Hindawi
Journal of Mathematics
Volume 2022, Article ID 8314977, 10 pages
https://doi.org/10.1155/2022/8314977

Research Article

Large Deviation Rates for the Continuous-Time Supercritical Branching Processes with Immigration

Juan Wang and Xiaojuan Wang

School of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

Correspondence should be addressed to Juan Wang; wangjuanrose@126.com

Received 21 February 2022; Revised 5 May 2022; Accepted 13 May 2022; Published 17 June 2022

Academic Editor: G. Muhiuddin

Let $\{Y(t); t \geq 0\}$ be a supercritical continuous-time branching process with immigration; our focus is on the large deviation rates of $Y(t)$ and thus extending the results of the discrete-time Galton–Watson process to the continuous-time case. Firstly, we prove that $Z(t) = e^{-mt}[Y(t) - ((e^{m(t+1)} - 1)/(e^m - 1))e^{a+m}]$ is a submartingale and converges to a random variable Z. Then, we study the decay rates of $P(|Z(t) - Z| > \varepsilon)$ as $t \longrightarrow \infty$ and $P(|(Y(t+v)/Y(t)) - e^{mv}| > \varepsilon | Z \geq \alpha)$ as $t \longrightarrow \infty$ for $\alpha > 0$ and $\varepsilon > 0$ under various moment conditions on $\{b_k; k \geq 0\}$ and $\{a_j; j \geq 0\}$. We conclude that the rates are supergeometric under the assumption of finite moment generation functions.

1. Introduction

Suppose that $\{Y(t); t \geq 0\}$ is a continuous-time branching process with immigration. Its generating matrix $Q = (q_{jk}; j, k \in Z_+)$ is defined as follows:

$$q_{jk} := \begin{cases} jb_{k-j+1} + a_{k-j}, & \text{if } j \geq 0, k \geq j, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\begin{aligned} &a_k \geq 0\,(k \neq 0), \quad 0 < -a_0 = \sum_{k \neq 0} a_k < \infty, \\ &b_k \geq 0\,(k \neq 1), \quad 0 < -b_1 = \sum_{k \neq 1} b_k < \infty. \end{aligned} \tag{2}$$

Throughout this paper, we always suppose that $b_0 = 0$, $m := \sum_{k=1}^{\infty} kb_k < \infty$, and $a := \sum_{k=0}^{\infty} ka_k < \infty$.
Set

$$\begin{aligned} &Y(0) = 1, \\ &Y(t) = M(t) + K(t), \quad t > 0, \end{aligned} \tag{3}$$

where $M(t)$ and $K(t)$ separately represent the total number of the original individual's offspring and the offspring of the individual from immigrants at the time t. If $K(t) \equiv 0$ for all $t \geq 0$ and $Y(0) = 1$, then the process degenerates to the continuous-time branching process $\{\widehat{Y}(t); t \geq 0\}$. It is known from Athreya and Ney [1] or Harris [2] that $\{e^{-mt}\widehat{Y}(t); t \geq 0\}$ is a martingale and converges to a r.v. W w.p.1 as $t \longrightarrow \infty$. Therefore, $\widehat{Y}(t+v)/\widehat{Y}(t)$ converges to e^{mv} w.p.1 as $t \longrightarrow \infty$ for arbitrary fixed $v \geq 0$, and hence

$$\lim_{t \longrightarrow \infty} P\left(\left|\frac{\widehat{Y}(t+v)}{\widehat{Y}(t)} - e^{mv}\right| > \varepsilon\right) = 0, \quad \forall \varepsilon > 0. \tag{4}$$

For $\{Y(t); t \geq 0\}$, define

$$\begin{aligned} Z(t) &= e^{-mt}\left[Y(t) - \frac{e^{m(t+1)} - 1}{e^m - 1} e^{a+m}\right] \\ &:= U(t) + I(t), \quad t \geq 0, \end{aligned} \tag{5}$$

where $U(t) = M(t)/e^{mt}$ and $I(t) = (K(t)/e^{mt}) - (e^{m(t+1)} - 1/(e^m - 1))e^{a+m-mt}$.

In this paper, let $a = m/(e^m - 1) < \infty$, we will demonstrate that $Z(t)$ is a submartingale and converges to a r.v. Z almost surely since $U(t)$ converges to a r.v. U a.s. and $I(t)$ converges to a r.v. I a.s.

The above discussion of the decay rates is not only of its own meaning but also of some significance to the algorithm tree structure in computer science (Karp and Zhang [3] and Miller [4]). Heyde [5] discussed that there exists a constant sequence $\{D_n\}$ which makes $W_n = Z_n/D_n$ converge to a random variable W w.p.1. Athreya [6] considered similar decay rates for the Galton–Watson process. Seneta [7] researched the supercritical Galton–Watson process with immigration, and Riotershtein [8] considered multitype branching processes with immigration in a random environment. Liu and Zhang [9] studied the decay rates of $P(|(Y_{n+1}/Y_n) - m| > \varepsilon)$ for the supercritical branching processes with immigration. Recently, Sun and Zhang [10] considered the convergence rates of harmonic moments for supercritical branching processes with immigration. Chen and He [11] investigated the lower deviation and moderate deviation probabilities for maximum of a branching random walk. Abraham et al. [12] analyzed the stationary continuous state branching processes. The model considered in this paper involves continuous-time and immigration, while the models considered in the above references do not involve continuous-time. Such change in conditions may affect the large deviation rates and thus is mathematically interesting. We need to find a new method to investigate the effect of continuous-time and immigration.

Based on the previous results, it is natural to develop the large deviation rates for the continuous-time branching processes with immigration. In this paper, we are aiming to discuss the decay rates of

$$\begin{aligned} &P(|Z(t) - Z| > \varepsilon),\\ &P\left(\left|\frac{Y(t+v)}{Y(t)} - e^{mv}\right| > \varepsilon | Z \geq \alpha\right), \quad \text{as } t \longrightarrow \infty, \end{aligned} \tag{6}$$

under various moment conditions on $\{b_k; k \geq 0\}$ and $\{a_k; k \geq 0\}$ for any $\alpha > 0$ and $\varepsilon > 0$. We conclude that the rates of (6) are always supergeometric under the assumption of finite moment generation functions. The specific conclusions are presented in Section 3.

