrow u\text{in}L^\infty(t-s-h,t;V), \quad (86)$$

and

$$u_n(T,x)\longrightarrow u(T,x)n\longrightarrow\infty\text{for}a.e.(T,x)\in[t-s-h,t]\times\Omega. \quad (87)$$

Note that when $f\in\mathscr{C}(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$, by (87), we have

$$\begin{aligned}&f(T,u_n(T,x))\longrightarrow f(T,u(T,x))\\ &n\longrightarrow\infty\text{for}a.e.(T,x)\in[t-s-h,t]\times\Omega.\end{aligned} \quad (88)$$

Applying the Lebesgue dominated convergence theorem, we infer that

$$\lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\left\|f(T,u_n(x,T-\rho(T)))-f(T,u_m(x,T-\rho(T))\right\|_{L^2(\Omega\times[t-s,t])}=0. \quad (89)$$

It follows from (83) and 89 that

$$\begin{aligned}&\|y_{nt}-y_{mt}\|^2_{C_{V,V}}\\ &=\|y_{nt}-y_{mt}\|^2_{C_V}+\|y_{nt}'-y_{mt}'\|^2_{C_V}\\ &=\sup_{s\in[-h,0]}\|\nabla y_n(t+\theta)-\nabla y_m(t+\theta)\|^2\\ &\quad+\sup_{s\in[-h,0]}\|\nabla y_n'(t+\theta)-\nabla y_m'(t+\theta)\|^2\\ &\leq 2\|y_n'(T)-y_m'(T)\|_{L^2(\Omega\times[t-s,t])}\\ &\times\|f(T,u_n(x,T-\rho(T)))-f(T,u_m(x,T-\rho(T)))\|_{L^2(\Omega\times[t-s,t])}.\end{aligned} \quad (90)$$

Combining (89) with (90), we get

$$\lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty}\|y_{nt}-y_{mt}\|^2_{C_{V,V}}=0. \quad (91)$$

This completes the proof. □

Theorem 5. *(Existence of $\mathfrak{D}_{C_{V,V}}$ pullback attractors) Under the assumptions of Lemma 3.2, the multivalued process $\{U(t,\tau)\}$ on $C_{V,V}$ possesses an unique pullback attractor $\left\{\mathscr{A}_{C_{V,V}}(t)\right\}_{t\in\mathbb{R}}$ in $\mathfrak{D}_{C_{V,V}}$.*

Proof. By Lemma 2, we know that the multivalued process $\{U(t,\tau)\}$ corresponding to Equation (1) is $\mathfrak{D}_{C_{V,V}}$ pullback asymptotically upper-semicompact on $C_{V,V}$. Furthermore, according to Lemma 1, the multivalued process $\{U(t,\tau)\}$ possesses a $\mathfrak{D}_{C_{V,V}}$ pullback absorbing set $\mathscr{Q}_{C_{V,V}}$ in $\mathfrak{D}_{C_{V,V}}$. In order to obtain the existence of $\mathfrak{D}_{C_{V,V}}$ pullback attractors, we only need to show the negative invariance of $\left\{\mathscr{A}_{C_{V,V}}(t)\right\}_{t\in\mathbb{R}}$, where

$$\mathscr{A}_{C_{V,V}}(t)=\omega_t\left(\mathscr{Q}_{C_{V,V}}\right)=\bigcap_{T\in\mathbb{R}^+}\overline{\bigcup_{s\geq T}U(t,t-s)Q(t-s)},\quad\forall t\in\mathbb{R}, \quad (92)$$

and $\mathscr{Q}_{C_{V,V}}=\{Q(t)\}_{t\in\mathbb{R}}$ is the $\mathfrak{D}_{C_{V,V}}$ pullback absorbing set of $\{U(t,\tau)\}$ in $C_{V,V}$.

Let $y \in \mathscr{A}_{C_{V,V}}(t)$. Then there exist sequences $s_n \in \mathbb{R}^+$, $s_n \longrightarrow +\infty (n \longrightarrow \infty)$, $x_n \in Q(t - s_n)$, and $y_n \in U(t, t - s_n)x_n$ such that

$$y_n \longrightarrow y \in C_{V,V} \text{as} n \longrightarrow \infty. \tag{93}$$

On the other hand, for n that is large enough, we get

$$y_n \in U(t, t - s_n)x_n = U(t, \tau)U(\tau, t - s_n)x_n. \tag{94}$$

Lemma 2 implies that the multivalued process $\{U(t, \tau)\}$ corresponding to Equation (1) is $\mathfrak{D}_{C_{V,V}}$ pullback asymptotically upper-semicompact on $C_{V,V}$, then there is a subsequence of $\tilde{x}_n \in U(\tau, t - s_n)x_n = U(t, \tau - (\tau + s_n - t))x_n$, which we still relabel as $\tilde{x}_n$ such that $y_n \in U(t, \tau)\tilde{x}_n$ and

$$\tilde{x}_n \longrightarrow x \in C_{V,V} \text{as} n \longrightarrow \infty. \tag{95}$$

Clearly, $x \in \mathscr{A}_{C_{V,V}}(\tau)$.

By slightly modifying the proof of the existence of solutions Theorem 4, we can see that

$$y_n(\cdot) \rightharpoonup u(\cdot + t, \tau, x) \text{in} L^2(-h, 0; V) \cap H^1(-h, 0; V), \tag{96}$$

where $u(\cdot)$ is a solution in Theorem 4. This together with 9495, we can deduce that

$$y \in U(t, \tau)x \subset U(t, \tau)\mathscr{A}_{C_{V,V}}(\tau). \tag{97}$$

This completes the proof. □

Remark 2. In this article, we showed the existence of pullback attractors in the Banach spaces $C_{H_0^1(\Omega), H_0^1(\Omega)}$ for the multivalued process generated by a class of second-order nonautonomous evolution equations with hereditary characteristics. Furthermore, it would be interesting to consider the long-time behavior for the following nonautonomous semilinear second-order evolution equation with the delay term and critical exponent as

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \frac{\partial u}{\partial t} - \Delta \frac{\partial^2 u}{\partial t^2} &= h(u) + f(t, u(x, t - \rho(t))) \\ &\quad + g(t, x), \quad \text{in} (\tau, \infty) \times \Omega, \end{aligned} \tag{98}$$

where the nonlinearity $h(\cdot)$ fulfills the critical exponential growth condition, and the nonlinearity $f(\cdot)$ and the external force $g(t, x)$ satisfy (A_1) and (A_2), respectively. Maybe we can extend the results presented here in this case and can possibly address them in forthcoming papers.

Data Availability

The data used to support the findings of this paper are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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