2. Preliminaries

Before the investigation, we present the generating functions $B(w)$ and $A(w)$ of the known sequences $\{b_k; k \geq 0\}$ and $\{a_k; k \geq 0\}$ as

$$\begin{aligned} B(w) &= \sum_{k=0}^{\infty} b_k w^k,\\ A(w) &= \sum_{k=0}^{\infty} a_k w^k. \end{aligned} \tag{7}$$

Clearly, $B(w)$ is well defined and finite at least on $[-1, 1]$ with $B(1) = 0$, so 1 is the solution of $B(w) = 0$. Then, $m = B'(1) = \sum_{k=0}^{\infty} k b_k$ is the mean birth rate of $M(t)$. Moreover, $A(w)$ is similar to $B(w)$.

We present some preliminaries in this section.

Lemma 1. *Let $\eta > 0$ and $j, k \geq 0$. If*

$$\lim_{t \longrightarrow \infty} e^{\eta t} p_{jk}(t) = \phi_{jk} \geq 0, \tag{8}$$

exists, then for arbitrary $b > 0$,

$$\lim_{t \longrightarrow \infty} e^{-bt} \int_0^t e^{(\eta + b)v} p_{jk}(v) \mathrm{d}v = b^{-1} \phi_{jk}. \tag{9}$$

Proof. Omitted. □

Lemma 2. *$lim_{t \longrightarrow \infty} e^{-(b_1 + a_0)t} p_{1k}(t) = \rho_k$ exists for any $i \geq 1$, and $\rho_k \leq \rho_1 = 1 (k \geq 0)$. Furthermore, $Q(w) = \sum_{k=1}^{\infty} \rho_k w^k$ satisfies the functional equation:*

$$(b_1 + a_0) Q(w) = B(w) Q'(w) + A(w) Q(w), \quad 0 \leq w \leq 1, \tag{10}$$

with condition $Q(0) = 0$.

Proof. By the Kolmogorov forward equation,

$$p'_{1k}(t) = \sum_{i=1}^{k} p_{1i}(t) (i b_{k-i+1} + a_{k-i}), \quad k \geq 1. \tag{11}$$

For $k = 1$,

$$p'_{11}(t) = p_{11}(t)(b_1 + a_0). \tag{12}$$

That is,

$$p_{11}(t) = e^{(b_1 + a_0)t}. \tag{13}$$

Hence,

$$\rho_1 = \lim_{t \longrightarrow \infty} e^{-(b_1 + a_0)t} p_{11}(t) = 1. \tag{14}$$

For $k = 2$,

$$p'_{12}(t) = p_{11}(t)(b_2 + a_1) + p_{12}(t)(2b_1 + a_0). \tag{15}$$

So,

$$e^{-(2b_1 + a_0)t} p_{12}(t) = (b_2 + a_1) \int_0^t p_{11}(v) e^{-(2b_1 + a_0)v} \mathrm{d}v. \tag{16}$$

By Lemma 1,

$$\begin{aligned} \rho_2 &= \lim_{t \longrightarrow \infty} e^{-(b_1 + a_0)t} p_{12}(t) = (b_2 + a_1) \lim_{t \longrightarrow \infty} e^{b_1 t}\\ &\cdot \int_0^t p_{11}(v) e^{-(2b_1 + a_0)v} \mathrm{d}v = -\frac{\rho_1((b_2 + a_1)}{b_1} \leq \rho_1. \end{aligned} \tag{17}$$

It follows from (11) that

$$\begin{aligned} e^{-(kb_1 + a_0)t} p_{1k}(t) &= \sum_{i=1}^{k-1} (i b_{k-i+1} + a_{k-i})\\ &\cdot \int_0^t p_{1i}(v) e^{-(kb_1 + a_0)v} \mathrm{d}v. \end{aligned} \tag{18}$$

Hence,

$$\rho_k = \lim_{t\longrightarrow\infty} e^{-(b_1+a_0)t} p_{1k}(t) = \frac{1}{-(k-1)b_1}\sum_{i=1}^{k-1}(ib_{k-i+1}+a_{k-i})\rho_i \le \rho_1. \tag{19}$$

Finally, we can obtain the following equality from (11):

$$\sum_{k=0}^{\infty} e^{-(b_1+a_0)t} p'_{jk}(t)\omega^k = B(\omega)\sum_{k=1}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)\omega^{k-1} + A(\omega)\sum_{k=0}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)\omega^k, \tag{20}$$

and moreover,

$$\left(\sum_{k=0}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)\omega^k\right)'_t + (b_1+a_0)\sum_{k=0}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)\omega^k = B(\omega)\sum_{k=1}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)k\omega^{k-1} + A(\omega)\sum_{k=0}^{\infty} e^{-(b_1+a_0)t} p_{jk}(t)\omega^k. \tag{21}$$

Taking the limit as $t\longrightarrow\infty$, we have

$$(b_1+a_0)Q(\omega) = B(\omega)Q'(\omega) + A(\omega)Q(\omega). \tag{22}$$

The proof is completed. □

Proposition 1. *Let ς_t be the σ-algebra generated by $\{Y(t); t\ge 0\}$ and $E[M(1)\log M(1)]<\infty$. Then, $\{Z(t); t\ge 0\}$ is a submartingale and converges to a r.v. Z almost surely.*

Proof. According to the definition of $Z(t)$, we conclude the following inequalities:

$$\begin{aligned}
E[Z(t+v)|Z(t)] &= E\left[e^{-m(t+v)}\left(Y(t+v) - \frac{e^{m(t+v+1)}-1}{e^m-1}e^{a+m}\right)\middle| Y(t)\right]\\
&= e^{-m(t+v)}E\left[\sum_{j=1}^{Y(t)}\xi_{t,j}(v) + K(v) - \frac{e^{m(t+v+1)}-1}{e^m-1}e^{a+m}\middle| Y(t)\right]\\
&= e^{-m(t+v)}\left(e^{mv}Y(t) + e^{(a+m)v} - \frac{e^{m(t+v+1)}-1}{e^m-1}e^{a+m}\right)\\
&= e^{-mt}\left(Y(t) + \frac{e^{av+m} - e^{av} - e^{mt+a+2m} + e^{a+m-mv}}{e^m-1}\right)\\
&\ge e^{-mt}\left(Y(t) - \frac{e^{m(t+1)}-1}{e^m-1}e^{a+m}\right) = Z(t),
\end{aligned} \tag{23}$$

which implies $\{Z(t); t\ge 0\}$ is a submartingale. We know that $E[e^{-mt}Y(t)]<\infty$ if and only if $E[M(1)\log M(1)]<\infty$. Hence, $E[Z(t)]<\infty$. Thus, $Z(t)$ is an integrable submartingale and converges to a r.v. Z almost surely.

Define $G(w,t) = E[w^{Y(t)}|Y(0)=1]$, $w\ge 0$ with initial condition $G(w,0)=w$, where $Y(t)$ is a continuous-time branching processes with immigration with $Y(0)=1$. Denote $k(F(w)) = w$ and

$$G(w) := G(w,1) = E\left[w^{Y(1)}|Y(0)=1\right] = H(w,1)F(w,1) := H(w)F(w). \tag{24}$$

We will study the properties of $k(w)$ and $G(w)$ in the subsequent contents. Let $\varpi = \sup\{w\ge 0; G(w)<\infty\}$. Apparently, $G(w)$ is strictly increasing with $w\in(0,\varpi)$ and $\varpi\ge 1$. Furthermore, we have $k(w)\ge w$ for $0\le w\le 1$ since $F(w)\le w$ and $k(w)\le w$ for $1\le w\le\varpi$ since $F(w)\ge w$. Therefore, the iterates $k(w,n)$ of k are nonincreasing with $w\in[1,\varpi]$ and nondecreasing with $w\in[0,1]$ (with respect to n). □

Proposition 2. *Let $v_0>1$. If $G(w_0,v_0)<\infty$ for some $w_0>1$, then for $1\le w\le F(w_0,v_0)$, $k(w,t)\downarrow 1$ as $t\uparrow\infty$ and*

$$R(w,t) \equiv e^{mt}(k(w,t)-1)\downarrow R(w), \quad \text{as } t\uparrow\infty, \tag{25}$$

where $R(w)$ is the unique solution of

$$R(F(w,v_0)) = e^{mv_0}R(w), \quad \text{for } 1 \le w \le F(w_0,v_0), \tag{26}$$

subject to

$$\begin{aligned} R(1) &= 0, \\ R'(1) &= 1, \end{aligned} \tag{27}$$

$$0 < R(w) < \infty, \quad \text{for } 1 < w \le F(w_0,v_0). \tag{28}$$

Proof. Note that $F(w,t) \ge w$ for $w \ge 1$, we know that $F(w,v_0+t) = F(F(w,t),v_0) \ge F(w,v_0)$ for $w \ge 1$, $t \ge 0$. Hence, $k(\cdot,v_0+t)$ is well defined on $[0,F(w_0,v_0)]$ for $t \ge 0$ and $k(w,v_0+t) \ge 1$ for $w \ge 1$.

Since $w = F(k(w,v_0+t),v_0+t) = F(F(k(w,v_0+t),t),v_0) \ge F(k(w,v_0+t),v_0)$, we have $k(w,v_0+t) \le k(w,v_0)$ for any $1 \le w \le F(w_0,v_0)$, which implying $k(w,t)\downarrow$ as $v_0 \le t\uparrow$. On the other hand,

$$F(k(\omega,v_0+t),v_0) = k(\omega,t), \tag{29}$$

that is,

$$k(\omega,v_0+t) = k(k(\omega,t),v_0). \tag{30}$$

Denote $c := \lim_{t\longrightarrow\infty}k(w,t)$. Putting $t\uparrow\infty$ yields $c = k(c,v_0)$. Thus, we obtain $c = 1$, i.e., $k(w,t)\downarrow 1$ as $t\uparrow\infty$.

Now, there exists $a \in (k(w,t),1)$ for $v,t > 0$ such that

$$\begin{aligned} \frac{R(w,v+t)}{R(w,t)} &= \frac{e^{mv}(k(k(w,t),v)-1)}{k(w,t)-1} \\ &= e^{mv}k_w'(a,v) = \frac{e^{mv}}{F_w'(a,v)} < 1. \end{aligned} \tag{31}$$

Hence, $R(w) = \lim_{t\longrightarrow\infty}R(w,t)$ exists.

Furthermore, $R(w)$ satisfies (26) and (27). For $w \in [F(w_0,v_0),1]$,

$$\begin{aligned} R(F(w,v_0),t+v_0) &= e^{m(t+v_0)}(k(F(w,v_0),t+v_0)-1) \\ &= e^{m(t+v_0)}(k(k(F(w,v_0),v_0),t)-1) \\ &= e^{m(t+v_0)}(k(w,t)-1). \end{aligned} \tag{32}$$

Letting $t\uparrow\infty$ yields

$$R(F(w,v_0)) = e^{m(t+v_0)}R(w). \tag{33}$$

For another,

$$\begin{aligned} R(1) &= \lim_{t\longrightarrow\infty}R(1,t) = 0, \\ R'(1) &= 1, \end{aligned} \tag{34}$$

since $R_w'(1,t) = 1$ for $t > v_0$. Thus, $0 < R(w) < \infty$.

Finally, it is easily seen that the solution of (26) and (27) is unique. □

Proposition 3. *Let $l \equiv G(w_0) < \infty$ for some $w_0 = e^{\theta_0} > 1$. Then,*

$$G(k(w_0),t) \le G(w_0)\prod_{l=1}^{[t-1]}H(k(w_0,l)), \quad t \ge 2, \tag{35}$$

where $[t]$ is the round of t.

Proof. For $t = 2$, noting $k(w_0) > 1$ yields

$$\begin{aligned} G(k(w_0),2) &= G(F(k(w_0)))H(k(w_0)) \\ &= G(w_0)H(k(w_0)). \end{aligned} \tag{36}$$

Assume that inequality (35) holds for $t-1$, i.e.,

$$G(k(w_0,t-2),t-1) \le G(w_0)\prod_{l=1}^{[t-2]}H(k(w_0,l)). \tag{37}$$

Then,

$$\begin{aligned} G(k(w_0,t-1),t) &= G(f(k(w_0,t-1)),t-1)H \\ &\quad \cdot (k(w_0,t-1)) \\ &= G(k(w_0,t-2),t-1)H(k(w_0,t-1)) \\ &\le G(w_0)\prod_{l=1}^{[t-1]}H(k(w_0,l)), \end{aligned} \tag{38}$$

holds for all $t \ge 2$.

Therefore, (35) is proved. □

3. Main Results and Proofs

Theorem 1. *Assume that $B(\theta_0) < \infty$ and $A(\theta_0) < \infty$ for some $\theta_0 > 0$. Then, for arbitrary small $\varepsilon > 0$,*

$$\begin{aligned} &\lim_{t\longrightarrow\infty} e^{-(b_1+a_0)t}\cdot P\left(\left|\frac{Y(t+v)}{Y(t)} - e^{mv}\right| > \varepsilon | Y(0) = 1\right) \\ &\quad = \sum_{l=1}^{\infty}\phi(v,l,\varepsilon)\rho_l < \infty, \end{aligned} \tag{39}$$

where $\phi(v,l,\varepsilon) = P(|\overline{M}_l(v) + (K(v)/l) - e^{mv}| > \varepsilon)$, $\overline{M}_l(v) = \sum_{j=1}^{l}(M_j(v)/l)$.

Proof. Note that $Y(t+v)$ could be defined as

$$Y(t+v) = \sum_{j=1}^{Y(t)}\xi_{t,j}(v) + K(v), \tag{40}$$

where $\{K(t); t \ge 0\}$ and $\{\xi_{t,j}(v); t \ge 0; j \ge 1\}$ are independent and identical distributed, $\{\xi_{t,j}(v); t \ge 0, j \ge 1\}$ are i.i.d. random variables with the same law as $\{\widehat{Y}(v); v \ge 0\}$. Hence,

$$\begin{aligned} P\left(\left|\frac{Y(t+v)}{Y(t)}-e^{mv}\right|>\varepsilon|Y(0)=1\right) &= P\left(\left|\frac{\sum_{j=1}^{Y(t)}\xi_{t,j}(v)+K(v)}{Y(t)}-e^{mv}\right|>\varepsilon|Y(0)=1\right) \\ &= \sum_{l=1}^{\infty}P(Y(t)=l|Y(0)=1)\cdot P\left(\left|\frac{\sum_{j=1}^{l}\xi_{t,j}(v)+K(v)}{l}-e^{mv}\right|>\varepsilon\right) \\ &= \sum_{l=1}^{\infty}P(Y(t)=l|Y(0)=1)\phi(v,l,\varepsilon). \end{aligned} \tag{41}$$

Indeed, for any fixed $v>0$,

$$\begin{aligned} \phi(v,l,\varepsilon) &= P\left(\frac{\sum_{j=1}^{l}\xi_{t,j}(v)+K(v)}{l}-e^{mv}>\varepsilon\right)+P\left(\frac{\sum_{j=1}^{l}\xi_{t,j}(v)+K(v)}{l}-e^{mv}<-\varepsilon\right) \\ &\le P\left(\sum_{j=1}^{l}\xi_{t,j}(v)>l\left(e^{mv}+\frac{\varepsilon}{2}\right)\right)+P\left(\frac{K(v)}{l}>\frac{\varepsilon}{2}\right)+P\left(\sum_{j=1}^{l}\xi_{t,j}(v)<l\left(e^{mv}-\varepsilon\right)\right) \\ &\le P\left(\alpha^{\sum_{j=1}^{l}\xi_{t,j}(v)}>\alpha^{l(e^{mv}+(\varepsilon/2))}\right)+P\left(\frac{K(v)}{l}>\frac{\varepsilon}{2}\right)+P\left(\beta^{\sum_{j=1}^{l}\xi_{t,j}(v)}>\beta^{l(e^{mv}-\varepsilon)}\right) \\ &:= I_1+I_2+I_3, \end{aligned} \tag{42}$$

where $\alpha\in(1,\theta_0)$ and $\beta\in(0,1)$ are arbitrary constants. Thus,

$$\phi(v,l,\varepsilon)\le\left[f(\alpha,v)\alpha^{-(e^{mv}+(\varepsilon/2))}\right]^{l}+I_2+\left[f(\beta,v)\beta^{-(e^{mv}-\varepsilon)}\right]^{l}, \tag{43}$$

where $f(s,v)=\sum_{k=0}^{\infty}p_{1k}(v)s^{k}$.

Since it is assumed that $B(\theta_0)<\infty$ and $A(\theta_0)<\infty$, we have $f(\alpha,v)<\infty$ and it follows that there exist $\alpha_0\in(1,\theta_0)$ and $\beta_0\in(0,1)$ such that

$$\begin{aligned} &0<f(\alpha_0,v)\alpha_0^{-(e^{mv}+\varepsilon/2)}<1, \\ &0<f(\beta_0,v)\beta_0^{-(e^{mv}-\varepsilon)}<1, \end{aligned} \tag{44}$$

for any $\varepsilon\in(0,1)$ and $v>0$. Hence, there exist $c_j,\lambda_j\in(0,1)\,(j=1,3)$ such that $I_j\le c_j\lambda_j(v,\varepsilon)^{l}$ for all $l\ge1$. By Markov's inequality, we can prove that there exist $c_2>0$ and $\lambda_2\in(0,1)$ such that $I_2\le c_2\lambda_2(v,\varepsilon)^{l}$. Therefore, by Lemma 2,

$$\begin{aligned} &\lim_{t\longrightarrow\infty}e^{-(b_1+a_0)t}\cdot P\left(\left|\frac{Y(t+v)}{Y(t)}-e^{mv}\right|>\varepsilon|Y(0)=1\right) \\ &\quad=\lim_{t\longrightarrow\infty}\sum_{l=1}^{\infty}e^{-(b_1+a_0)t}\cdot P(Y(t)=1|Y(0)=1)\phi(v,l,\varepsilon) \\ &\quad=\sum_{l=1}^{\infty}\phi(v,l,\varepsilon)\rho_l<\infty. \end{aligned} \tag{45}$$

The proof is completed. □

Theorem 2. *Assume that for fixed $\varepsilon>0$ and $v>0$, there exist constants $C_\varepsilon(v)$ and $r>0$ such that $mr>-(b_1+a_0)$ and $\phi(v,l,\varepsilon)\le l^{-r}C_\varepsilon(v)$ for all $l\ge1$. Then, (39) holds.*

Proof. Under the above assumptions, there exists another $\widetilde{C}_\varepsilon>0$ such that $\phi(v,l,\varepsilon)\le\widetilde{C}_\varepsilon(l+1)^{-r}$ for all $l>0$ since l^{-r} is equivalent to $(l+1)^{-r}$ as $l \longrightarrow \infty$. In general, we denote the positive constants $\widetilde{C}_\varepsilon$ by C_ε. Therefore,

$$\begin{aligned} k(l,t) &:= \frac{\phi(v,l,\varepsilon)}{e^{(b_1+a_0)t}} \cdot P(Y(t)=l) \\ &\leq \frac{C_\varepsilon}{(l+1)^r} \cdot \frac{P(Y(t)=l)}{e^{(b_1+a_0)t}} =: \widetilde{k}(l,t). \end{aligned} \tag{46}$$

With a simple modification of the convergence theorem, it can be proved that

$$\lim_{t\longrightarrow\infty} \sum_{l=0}^{\infty} \widetilde{k}(l,t) = \lim_{t\longrightarrow\infty} \frac{C_\varepsilon}{e^{(b_1+a_0)t}} E\left[(Y(t)+1)^{-r}\right] < \infty, \tag{47}$$

since

$$\begin{aligned} & e^{-(b_1+a_0)t} \cdot P\left(\left|\frac{Y(t+v)}{Y(t)} - e^{mv}\right| > \varepsilon\right) \\ &= \sum_{l=1}^{\infty} \frac{\phi(v,l,\varepsilon) P(Y(t)=l)}{e^{(b_1+a_0)t}}. \end{aligned} \tag{48}$$

For all $Y \geq 0$,

$$E\left[Y^{-r}\right] = \frac{1}{\Gamma(r)} \int_0^\infty E\left[e^{-tY}\right] t^{r-1} \mathrm{d}t, \tag{49}$$

and hence

$$\begin{aligned} \frac{E\left[(Y(t)+1)^{-r}\right]}{e^{(b_1+a_0)t}} &= \frac{1}{\Gamma(r)} \int_0^\infty \frac{E\left[e^{-t(Y(t)+1)}\right] t^{r-1}}{e^{(b_1+a_0)t}} \mathrm{d}t \\ &= \frac{1}{\Gamma(r)} \int_0^1 \frac{G(v,t)(-\log v)^{r-1}}{e^{(b_1+a_0)t}} \mathrm{d}v. \end{aligned} \tag{50}$$

Since $Q(v,t) = \left(G(v,t)/e^{(b_1+a_0)t}\right) \uparrow Q(v)$, the proof shall be completed if we prove

$$\int_0^1 Q(v) k(v) \mathrm{d}v < \infty, \tag{51}$$

where $k(v) = ((\log v)^{r-1}$. Indeed, for any fixed $v_0 \in (0,1)$, let $v(t)$ be the inverse function of $F(v_0,t)$ at v_0, then we have $v(t)\uparrow 1$ as $t \uparrow \infty$. Note that $H(v,1) Q(F(v,1)) = e^{b_1+a_0} Q(v)$, thus we can write

$$\begin{aligned} I_n &= \int_{v_n}^{v_{n+1}} Q(v) k(v) \mathrm{d}v \\ &= \int_{v_n}^{v_{n+1}} \frac{Q(F(v,1)) H(v,1)}{e^{b_1+a_0}} k(v) \mathrm{d}v \\ &= \int_{v_{n-1}}^{v_n} Q(w) k(w) \frac{H\left(F^{-1}(w)\right) k\left(F^{-1}(w)\right) \left(F^{-1}(w)\right)'}{k(w) e^{b_1+a_0}} \mathrm{d}w \\ &:= \int_{v_{n-1}}^{v_n} Q(w) k(w) D(w) \mathrm{d}w, \end{aligned} \tag{52}$$

where

$$D(w) = \frac{h\left(F^{-1}(w)\right) k\left(F^{-1}(w)\right) \left(F^{-1}(w)\right)'}{k(w) e^{b_1+a_0}}. \tag{53}$$

Since $\lim_{w\longrightarrow 1} \left(F^{-1}(w)\right)' = m^{-1}$, it follows promptly that $D(w)$ satisfying

$$\lim_{w\longrightarrow 1} D(w) = e^{-(b_1+a_0+mr)}. \tag{54}$$

By hypothesis $e^{b_1+a_0+mr} > 1$ and hence for any $\lambda \in \left(e^{-(b_1+a_0+mr)}, 1\right)$, we can pick an n_0 such that $D(w) < \lambda$ for all $w \geq t_{n_0}$. Hence, for $n \geq n_0$,

$$I_n \leq \lambda I_{n-1}, \tag{55}$$

which implies

$$\sum_{n=n_0}^{\infty} I_n \leq I_{n_0} \sum_{j=0}^{\infty} \lambda^j < \infty, \tag{56}$$

and hence

$$\int_{t_{n_0}}^1 Q(v) k(v) \mathrm{d}v < \infty. \tag{57}$$

Applying $\int_0^t Q(v) k(v) \mathrm{d}v < \infty$ for $0 < t < 1$ together with (57) implies (51). □

Corollary 1. *Assume that* $E[Y^{2r+\delta}(1)] < \infty$ *for some* $r \geq 1$ *and* $\delta > 0$ *such that* $mr > -(b_1+a_0)$. *Then, (39) holds.*

Proof. By Markov's inequality, we have

$$\begin{aligned} ((v,l,\varepsilon) &= P\left(\left|\overline{Y}_l(v) + \frac{K(v)}{l} - e^{mv}\right| > \varepsilon\right) \\ &\leq \frac{E\left[\sqrt{l}\,\left(\overline{Y}_l(v) + (K(v)/l) - e^{mv}\right)\right]^{2r}}{\varepsilon^{2r} l^r}. \end{aligned} \tag{58}$$

According to the assumption,

$$\widetilde{C}_\varepsilon = \sup_l E\left[\sqrt{l}\left(\overline{Y}_l(v) + \frac{K(v)}{l} - e^{mv}\right)\right]^{2r}, \tag{59}$$

is finite, and then there exists C_ε s.t. $\phi(v,l,\varepsilon) \leq C_\varepsilon l^{-r}$ for all $l \geq 1$. Applying Theorem 2, the proof is completed. □

Theorem 3. *Assume that* $E[e^{\theta_0 Y(v)} | Y(0) = 1] < \infty$ *for some* $v > 0$ *and* $\theta_0 > 0$. *Then, there exists* $\theta_1 > 0$ *satisfying*

$$C_1 = \sup_{t\geq 0} E\left[e^{\theta_1 Z(t)}\right] < \infty. \tag{60}$$

Proof. In general, we suppose that $K = G(w_0) < \infty$ for $w_0 = e^{\theta_0} > 1$. Firstly, we prove that

$$L = \prod_{l=1}^{\infty} h(k(w_0,l)) < \infty. \tag{61}$$

Since $\sum_{l=1}^{\infty} [h(k(w_0,l)) - 1] < \infty$ is equivalent to $\prod_{l=1}^{\infty} h(k(w_0,l)) < \infty$, we have

$$\sum_{l=1}^{\infty}[h(k(w_0,l))-1]\le\sum_{l=1}^{\infty}h'(k(w_0))(k(w_0,l)-1)$$
$$=h'(k(w_0))\cdot\sum_{l=1}^{\infty}[k(w_0,l)-1]\coloneqq C_5<\infty. \tag{62}$$

Hence, (61) is proved since $k(w_0,l)-1\sim(R(w_0)/e^{ml})$. By Proposition 3, note that for any $t\ge 1$,

$$G(v,t)\le \mathrm{KL},\quad \text{if } 0\le v\le k(w_0,t-1). \tag{63}$$

Indeed, if $0\le v\le k(w_0,t-1)$, by Proposition 3,

$$G(v,t)\le G(k(w_0,t-1),t)$$
$$\le G(w_0)\cdot\prod_{l=1}^{[t-1]}h(k(w_0,l))\le \mathrm{KL}. \tag{64}$$

But, by definition of $Z(t)$,

$$E\left[e^{\theta Z(t)}\middle|Y(0)=1\right]=E\left[e^{\theta e^{-mt}[Y(t)-((e^m(t+1))1)/e^m-1)e^{a+m}]}\middle|Y(0)=1\right]\le G\left(e^{\theta e^{-mt}},t\right), \tag{65}$$

and hence the above inequality becomes

$$E\left[e^{\theta Z(t)}|Y(0)=1\right]\le \mathrm{KL}, \tag{66}$$

since $\theta\le e^{mt}\log k(w_0,t-1)$. Hence, $k(w_0,t)\downarrow 1$ as $t\uparrow\infty$ since $w_0>1$. Finally, by Proposition 2,

$$\lim_{t\longrightarrow\infty}e^{mt}\log k(w_0,t-1)$$
$$=\lim_{t\longrightarrow\infty}e^{mt}\left(k(w_0,t-1)-1\right)=e^mR(w_0), \tag{67}$$

which is positive and finite. Therefore, we can find $\theta_1>0$ such that

$$C_1=\sup_{t\ge 0}E\left[e^{\theta_1 Z(t)}\right]<\infty. \tag{68}$$

□

Theorem 4. *Assume that $E[e^{\theta_0 Y(v)}|Y(0)=1]<\infty$ for some $v>0$ and $\theta_0>0$. Then, there exist constants C_2 and $\lambda>0$ satisfying*

$$P(|Z(t)-Z|>\varepsilon)\le C_2e^{-\lambda\varepsilon^{2/3}e^{mt/3}}. \tag{69}$$

Proof. According to Theorem 3 and the construction of $Z(t)$, we know that there exists $\theta_1>0$ such that $\phi(\theta)=E[e^{\theta Z}]=E[e^{\theta(U+I)}]<\infty$ for all $\theta\le\theta_1$ which implies $\phi_1(\theta)=E[e^{\theta U}]<\infty, \phi_2(\theta)=E[e^{\theta I}]<\infty$ for all $\theta\le\theta_1$. If $\{U^{(i)};i\ge 1\}$ are i.i.d. copies of U and $S_n=\sum_{i=1}^n[U^{(i)}-1]$, I and $I^{(1)}$ have the identical distribution, then for any $\theta\le\theta_1$,

$$E\left[e^{\theta((S_n+I^{(1)}+e^{a+m})/\sqrt{n})}\right]=E\left[e^{\theta\left(\sum_{i=1}^n[U^{(i)}-1]+I^{(1)}+e^{a+m}/\sqrt{n}\right)}\right]$$
$$=\left(\phi_1\left(\frac{\theta}{\sqrt{n}}\right)e^{-(\theta/\sqrt{n})}\right)^n$$
$$\cdot E\left[e^{\theta\left(I^{(1)}+e^{a+m}/\sqrt{n}\right)}\right]$$
$$=\left(1+\frac{1}{n}\frac{\phi_1(\theta/\sqrt{n})e^{-(\theta/\sqrt{n})}-1}{\left(\theta^2/n\right)}\theta^2\right)^n\cdot E\left[e^{\theta\left(I^{(1)}+e^{a+m}/\sqrt{n}\right)}\right]. \tag{70}$$

Note that

$$\lim_{w\longrightarrow 0}\frac{\phi_1(w)e^{-w}-1}{w^2}=\frac{1}{2}\mathrm{Var}(U)<\infty, \tag{71}$$

we write

$$\lim_{|w|\le 1}\left|\frac{(\phi_1(w)e^{-w}-1)}{w^2}\right|=:c,$$
$$E\left[e^{\theta\left(I^{(1)}+e^{a+m}\right)/\sqrt{n}}\right]\le E\left[e^{\theta\left(I^{(1)}+e^{a+m}\right)}\right]=:C. \tag{72}$$

Now, let $\theta_2=\min(\theta_1,1)$, then

$$\sup_{|\theta|\le\theta_2}\left[\phi_1\left(\frac{\theta}{\sqrt{n}}\right)e^{-(\theta/\sqrt{n})}\right]^n\le e^c=:L<\infty, \tag{73}$$

since $(1+x/n)^n\le e^x$ for $x>0$. Therefore,

$$E\left[e^{\theta((S_n+I^{(1)}+e^{a+m})/\sqrt{n})}\right]\le \mathrm{CL}=:Q<\infty. \tag{74}$$

Observe that

$$\begin{aligned}
Z - Z(t) &= \lim_{v\longrightarrow\infty}(Z(t+v) - Z(t)) = \lim_{v\longrightarrow\infty} e^{-mt}\left[\frac{Y(t+v)}{e^{mv}} - Y(t) - e^{a+m}\left(\frac{1}{e^{m}} + \frac{1}{e^{2m}} + \cdots + \frac{1}{e^{vm}} + \frac{1}{e^{(v+1)m}} + \cdots\right)\right.\\
&\quad\left. + e^{a+m}\left(\frac{1}{e^{m(v+1)}} + \cdots\right)\right] = \lim_{v\longrightarrow\infty} e^{-mt}\left[\frac{Y(t+v)}{e^{mv}} + e^{a+m}\left(\frac{1}{e^{m(v+1)}} + \cdots\right)\right]\\
&\quad + \lim_{v\longrightarrow\infty} e^{-mt}\left[-Y(t) - e^{a+m}\left(\frac{1}{e^{m}} + \frac{1}{e^{2m}} + \cdots + \frac{1}{e^{vm}} + \frac{1}{e^{m(v+1)}} + \cdots\right)\right]\\
&= \lim_{v\longrightarrow\infty} e^{-mt}\left[\frac{\sum_{j=1}^{Y(t)}\xi_{t,j}(v) + K(v)}{e^{mv}} - \frac{e^{a+m}\left(e^{m(v+1)} - 1\right)}{e^{vm}\left(e^{m} - 1\right)}\right]\\
&\quad + e^{-mt}\left(-Y(t) - \frac{e^{a+m}}{e^{m} - 1} + \frac{e^{a+m}e^{m}}{e^{m} - 1}\right) = e^{-mt}\left(\sum_{j=1}^{Y(t)} U^{(j)} + I^{(1)} - Y(t) + e^{a+m}\right),
\end{aligned} \tag{75}$$

where $\xi_{t,j}(v)$ is the population size at time $s+t$ of the jth particle among the $Y(t)$ particles, $U^{(j)}$ is the limit random variable in the descent line of the jth original parent at time t, and $I^{(1)}$ and I are identically distributed. Putting the conditional independence into consideration,

$$P\left(|Z - Z(t)| \geq \varepsilon | \xi_t\right) = \psi\left(Y(t), e^{mt}\varepsilon\right), \tag{76}$$

where

$$\psi(l, r) = P\left(\sum_{j=1}^{l}\left(U^{(j)} - 1\right) + I^{(1)} + e^{a+m} \geq r\right). \tag{77}$$

However,

$$\begin{aligned}
P\left(S_l + I^{(1)} + e^{a+m} \geq r\right) &= P\left(\frac{S_l + I^{(1)} + e^{a+m}}{\sqrt{l}} \geq \frac{r}{\sqrt{l}}\right)\\
&\leq E\left[e^{\theta_2\left(\left(S_k + I^{(1)} + e^{a+m}\right)/\sqrt{l}\right)}\right] e^{-\theta_2(r/\sqrt{l})} \leq C_5 e^{-\theta_{-\theta_2(r/\sqrt{l})}\frac{r}{\sqrt{l}}}.
\end{aligned} \tag{78}$$

Thus,

$$\begin{aligned}
P(Z - Z(t) \geq \varepsilon) &= E\left[\psi\left(Y(t), e^{mt}\varepsilon\right)\right]\\
&\leq C_5 E\left[e^{-\theta_2\left(e^{mt}\varepsilon/\sqrt{Y(t)}\right)}\right] = C_5 E\left[e^{-\theta_2 e^{(mt/2)\varepsilon}\left(1/\sqrt{Z(t) + e^{a+m}\left(1 + (1/e^{m}) + \left(1/e^{2m}\right) + \cdots + (1/e^{m})\right)}\right)}\right].
\end{aligned} \tag{79}$$

For arbitrary $\lambda > 0$,

$$\begin{aligned}
E\left[e^{-\lambda\left(1/\sqrt{Z(t) + e^{a+m}\left(1 + (1/e^{m}) + \left(1/e^{2m}\right) + \cdots + (1/e^{m})\right)}\right)}\right] &= \lambda\int_0^{\infty} e^{-\lambda y} P\left(\frac{1}{\sqrt{Z(t) + e^{a+m}\left(1 + (1/e^{m}) + \left(1/e^{2m}\right) + \cdots + (1/e^{m})\right)}} \leq y\right) \mathrm{d}y\\
&= \lambda\int_0^{\infty} e^{-\lambda y} P\left(Z(t) + e^{a+m}\left(1 + \frac{1}{e^{m}} + \frac{1}{e^{2m}} + \cdots + \frac{1}{e^{tm}}\right) \geq y^{-2}\right) \mathrm{d}y\\
&= \lambda\int_0^{\infty} e^{-\lambda y} P\left(e^{\theta_1\left[Z(t) + te^{a+m}n\left(1 + (1/e^{m}) + \left(1/e^{2m}\right) + \cdots + (1/e^{m})\right)\right]} \geq e^{\theta_1 y^{-2}}\right) \mathrm{d}y\\
&\leq \lambda C_6 \int_0^{\infty} e^{-\lambda y} e^{-\theta_1 y^{-2}} \mathrm{d}y\\
&= C_6 \int_0^{\infty} e^{-y} e^{-\theta_1 \lambda^2 y^{-2}} \mathrm{d}y.
\end{aligned} \tag{80}$$

Hence,

$$P(Z - Z(t) \geq \varepsilon) \leq C_5 C_6 \int_0^{\infty} e^{-y} e^{-\theta_1 \lambda_t^2 y^{-2}} \mathrm{d}y, \tag{81}$$

where $\lambda_t = \theta_2 \varepsilon e^{mt/2}$. But, for any $\lambda > 0$,

$$\begin{aligned} I(\lambda) &= \int_0^{\infty} e^{-y} e^{-\lambda^2 y^{-2}} \mathrm{d}y \\ &= \int_0^{l(\lambda)} e^{-y} e^{-\lambda^2 y^{-2}} \mathrm{d}y + \int_{l(\lambda)}^{\infty} e^{-y} e^{-\lambda^2 y^{-2}} \mathrm{d}y \\ &\leq e^{-(\lambda^2 / l(\lambda))} + e^{-l(\lambda)}. \end{aligned} \tag{82}$$

Now, choose $l(\lambda) = \lambda^{(2/3)}$, we have $I(\lambda) \leq 2e^{-\lambda^{(2/3)}}$. Therefore,

$$\begin{aligned} P(|Z - Z(t)| \geq \varepsilon) &\leq 2C_5 C_6 e^{-\left(\sqrt{\theta_1}\, \theta_2 \varepsilon e^{(mt/2)}\right)^{(2/3)}} \\ &= C_7 e^{-\lambda \varepsilon^{(2/3)} e^{(mt/2)}}, \end{aligned} \tag{83}$$

where $\lambda = (\sqrt{\theta_1}\, \theta_2)^{(2/3)}$. Similar arguments still hold for $P(Z - Z(t) \leq \varepsilon)$.

Under the condition $Z \geq \alpha\, (\alpha > 0)$, the following theorem proves that the decay rate of $P(|(Y(t+v)/Y(t)) - e^{mv}| > \varepsilon)$ is also supergeometric. □

Theorem 5. *Assume that* $E[e^{\theta_0 Y(v)} | Y(0) = 1] < \infty$ *for some* $v > 0$ *and* $\theta_0 > 0$. *Then, there exist constants* $\lambda > 0$ *and* C_3 *such that for any* $\alpha > 0$, $\varepsilon > 0$, *we have* $0 < I(\varepsilon) < \infty$ *satisfying*

$$\begin{aligned} P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon | Z \geq \alpha \right) &\leq C_3 e^{-\alpha \eta I(\varepsilon) e^{mt}} \\ &+ C_2 e^{-\lambda [\alpha(1-\eta)]^{(2/3)} e^{(mt/3)}}, \end{aligned} \tag{84}$$

for $0 < \eta < 1$.

In particular, when $\eta = 1/2$,

$$P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon | Z \geq \alpha \right) \leq C_4 e^{-\lambda (\alpha/2)^{(2/3)} e^{(mt/3)}}. \tag{85}$$

Proof. Note that

$$\begin{aligned} &P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon | Z \geq \alpha \right) \\ &= P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon, Z \geq \alpha \right) \cdot \frac{1}{P(Z \geq \alpha)} \\ &:= (\alpha_{t1} + \alpha_{t2}) p_\alpha, \end{aligned} \tag{86}$$

where $0 < \eta < 1$, $p_\alpha = (1/P(Z \geq \alpha))$, $\alpha_{t1} = P(|(Y(t+v)/Y(t)) - e^{mv}| > \varepsilon, Z(t) \leq \alpha\eta, Z \geq \alpha)$, and $\alpha_{t2} = P(|(Y(t+v)/Y(t)) - e^{mv}| > \varepsilon, Z(t) \geq \alpha\eta, Z \geq \alpha)$. Clearly, as shown in Theorem 4,

$$\alpha_{t1} \leq P(Z - Z(t) \geq \alpha(1-\eta)) \leq C_7 e^{-\lambda [\alpha(1-\eta)]^{(2/3)} e^{(mt/3)}}. \tag{87}$$

On the other hand,

$$\begin{aligned} \alpha_{t2} &\leq P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon, Z(t) \geq \alpha\eta \right) \\ &= \sum_l P\left(\left| \frac{Y(t+v)}{Y(t)} - e^{mv} \right| > \varepsilon, Z(t) \geq \alpha\eta | Y(t) = l \right) \cdot P(Y(t) = l) \\ &= \sum_l P\left(\left| \frac{Y(t+v)}{l} - e^{mv} \right| > \varepsilon, l \geq \alpha\eta e^{mt} + \frac{e^{m(t+1)} - 1}{e^m - 1} e^{a+m} \right) \cdot P(Y(t) = l) \\ &\leq \sum_{l \geq \alpha\eta e^{mt}} P\left(\left| \frac{\sum_{j=1}^{l} \xi_{t,j}(v) + K(v)}{l} - e^{mv} \right| > \varepsilon \right) \cdot P(Y(t) = l) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l\geq \alpha\eta e^{mt}} P\left(\left|\frac{\sum_{j=1}^{l}\left(\xi_{t,j}(v)-e^{mv}\right)}{l}\right|+\frac{K(v)}{l}>\varepsilon\right)\cdot P(Y(t)=l)\\
&\leq \sum_{l\geq \alpha\eta e^{mt}} \left[P\left(\left|\frac{\sum_{j=1}^{l}\left(\xi_{t,j}(v)-e^{mv}\right)}{l}\right|>\frac{\varepsilon}{2}\right)\cdot P(Y(t)=l)+P\left(\frac{K(v)}{l}>\frac{\varepsilon}{2}\right)\cdot P(Y(t)=l)\right]\\
&\leq C_8 e^{-\alpha\eta I_1(\varepsilon)e^{mt}}+\sum_{l\geq \alpha\eta e^{mt}} P\left(\frac{K(v)}{l}>\frac{\varepsilon}{2}\right)\cdot P(Y(t)=l).
\end{aligned} \tag{88}$$

The existence of C_8 and $I_1(\varepsilon)$ follows from Chernoff type bounds since $E[e^{\theta_1 Y(v)}]<\infty$. Moreover,

$$P\left(\frac{K(v)}{l}>\frac{\varepsilon}{2}\right)\leq E\left[e^{\theta_0 K(v)}\right]\cdot e^{-\left(\theta_0 l\varepsilon/2\right)}\leq C_9 e^{-\left(\theta_0\varepsilon\alpha\eta e^{mt}/2\right)}. \tag{89}$$

Therefore, we can choose $I(\varepsilon)$ and constant C_{10} satisfying

$$\alpha_{t2}\leq C_{10}e^{-\alpha\eta I(\varepsilon)e^{mt}}. \tag{90}$$

Hence,

$$\begin{aligned}
&P\left(\left|\frac{Y(t+v)}{Y(t)}-e^{mv}\right|>\varepsilon|Z\geq\alpha\right)\\
&\leq p_\alpha\left(C_7 e^{-\lambda[\alpha(1-\eta)]^{(2/3)}e^{(mt/3)}}+C_{10}e^{-\alpha\eta I(\varepsilon)e^{mt}}\right).
\end{aligned} \tag{91}$$

In particular, for $\eta=(1/2)$, there exists C_4 such that

$$P\left(\left|\frac{Y(t+v)}{Y(t)}-e^{mv}\right|>\varepsilon|Z\geq\alpha\right)\leq C_4 e^{-\lambda(\alpha/2)^{(2/3)}e^{mt/3}}, \tag{92}$$

since the first term α_{t1} tends to 0 faster than α_{t2}. □

4. Concluding Remarks

In this paper, we studied the evolutionary structure of a supercritical branching process with immigration through the behavior of conditional limits under various notions of "information" about the current population size. A natural next question concerns the large deviation rates of high-dimensional branching processes under other "information" notions. These and other related issues will be investigated in subsequent papers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was substantially supported by the National Natural Sciences Foundation of China (No. 11901392).

References

[1] K. B. Athreya and P. E. Ney, *Branching Processes*, Springer, Berlin, Germany, 1972.

[2] T. E. Harris, *The Theory of Branching Processes*, Springer, Berlin, Germany, 1963.

[3] R. Karp and Y. Zhang, "Tail probabilities for finite supercritical braching processes," Technical Report, Southern Methodist University, Dallas, TX, USA, 1983.

[4] H. I. Miller and J. A. O'sullivan, "Entropies and conbinatorics of random branching processes and context free languages," *IEEE Transaction on Information Theory*, vol. 38, pp. 1292–1311, 1992.

[5] C. C. Heyde, "Extension of a result of Senata for the supercritical Galton-Watson processes," *Annals of Mathematical Statistics*, vol. 41, no. 2, pp. 739–742, 1970.

[6] K. B. Athreya, "Large deviation rates for branching processes–I. Single type case," *Annals of Applied Probability*, vol. 4, no. 3, pp. 779–790, 1994.

[7] E. Seneta, "A note on the supercritical Galton-Watson process with immigration," *Mathematical Biosciences*, vol. 6, pp. 305–311, 1970.

[8] A. Roitershtein A, "A note on multitype branching processes with immigration in a random environment," *Annals of Probability*, vol. 35, no. 4, pp. 1573–1592, 2007.

[9] J. N. Liu and M. Zhang, "Large deviation for supercritical branching processes with immigration," *Acta Mathematica Sinica, English Series*, vol. 32, no. 8, pp. 893–900, 2016.

[10] Q. Sun and M. Zhang, "Harmonic moments and large deviations for supercritical branching processes with immigration," *Frontiers of Mathematics in China*, vol. 12, no. 5, pp. 1201–1220, 2017.

[11] X. Chen and H. He, "Lower deviation and moderate deviation probabilities for maximum of a branching random walk," *Annales de l'Institut Henri Poincare Probabilites et Statistiques*, vol. 56, no. 4, pp. 2507–2539, 2020.

[12] R. Abraham, J. F. Delmas, and H. He, "Some properties of stationary continuous state branching processes," *Stochastic Processes and their Applications*, vol. 141, pp. 309–343, 2021.